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BAIRE ONE STAR FUNCTIONS

1. Introduction

The class of Baire-one-star functions has already been studied in real analysis for a long time, under various names and from different points of view. The impetus for writing this paper mainly comes from two sources. First, the important role these functions played in the recent studies of finely continuous functions. (See [1], [2] and the references there.) Secondly, it stems from the fact that the connections between different notions of a function "being better than a Baire-one function" seem not to be known to the extend they should be. Before explaining this impression in detail, we must introduce the precise definitions:

Definition 1 *Let (X,ρ) and (Y,σ) be metric spaces. A map $f : X \to Y$ is said to be*

(i) *a **first Borel class function** (written $f \in \mathcal{B}_1(X,Y)$) if the f-preimage of any* open *subset of Y is of type $\mathbf{F}_\sigma$ in X , see [6].*

(ii) *a **first level Borel function** if the f-preimage of any* closed *subset of Y is of type $\mathbf{F}_\sigma$ in X , see [6].*

(iii) *a **Baire-one-star** function (written $f \in \mathcal{B}_1^*(X,Y)$) if for any $F \subset X$ nonempty closed there is a nonvoid $U \subset F$ relatively open in F such that the restriction of f to F is continuous on U, see [10].*

(iv) *a **piecewise continuous** function if there exist closed subsets $\{F_i\}_{i=1}^\infty$ of X such that $\bigcup_{i=1}^\infty F_i = X$ and that any of the restrictions $f_{|F_i}$ is continuous, see [6]. (Of course, one can always assume $F_i \subset F_{i+1}$.)*

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It is easy to see, but worth mentioning, that in (iii) the assumption that F is closed can be dropped. Moreover, using, for example, some ideas from below, it can be seen immediately that $f \in \mathcal{B}_1^*$ iff for any $\emptyset \neq F \subset X$ the restriction $f_{|F}$ has at least one point of continuity.

In [11] the author considered first level Borel functions $f : [0,1] \to \mathbb{R}$ having the Darboux property. A special case of [6] implies that on $[0,1]$ first level Borel functions are piecewise continuous and it follows rather easily that the functions from [11] are Darboux Baire-one-star functions. (See, for example, Theorem 13 in [3].) Consequently, the result of [11] is, in a stronger form, contained in Theorem 4 of [10]. We will return to this question in the last part of our paper in more generality.

As already indicated, the paper [6] contains several very general results about the relationship between the notions just defined, we only quote the main one.

Theorem 1.1 [6] *A first level Borel function, mapping a metric space, X, that is an absolute Souslin-$\mathcal{F}$ set, into a metric space, Y, is piecewise continuous.*

Note, that any Borel (or even analytic) subset of an arbitrary complete metric space is, considered as a metric subspace, an absolute Souslin-$\mathcal{F}$ set. However, the generality of the situation considered in [6] demands a rather complicated proof based on techniques from descriptive set theory. We shall give here a self contained and relatively easy proof of this result for the case that the domain space X is complete, see Theorem 5 below. Moreover, a minor modification of our proof gives a very general version of the results from [1] and [2] about the Baire-one-starness of continuous functions between fine topologies.

Besides the paper [6], there are many other works containing valuable information about Baire-one-star functions, we mention here only [9] including some remarks about the history of the subject (see chap.2.D. Exercises and Remarks), further [3], [4], and [10] where special properties of Darboux functions in $\mathcal{B}_1^*(\mathbb{R},\mathbb{R})$ are derived. Our whole exposition is probably related to further papers, however, we did our best in referring to all significant sources.

We shall use the following notations. For A a set in a metric space $B(A,r)$ $[U(A,r)]$, $r \geq 0$, denote the set of all points having distance from A at most r [less than r]. So $B(x,r), U(x,r)$ are the closed and open balls. If f is a mapping between topological spaces, we denote by $C(f)$ the set of all points in the domain space at which f is continuous and by $D(f)$ the set of all such $x \in X$ for which the restriction $f_{|\{x\}\cup C(f)}$ is discontinuous. Let $\{z_n\}_{n\geq 1}$ be a sequence in a topological space (Z,τ), then we define clus $(\{z_n\}) = \bigcap_{m\geq 1} \mathrm{cl}\ \{z_n \mid n \geq m\}$

which is of course nonvoid provided Z is compact. The most advanced topological topics we will use are paracompactness and Čech-Stone-compactification, however, see Remark 6. Let (Y, σ) be a bounded metric space and βY its Čech-Stone-compactification, see [8]. Then we uniquely define the extension $\tilde{\sigma} : \beta Y \times Y \to \mathbb{R}$ of its metric σ by the request that $\tilde{\sigma}$ has to be separately continuous in the first entry. Since βY is an Hausdorff space, one easily finds $\tilde{\sigma}(x, y) > 0$ whenever $x \in \beta Y \setminus Y$.

2.

Lemma 2.1 *Let (X, ρ) and (Y, σ) be metric spaces, the first of them being complete, and let $f : X \to Y$ belong to the first Borel class.*

(i) *Then $C(f)$ is dense and hence residual in X.*

(ii) *If $\text{int}\, C(f) = \emptyset$, then the set $f(U \cap D(f))$ is infinite whenever $U \subset X$ is open and nonvoid.*

PROOF.

(i) This statement is a classical one. (See for example, [12] for a stronger version, where it is proved that f is Baire one provided that Y is a Banach space. Note that we can always embed Y in such a space.), But for the sake of completeness, we will give an independent proof. If $X \setminus \text{cl}\, C(f) \neq \emptyset$, then Baire's theorem ensures the existence of an $\varepsilon > 0$ such that the closed $\{x \mid (f, x) \geq 3\varepsilon\}$ is somewhere dense. Hence, $(f, \cdot) \geq 3\varepsilon$ on some U which is nonvoid and open. Therefore, for any $y \in Y$ the $\mathbf{F}_\sigma$-set $U \cap f^{-1}(U(y, \varepsilon))$ has empty interior and is of the first category. Again the category argument implies that for any countable set $M \subset X$ and for any $\emptyset \neq V \subset U$ which is open, there is an $x \in V$ with $(f(x), f(M)) \geq \varepsilon$. This ensures, that one can easily find a sequence $\{x_i\}_{i=1}^{\infty} \subset U$ such that for all $i \neq j$ $\sigma(f(x_i), f(x_j)) \geq \varepsilon$ and $\rho(x_i, x_{i+2^n}) \leq 2^{-n}$ if $1 \leq i \leq 2^n, n \geq 0$. Then $S = \{x_i \mid i \geq 1\}$ is countable without isolated points and $f(S)$ is closed in Y. Therefore,

$$S = f^{-1}(f(S)) \setminus \bigcup_{x \in S} [f^{-1}(U(f(x), \frac{\varepsilon}{2})) \setminus \{x\}]$$

is a $\mathbf{G}_\delta$-set minus an $\mathbf{F}_\sigma$-set, and hence of type $\mathbf{G}_\delta$. But then S is simultaneously residual and of the first category in the perfect and complete metric subspace $\text{cl}\, S$ of X.

(ii) First, we show that $\text{cl}\, D(f) = X$. Else, we have $B(x, \varepsilon) \cap D(f) = \emptyset$ for some x and ε with $0 < \varepsilon < (f, x)/2$. Since $x \notin D(f)$, we get $f(C(f) \cap B(x, \delta)) \subset B(f(x), \varepsilon)$ for suitable $\delta > 0$. However, there is a $y \in U(x, \delta)$ with

$(f(y), B(f(x), \varepsilon)) > 0$. Because y is a cluster point of $C(f) \cap U(x, \delta)$, we infer $y \in D(f)$.

Next, let $f(D(f) \cap U(x, \varepsilon)) = \{y_1, \ldots, y_n\}$ for some $x \in X, \varepsilon > 0$. Since due to the foregoing $f(C(f) \cap U(x, \varepsilon)) \subset \text{cl } (f(D(f) \cap U(x, \varepsilon)))$ and $f(U(x, \varepsilon) \setminus D(f)) \subset \text{cl } f(C(f) \cap U(x, \varepsilon))$ by (i) and the definition of $D(f)$, we infer $f(U(x, \varepsilon)) \subset \{y_1, \ldots, y_n\}$. But then any level set $U(x, \varepsilon) \cap f^{-1}(y_k)$ is not only of type $\mathbf{G}_\delta$, but also an $\mathbf{F}_\sigma$-set. We find a $k \in \{1, \ldots, n\}$ such that $f^{-1}(y_k) \cap U(x, \varepsilon)$ is, like $C(f) \cap U(x, \varepsilon)$ itself, a second category set in X. Therefore, int $C(f) \supset \text{int } f^{-1}(y_k) \neq \emptyset$, which is a contradiction, finishing the proof.

Proposition 2.2 *Let (X, ρ) and (Y, σ) be metric spaces, the first of them being nonempty and complete, and the second being bounded. If $f : X \to Y$ is a first level Borel function, then $C(f)$ has nonvoid interior.*

PROOF. Assume the opposite to be true, i.e. int $C(f) = \emptyset$. We will inductively construct a sequence of quadruples

$$\{(x_i, \{x_{i,n}\}_{n=1}^{\infty}, \varepsilon_i, a_i)\}_{i=1}^{\infty}$$

such that

(i) always $x_i \in D(f)$; $x_{i,n} \in C(f)$; $\varepsilon_i \in (0, \infty)$; and $a_i \in \beta Y$.

(ii) If $i \geq 1$, then $\lim_{n \to \infty} x_{i,n} = x_i$, $a_i \in \text{clus}_{\beta Y} (\{f(x_{i,n})\}_{n=1}^{\infty})$, and $\sigma(f(x_{i,n}), f(x_i)) \geq 2\varepsilon_i$ whenever $n \geq 1$.

(iii) For $1 \leq i \neq j$ both $\sigma(f(x_i), f(x_j)) > \varepsilon_i + \varepsilon_j$ and $\tilde{\sigma}(a_j, f(x_i)) > \varepsilon_i$ hold.

(iv) $\rho(x_i, x_{i+2^k}) \leq 2^{-k}$ whenever $k \geq 0$ and $1 \leq i \leq 2^k$.

For this purpose, suppose that such quadruples fulfilling (i), ... ,(iv)[1] have already been chosen for $1 \leq i < 2^k + j$, where $k \geq 0$ and $1 \leq j \leq 2^k$. Indeed, due to Lemma 2.1 there are no problems in finding suitable $x_1, \{x_{1,n}\}_{n=1}^{\infty}, \varepsilon_1$, and a_1. We put $M = \bigcup\{B(f(x_i), \varepsilon_i) \mid i < 2^k + j\}$, then $a_j \notin cl_{\beta Y} M$ according to (ii),(iii) and the continuity of the maps $\tilde{\sigma}(\cdot, f(x_i))$ on βY. Therefore, we can find $x = x_{j,n} \in U(x_j, 2^{-k-1}) \cap C(f)$ with $f(x) \notin M$. We fix a $\delta \in (0, 2^{-k-1})$ such that $\delta < (f(B(x, \delta)), M)$[2]. Again Lemma 2.1 ensures the existence of a point $x_{2^k+j} \in D(f) \cap U(x, \delta) \subset B(x_j, 2^{-k})$ satisfying $f(x_{2^k+j}) \notin \{a_i \mid i < 2^k + j\}$. Next, we select an $\varepsilon_{2^k+j} \in (0, \delta)$ such that

- $\varepsilon_{2^k+j} < \tilde{\sigma}(a_i, f(x_{2^k+j}))$ for $1 \leq i < 2^k + j$; and that

[1] whenever all symbols which appear are well defined

[2] In the second part of the paper we will deal with certain modifications of this proof putting some additional conditions on x and δ.

- there is a sequence $\{x_{2^k+j,n}\}_{n=1}^{\infty} \subset C(f) \cap U(x,\delta)$ with $\lim_{n\to\infty} x_{2^k+j,n} = x_{2^k+j}$ and $\sigma(f(x_{2^k+j}), f(x_{2^k+j,n})) \geq 2\varepsilon_{2^k+j}$ for $n \geq 1$.

Finally, we pick up any $a_{2^k+j} \in \text{clus}_{\beta Y}(\{f(x_{2^k+j,n})\}_{n=1}^{\infty})$. It is easy to see that the quadruples $(x_i, \{x_{i,n}\}_{n=1}^{\infty}, \varepsilon_i, a_i)$, now defined for $1 \leq i \leq 2^k + j$, fulfill (i)–(iv) again. (Notice that $a_{2^k+j} \in cl_{\beta Y} f(U(x,\delta)) \subset \bigcap_{i<2^k+j}\{z \in \beta Y \mid \tilde{\sigma}(z, f(x_i)) \geq \varepsilon_i + \delta\}$.) Consequently, we may assume the existence of the desired sequence of quadruples. (It is worth mentioning[3] that until now we have only used $f \in \mathcal{B}_1$, but never that f is a first level function.)

Let $G = \bigcup_{i=1}^{\infty} U(f(x_i), \varepsilon_i)$. Using (iii) one easily verifies that

$$\{x_i \mid i \geq 1\} = f^{-1}(G) \setminus \bigcup_{i=1}^{\infty}[f^{-1}(U(f(x_i),\varepsilon_i)) \setminus \{x_i\}].$$

As in the proof of Lemma 2.1 , but using the definition of first level Borel functions, we infer that $\{x_i \mid i \geq 1\}$ is a first category dense $\mathbf{G}_\delta$-subset of the perfect and complete metric space $\text{cl}(\{x_i \mid i \geq 1\})$. This contradiction shows that $\text{int}\, C(f) \neq \emptyset$.

Theorem 2.3 *Let (X,ρ) and (Y,σ) be metric spaces and $f : X \to Y$ a map. Consider the following statements:*

(i) *f is a first level Borel function.*

(ii) *f is a $\mathcal{B}_1^*$-function.*

(iii) *f is piecewise continuous.*

Then (ii)⇒(iii)⇒(i) and in case that (X,ρ) is complete, we have (i)⇒(ii).

PROOF. As before, we may assume σ to be bounded.

(i)⇒(ii) Since the restriction of f to any subspace of X is again a first level function there, the implication follows from Proposition 2.2.

(ii)⇒(iii) Let $\mathcal{G}$ be the family of all open U in X such that $f_{|U}$ is piecewise continuous. First, we show $\bigcup \mathcal{G} \in \mathcal{G}$. Since the metric space (G,ρ), where $G = \bigcup \mathcal{G}$, is paracompact ([8]), there is a family $\mathcal{F}$ of open sets with $\bigcup \mathcal{F} = G$, each $U \in \mathcal{F}$ is contained in some $V \in \mathcal{G}$, and for any $x \in G$ there is an $\varepsilon_x > 0$ such that the subfamily $\mathcal{F}_x = \{U \in \mathcal{F} \mid U \cap U(x,\varepsilon_x) \neq \emptyset\}$ is finite. Obviously, for any $U \in \mathcal{F}$ there are closed sets $F(U,n)$, $n \geq 1$, with $\bigcup_{n=1}^{\infty} F(U,n) = U$, $(F(U,n), X \setminus U) \geq 1/n$, and $f_{|F(U,n)}$ is continuous for each $n \geq 1$. Then each of the sets $F(n) = \bigcup\{F(U,n) \mid U \in \mathcal{F}\}$ is closed in X and $f_{|F(n)}$ is always continuous. Indeed, if $x \in G \setminus F(n)$, then $(x, F(n)) \geq$

[3]and will be used in the proof of Theorem 7

$\min(\{\varepsilon_x\}\cup\{ (x,F(U,n)) \mid U\in\mathcal{F}_x\})>0$. Furthermore, if $M\subset Y$ is closed and $x\in F(n)\setminus f^{-1}(M)$, then $(x,f^{-1}(M)\cap F(n))\geq\min(\{\varepsilon_x\}\cup\{ (x,F(U,n)\cap f^{-1}(M)) \mid U\in\mathcal{F}_x\})>0$. Hence, $(f_{|F(n)})^{-1}(M)$ is closed in $F(n)$. Since $\bigcup_{n=1}^{\infty}F(n)=G$, $G\in\mathcal{G}$.

Let $F=X\setminus G$. If $F\neq\emptyset$, then $f\in\mathcal{B}_1{}^*(X,Y)$ implies that there are $x\in F$ and $r>0$ such that $f_{|F\cap U(x,r)}$ is continuous. Obviously, $G\cup U(x,r)=\bigcup_{n=1}^{\infty}F(n)\cup[F\cap B(x,r-\frac{1}{n})]\in\mathcal{G}$. Hence $x\in G$. This contradiction shows that $F=\emptyset$, $X\in\mathcal{G}$, and we are done.

(iii)$\Rightarrow$(i) If $X=\bigcup_{n=1}^{\infty}F_n$ with each F_n closed and $f_{|F_n}$ continuous, then for any $F\subset Y$ closed $f^{-1}(F)=\bigcup_{n=1}^{\infty}(F_n\cap f^{-1}(F))$ is, as required in the definition of first level Borel functions, an $\mathbf{F}_\sigma$-set.

Remark. • As already mentioned, the implication (i)$\Rightarrow$ (iii) is due to [6]. The relation between (ii) and (iii) was investigated first in [3] and in a more general setting in [9]. Our proof of (ii) $\Rightarrow$ (iii) merely carries out Exercise 2.D.14.d of [9].

• Note that the "advanced" topological tools used here can be avoided in the separable cases. Indeed, if the space Y is separable, then we use the fact that it is homeomorphic to a completely bounded metric space $\tilde{Y}$ ([8]) and obviously Y can be replaced by $\tilde{Y}$ without changing any statement. But now, we can simply use the completion of $\tilde{Y}$ instead of $\beta\tilde{Y}$. The case of a separable X is even more simple. In this case the paracompactness, used only in the proof of (ii) $\Rightarrow$ (iii), can be replaced by the Lindelöf property of G.

• Using the foregoing ideas, mainly from the proof of Proposition 2.2, it is not difficult to show that $f:(X,\rho)\to(Y,\sigma)$ is in $\mathcal{B}_1^*(X,Y)$ iff for any (closed and) totally bounded $\emptyset\neq M\subset X$, $f_{|M}$ is continuous on some nonvoid relatively open subset of M. (Note, that one can show also that the restriction of f to any nonvoid closed subset has a point of continuity iff its restriction to any nonempty closed and totally bounded subset does so. However, the proof is a little bit longer.) Therefore, if we have (X,ρ) complete, then Theorem 2.3 implies that f is piecewise continuous iff its restriction to any compact subset is also. (Compare also with page 182 of [6].)

3.

Theorem 3.1 *Let (X,ρ) and (Y,σ) be metric spaces, the first of them being complete. Assume that both X and Y are equipped with topologies τ and η, respectively, finer than those induced by the metrics. Suppose that:*

(i) *For any $M\subset X$ there is a $\mathbf{G}_\delta$-set A with $\operatorname{int}_\tau M\subset A\subset\operatorname{cl}_\tau M$, and*

(ii) *For any disjoint $F,A\subset Y$ such that F is closed, A is countable and discrete (in the metric topology), there is a continuous $g:(Y,\eta)\to[0,1]$*

satisfying $g(F) \subset \{0\}$, *and* $g(A) \subset \{1\}$.

Then any continuous map $f : (X, \tau) \to (Y, \eta)$ *belongs to* $\mathcal{B}_1^*((X, \rho), (Y, \sigma))$.

PROOF. First, we show that $h \in \mathcal{B}_1((X, \rho), (Z, d))$ whenever (Z, d) is a metric space and $h : (X, \tau) \to (Z, d)$ is continuous. Indeed, let $F \subset Z$ be an arbitrary closed set. Then due to (i) we find for any $n \geq 1$ a G_δ-set $A_n \subset X$ with

$$\begin{aligned} h^{-1}(\{z \mid d(z, F) < \frac{1}{n}\}) &\subset \operatorname{int}_\tau h^{-1}(\{z \mid d(z, F) \leq \frac{1}{n}\}) \\ &\subset A_n \subset h^{-1}(\{z \mid d(z, F) \leq \frac{1}{n}\}). \end{aligned}$$

Obviously, $A = \bigcap_{n=1}^{\infty} A_n$ is G_δ and fulfills $A = h^{-1}(F)$, which was to be shown.

Now, let $f : (X, \tau) \to (Y, \eta)$ be continuous but not in $\mathcal{B}_1^*((X, \rho), (Y, \sigma))$. Then there is an $\tilde{X} \subset X$ nonvoid and closed such that $\tilde{f} = f_{|\tilde{X}}$ $\operatorname{int}_{\tilde{X}} C(\tilde{f}) = \emptyset$. We know from the foregoing that $\tilde{f}$ as well as $g \circ \tilde{f}$ (whenever $g : (Y, \eta) \to \mathbb{R}$ is continuous) is in the first Borel class (w.r.t. the metrics). Therefore, we can follow the proof of Proposition 2.2 (Compare with the last footnote there.) in order to find sequences $\{x_i\}_{i=1}^{\infty} \subset \tilde{X}$ and $\{\varepsilon_i\}_{i=1}^{\infty} \subset (0, \infty)$ such that $\rho(x_i, x_{i+2^k}) \leq 2^{-k}$ for $k \geq 0, 1 \leq i \leq 2^k$ and $\sigma(\tilde{f}(x_i), \tilde{f}(x_j)) \geq \varepsilon_i + \varepsilon_j$ if $i \neq j$. Now our assumption (ii) ensures that there is a continuous $g : (Y, \eta) \to [0, 1]$ with $g(\tilde{f}(x_i)) = 1$ for all $i \geq 1$ and $g(y) = 0$ whenever $y \in Y \setminus \bigcup_{i \geq 1} U(\tilde{f}(x_i), \varepsilon_i)$. Hence, the set $S = (g \circ \tilde{f})^{-1}[1/2, 1]$ is G_δ. Since $\{x_i \mid i \geq 1\} = S \setminus \bigcup_{i \geq 1}(\tilde{f}^{-1}(U(\tilde{f}(x_i), \varepsilon_i)) \setminus \{x_i\})$, we again obtain that $\{x_i \mid i \geq 1\}$ is a countable G_δ-set without isolated points. This leads to the already known contradiction and finishes the proof.

This theorem, although having a rather simple proof, almost directly applies to many different situations including the higher dimensional variants of the $\mathcal{B}_1^*$-results from [1],[2]. However, in the following discussion of the two assumptions made above we will see that a few modifications are still necessary. Assumption (i) is of course the *G_δ-insertion property* from Chap.2.D of [9], and is implied e.g. by the *essential radius condition* see Exercise 2.D.16 in [9]. In particular, (X, τ) can be any of the following spaces (for definitions see again [9]):

- the fine topology generated by an ideal not containing any open nonvoid set,
- the ordinary density topology (e.g. for $X = \mathbb{R}^n$) or any of its (coarser) modifications, e.g. O'Malley's topologies, strong density topology etc.,

- the *-porosity or porosity topology on (X, ρ), see [13] Theorem 3, (These are higher dimensional variants of the $\mathcal{I}$-density and deep-$\mathcal{I}$-density topology. See Corollaries 2.4.6. and 2.7.10 in [2].) and the *-strong porosity or strong porosity topology on (X, ρ) being an Hilbert space. (See [7] Theorem 2b.)

Assumption (ii) causes many more troubles. Actually, it is formulated in a way unnecessarily strong. Of course, it is related to the Lusin-Menchoff-Property of the fine topology η. (See Chap.3 of [9].) In fact, it is a consequence of the latter provided that any countable and metrically discrete set is η-closed. Therefore, we see that (Y, η) can be $\mathbb{R}^n$, $n \geq 1$, with the *ordinary density topology*. Note that we just obtained Theorem 3 of [1]. Unfortunately, the other topologies mentioned in the previous paragraph do not meet assumption (ii), even those having the Lusin-Menchoff-Property as O'Malley's a.e., or r-topology, or the porosity topology. (See Chap. 7.1 – 2. of [9] and Theorem 1b in [7].) In the sequel we will show how to modify the proof of Proposition 2.2 in order to apply also to these topologies.

Proposition 3.2 *Proposition Let (X, ρ) and τ be as in Theorem 3.1 and let (Y, η) be $\mathbb{R}^n$, $n \geq 1$, with the O'Malley a.e.- or r-topology. Then any continuous $f : (X, \tau) \to (\mathbb{R}^n, \eta)$ is in $\mathcal{B}_1^*((X, \rho), \mathbb{R}^n)$.*

PROOF. We proceed almost as before. Since f is in the first Borel class, we may assume that $f(\tilde{X})$ is bounded in $\mathbb{R}^n$, and hence that all a_j, $j \geq 1$, are chosen from $\mathbb{R}^n$. (See Remark 1.) An easy modification of the proof of Proposition 2.2 ensures that

$$\|f(x_{i+2^k}) - a_i\|, \|a_{i+2^k} - a_i\| \leq 3^{-k} \text{ for } k \geq 0 \text{ and } 1 \leq i \leq 2^k.$$

From this one derives by induction with respect to the integer part of $\log_2(l)$ that $\|a_{i+l\cdot 2^k} - a_i\| < 2 \cdot 3^{-k}$ for $l \geq 1$ (We will carry out similar estimates in more detail in the next section.) and in the very same manner one sees that for any $k \geq 0$

$$\tilde{A} = \{f(x_i)\}_{i=1}^{\infty} \subset \{f(x_j)\}_{j=1}^{2^k} \cup \bigcup_{j=1}^{2^k} B(a_j, 2 \cdot 3^{-k}).$$

Consequently, cl $\tilde{A}$ has Lebesgue measure zero. In the case of the a.e.-topology we simply write $A = \tilde{A}$ and for η being the r-topology we put $A = \text{cl}\,(\tilde{A}) \cap \bigcup_{i=1}^{\infty} U(f(x_i), \varepsilon_i)$. In any case, A is an η-closed set containing all $f(x_i)$'s and disjoint from $\mathbb{R}^n \setminus \bigcup_{i=1}^{\infty} U(f(x_i), \varepsilon_i)$. Since η has the Lusin-Menchoff-Property w.r.t. $(\mathbb{R}^n, \|\ \|)$, it now suffices to follow the proof of Theorem 3.1.

The remaining (*-)porosity topology demands a more careful study, and we will devote the following section to it.

4.

We will prove here that Theorem 3.1 is valid also in the case that (Y,η) is a normed linear space $(Y,\|\ \|)$[4] equipped with the porosity topology p. Indeed, since the *-porosity topology is a refinement of p, it suffices to consider p. We briefly recall that a set $M \subset (Y,\sigma)$ is porous at $x \in Y$ iff there is an $\varepsilon > 0$ such that for all $R > 0$ we can find an $r \in (0,R)$ and a $y \in Y$ with $B(y,\varepsilon r) \subset B(x,r) \setminus M$. A set E is superporous at x if $E \cup M$ is porous at x whenever M is so, and p is defined to be the family of all $G \subset Y$ such that $Y \setminus G$ is superporous at each $y \in G$. Then p is a topology on Y having the Lusin-Menchoff property w.r.t. the metric topology. (See Theorem 1a in [7].) This means, whenever $A \subset Y$ is p-closed and $F \subset Y \setminus A$ is closed, then there exists a continuous $g : (Y,p) \to [0,1]$ with $g(F) \subset \{0\}$ and $g(A) \subset \{1\}$. We shall use the following simple consequence of Proposition $2a'$ in [7]:

Corollary 4.1 *Let $(Y,\|\ \|)$ be a normed linear space, $\varepsilon > 0$ and $M \subset Y$ such that for all $x \in Y$ and $R > 0$ there is an $y \in Y$ with $B(y,\varepsilon R) \subset B(x,R) \setminus M$. Then M is p-closed.*

Let us first consider the (main) case that $(Y,\|\ \|)$ is finite dimensional. As before, we start with the proof of Theorem 3.1, and we can focus on the case that f is bounded on the "bad" subspace $\tilde{X}$. Since Y is boundedly compact, we may again carry out the whole construction from Proposition 2.2 in cl $f(\tilde{X})$; in particular, $a_i \in Y$. We show now how to achieve the p-closedness of the set $\{f(x_i)\}_{i=1}^{\infty}$.

For this purpose we have to introduce a few auxiliary notions which allow a more detailed study of the construction provided in the proof of Proposition 2.2. For $n \geq 1$ denote $\min\{k \mid 2^k \geq n\}$ by $\deg(n) \geq 0$. If $n \geq 2$ we define two kinds of predecessors. First, $pr(n) = n - 2^{\deg(n)-1}$. Note that $pr(n)$ is the unique j with $k \geq 1, 1 \leq j \leq 2^k$ and $n = 2^k + j$. Secondly, for $k \geq 1$ let $n[k] = n'$ where $2^{k-1} < n' \leq 2^k$ and 2^{k-1} divides $n - n'$. Obviously, $n[\deg(n)] = n$, $\deg(n[k]) = k$ and $n[k] \geq 2$ hold whenever these expressions are well defined Moreover, the following properties can be easily verified:

(1) $$n[l] = (n[k])[l] \text{ whenever } 1 \leq l \leq k, \text{ and}$$

(2) $$pr(n) \in \{n[\deg(n)-1], pr(n[\deg(n)-1])\} \text{ for all } n \geq 3.$$

[4] This is surely not the most general setting possible!

Now define for $n \geq 2$

(3)
$$r_n = \frac{1}{65}\min(\{\|a_i - f(x_j)\| \mid 1 \leq i,j < n\} \cup \{\|a_i - a_j\| \mid 1 \leq i,j < n\} \setminus \{0\}),$$

and select the x, δ in the proof of Proposition2.2 such that for all $z \in B(x,\delta)$ the inequality $\|a_{pr(n)} - f(z)\| \leq r_n$ holds. Again, this is the only modification needed. Next, we denote $a_{pr(n)}$ by b_n, then due to (2)

(4) $$b_n \in \{a_{n[\deg(n)-1]}, b_{n[\deg(n)-1]}\} \text{ whenever } n \geq 3.$$

Hence, we infer

(5) $$\|b_n - f(x_n)\|, \|b_n - a_n\| \leq r_n \text{ for } n \geq 2.$$

This implies

(6) $$0 < r_n \leq r_{n-1}/65 \text{ whenever } n \geq 3.$$

Now we derive a few more basic properties of the objects just defined.

(7) $$f(x_n) \in B(b_n, 64r_n) \subset U(b_{n[k]}, 2r_{n[k]}), \forall k \geq 1, n[k] \geq 2.$$

Indeed, due to (5) and (1) it suffices to prove $U(b_{n[\deg(n)-1]}, 2r_{n[\deg(n)-1]}) \supset B(b_n, 64r_n)$. But this follows from $64r_n < r_{n[\deg(n)-1]}$, (See 6.), and $b_n \in B(b_{n[\deg(n)-1]}, r_{n[\deg(n)-1]})$, according to (4), (5).

(8) For $2 \leq m < n$ either $B(b_m, 32r_m) \cap B(b_n, 32r_n) = \emptyset$ or $B(b_n, 64r_n) \subset B(b_m, 2r_m)$. If $\deg(n) = \deg(m)$, then either $b_n = b_m$ or $B(r_n, 32r_n) \cap B(r_m, 32r_m) = \emptyset$.

To see this, let $n' = n[\deg(n) - 1]$. If $n' = m$, then due to (4) $b_n = b_m$ or $b_n = a_m \in B(b_m, r_m)$ and $64r_n < r_m$. In case $n' < m$ (e.g. if $\deg(n) = \deg(m)$) and $b_n \neq b_m$ we have $\|a_{pr(n)} - a_{pr(m)}\| \geq 65r_m > 64r_n$ since $pr(m) < m$, (3) and $pr(n) \in \{pr(n'), n'\} \subset \{2, \ldots, m-1\}$. Therefore, $B(b_n, 32r_n) \cap B(b_m, 32r_m) = \emptyset$. Finally, if $n' > m$ we use $B(b_n, 64r_n) \subset B(b_{n'}, 64r_{n'})$ from (7) and induction with respect to n.

(9) For $m \geq 2$ we have $f(x_m) \in B(b_m, r_m) \setminus \bigcup_{n>m} B(b_n, 64r_n)$.

Indeed, if $n > m$ then (3) implies $\|f(x_m) - b_n\| = \|f(x_m) - a_{pr(n)}\| > 64r_n$.

Lemma 4.2 *Let $\{f(x_n) \mid n \geq 1\}$ be constructed above according to (1), ... ,(5) and let $B(x, R)$, $x \in Y$ and $R > 0$, be arbitrary. If for some $N \geq 1$, $b_N \in B(x, R/3)$, $r_N \geq R/20$, and $r_m \leq R/30$ whenever $m > N$, then there is a y with*

$$B(y, \frac{R}{6}) \subset B(x, R) \setminus \{f(x_n) \mid n \geq 1\}.$$

PROOF. We consider the case $N \geq 2$ only. The case $N = 1$ is much simpler since due to (7), $f(x_n) \in B(a_1, 2r_2)$ for all n and $2r_2 \leq R/30$. So the first part of the proof can be skipped. First suppose there is an $n \geq 1$ such that

$$(10) \qquad f(x_n) \in B(x, R) \setminus \left[B(b_N, \frac{R}{15}) \cup \{f(x_N)\} \cup B(a_N, \frac{R}{15})\right].$$

Since $B(b_N, 32r_N) \supset B(x, R)$, we infer from (9) that $n > N$. According to (5) and (7) $f(x_n) \in B(b_{n'}, 2r_{n'})$ where $n' = n[\deg(N)]$. Now (8) implies $b_{n'} = b_N$, and hence $\|b_{n'} - f(x_n)\| > R/15$ guarantees $n' \leq N$. Therefore, $n'' = n[\deg(N) + 1]$ fulfills $N < n'' \leq n$, $f(x_n) \in B(b_{n''}, 2r_{n''})$, $2r_{n''} \leq R/15$ and $b_{n''} \in \{a_{n'}, b_{n'}\}$. Consequently, (10) shows that $n' < N$ and from (3) we obtain that $R/20 \leq r_N \leq \|a_{n'} - b_{n'}\|/65$. Hence, $\|a_{n'} - b_N\| = \|a_{n'} - b_{n'}\| > 3R$ and $b_{n''} \in \{b_N\} \cup (Y \setminus B(x, 2R))$. We conclude $f(x_n) \in B(b_{n''}, 2r_{n''}) \subset B(b_N, R/15) \cup (Y \setminus B(x, 2R - (R/15))$, a contradiction to (10). We have just shown that

$$\{f(x_n) \mid n \geq 1\} \cap B(x, R) \subset B(b_N, \frac{R}{15}) \cup \{f(x_N)\} \cup B(a_N, \frac{R}{15}).$$

Finally, we choose $y_1, \ldots, y_4 \in B(x, 3R/4)$ with $\|y_i - y_j\| \geq R/2$ for $i \neq j$. Then the balls $B(y_i, R/6) \subset B(x, R)$ are at least $R/6$ apart each from each other, and hence, any of the sets $B(b_N, R/15)$, $\{f(x_N)\}$, $B(a_N, R/15)$ intersects at most one of them. So, we can choose y to be the center y_i of one of the nonintersected $B(y_i, R/6)$.

Theorem 4.3 *Let $\{f(x_n) \mid n \geq 1\}$ be constructed above according to (1), ... ,(5) and let $B(x, R)$ be an arbitrary ball with $x \in Y$ and $R > 0$. Then we find a point y such that*

$$B(y, R/48) \subset B(x, R) \setminus \{f(x_n) \mid n \geq 1\}.$$

PROOF. Let $n_0 = \min\{n \mid f(x_n) \in B(x, R/4)\}$. If this is not well defined, then there is nothing to prove.

First, suppose that

$$\{f(x_n) \mid n \geq 1\} \cap B(x, \frac{R}{4}) \subset B(f(x_{n_0}), \frac{R}{6}) \cup B(a_{n_0}, \frac{R}{60}).$$

We choose a unit vector e with $\|f(x_{n_0})-x\|e = f(x_{n_0})-x$. Obviously, the set $[-R/4, R/4]\setminus[\|f(x_{n_0})-x\|-R/6, \|f(x_{n_0})-x\|+R/6]$ contains closed intervals I_1, I_2 of length $R/24$ and of distance at least $R/25$. Denote by $\tilde{p}_1, \tilde{p}_2$ the middle points of I_1, I_2 and put $p_i = x+\tilde{p}_i e$. Then $B(p_i, R/48) \subset B(x, R/4)\setminus B(f(x_{n_0}), R/6)$ and $(B(p_1, R/48), B(p_2, R/48)) > 2\cdot R/60$. Therefore, at most one $B(p_i, R/48)$ intersects $B(a_{n_0}, R/60)$, and there remains at least one $B(p_j, R/48)$ making the conclusion of the theorem true.

Therefore, we may suppose that

(11)
$$f(x_m) \in B(x, R/4)\setminus\left(B(f(x_{n_0}), \frac{R}{6}) \cup B(a_{n_0}, \frac{R}{60})\right) \text{ for some } m > n_0$$

Put $m'' = m[\deg(n_0)]$. First, consider the case $b_{n_0} \neq b_{m''}$. We denote $K = \min\{k \mid b_{n_0[k]} \neq b_{m[k]}\} - 1$. Obviously $1 \leq K < \deg(n_0)$. Put $n_0' = n_0[K+1] \leq n_0$, $m' = m[K+1] \leq m$. Since $pr(n_0'), pr(m') \leq 2^K < m, n_0$, we have $r_{n_0'}, r_{m'} \leq \|b_{n_0'} - b_{m'}\|/65$ Consequently, $61/65 \leq \|f(x_{n_0}) - f(x_m)\|/\|b_{n_0'} - b_{m'}\| \leq 69/65$, because due to (7) $\|f(x_l) - b_{l[k]}\| \leq 2r_{l[k]}$ whenever $1 \leq k \leq \deg(l)$. This shows $3R/4 > \|b_{n_0'} - b_{m'}\| \geq (65/69)(R/6) > 2R/13$ and $r_{n_0'}, r_{m'} < 3R/260$. Therefore, $b_{n_0'}, b_{m'} \in B(x, 7R/24)$. Obviously, there is an $l \in \{m, n_0\}$ such that $\|b_{l[K+1]} - b_{l[K]}\| > R/13$. We are going to prove that the assumptions of Lemma 4.2 are fulfilled for $N = l[K]$. If $b_N = b_{m'}$ or $b_N = b_{n_0}$ then obviously $b_N \in B(x, R/3)$. Else, (4) implies that $b_{n_0'} = a_{n_0[K]}$, $b_{m'} = a_{m[K]}$ and $n_0[K] \neq m[K]$. Hence, $l_- := \min\{n_0[K], m[K]\} < l_+ := \max\{n_0[K], m[K]\}$. Because $b_{l_-} = b_{l_+} = b_N$, we infer $\|a_{l_-} - b_N\|/65 \geq \|a_{l_+} - b_N\|$. Consequently, $2R/3 \geq \|a_{l_-} - a_{l_+}\| \geq (64/65)\|a_{l_-} - b_N\|$ and $\|a_{l_+} - b_N\| \leq R/65$. Since $a_{l_+} \in B(x, 7R/24)$, $b_N \in B(x, R/3)$ follows in any case. Furthermore, $R/13 \leq \|a_N - b_N\| \leq r_N$ since according to (4) $b_{l[K+1]} = a_{l[K]}$. Finally, for $k > N$ we estimate $r_k \leq \|a_N - b_N\|/65 \leq (2R/3)/65 < R/30$.

Observe, that we can also suppose

$$\|f(x_m) - b_l\| > 2r_l \text{ whenever } l > n_0 \text{ and } b_l = b_{n_0}. \tag{12}$$

Else, $r_l \leq \|f(x_{n_0}) - b_{n_0}\|/65$ and $\|f(x_m) - b_{n_0}\| \leq 2r_l$. Hence, $(67/65)\|b_{n_0} - f(x_{n_0})\| \geq \|f(x_{n_0}) - f(x_m)\| \geq (63/65)\|b_{n_0} - f(x_{n_0})\|$, which implies $3R/4 > (R/2)(65/63) \geq \|b_{n_0} - f(x_{n_0})\| \geq (R/6)(65/67) > R/16$. Consequently, $r_{n_0} > R/20$, $r_k \leq (1/65)(3R/4) < R/30$ for all $k > n_0$ and therefore, $\|b_{n_0} - f(x_m)\| < R/15$ and $b_{n_0} \in B(x, R/3)$. Now, the conclusion would follow from Lemma 4.2 for $N = n_0$.

In the remaining case we have therefore, $b_{m''} = b_{n_0}$, $\|f(x_m) - b_{m''}\| \leq 2r_{m''}$ due to (7) and consequently $m'' \leq n_0$. Denote $m_1 = m[\deg(n_0)+1]$. Then $n_0 < m_1 \leq m$ and again (7) and (12) imply that $b_{m_1} \neq b_{m''}$, i.e. $b_{m_1} =$

$a_{m''}$ holds. We may exclude also the case $m'' = n_0$, since then $\|f(x_m) - a_{n_0}\| = \|f(x_m) - b_{m_1}\| \leq 2r_{m_1} \leq (2/65)\|f(x_{n_0}) - a_{n_0}\|$ implies $\|f(x_m) - f(x_{n_0})\| \geq (63/65)\|f(x_{n_0}) - a_{n_0}\|$. Because $f(x_{n_0}), f(x_m) \in B(x, R/4)$, we conclude $\|f(x_m) - a_{n_0}\| \leq R/63$, a contradiction to (11).

Hence, $m'' < n_0$, and we finish the proof of the theorem by showing that $N = m''$ fulfills the assumption of the Lemma 4.2. Indeed, according to (5) and (7) we can estimate $\|b_{n_0} - f(x_{n_0})\| \leq \|a_N - b_{n_0}\|/65$ and $\|f(x_m) - a_N\| = \|f(x_m) - b_{m_1}\| \leq 2(\|a_N - b_{n_0}\|/65)$. Therefore, $(62/65)\|a_N - b_{n_0}\| \leq \|f(x_m) - f(x_{n_0})\| \leq R/2$ which gives $\|a_N - b_{n_0}\| \leq 65R/124$ and $\|b_{n_0} - f(x_{n_0})\| \leq R/124$. We infer, as required, $b_N = b_{n_0} \in B(x, R/3)$, $r_N \geq \|a_N - b_N\| \geq (65/68)\|f(x_{n_0}) - f(x_m)\| \geq (65/68)(R/6) \geq R/20$, and $r_{\tilde{m}} \leq \|a_N - b_N\|/65 \leq R/30$ whenever $\tilde{m} > N$.

Proposition 4.4 *Proposition Let (X, ρ) and τ be as in Theorem 3.1 and let (Y, p) be a linear normed space with the porosity topology. Then any continuous function $f : (X, \tau) \to (Y, p)$ is in $\mathcal{B}_1^*((X, \rho), (Y, \| \ \|))$.*

PROOF. The finite dimensional case was considered above. Here the crucial result, namely the p-closedness of the modified $\{f(x_i)\}_{i=1}^{\infty}$ follows from Corollary 4.1 and Theorem 4.3. The infinite dimensional case is much simpler. A short look at the proof of Proposition 2.2 shows that we can also in this case force the set $\{f(x_i)\}_{i=1}^{\infty}$ to be totally bounded. Indeed, make all the occurring $f(U(x, \delta))$ sufficiently small and ensure that on each step n of the construction the set $U(x, \delta)$ is contained in the corresponding set of its predecessor $pr(n)$. Now it follows from Riesz' theorem about almost orthogonality that each ball $B(x, R)$, $R > 0$, in the infinite dimensional Y contains infinitely many balls of radius $R/4$ and of distance at least $R/4$ each from each other. Of course, a totally bounded set can never meet all these balls and therefore Theorem 4.3 as well as this Proposition follow again.

5.

Here we discuss the question, as to how big the f-image of the set of points of continuity of a first level Borel function f is. The following result is a generalization of [11].

Proposition 5.1 *Let X and Y be complete metric spaces, the second being locally compact and let $f : X \to Y$ be a first level Borel function. Suppose that for each $x \in X$ and $\varepsilon > 0$ there is a neighborhood $U \subset B(x, \varepsilon)$ of x with $\mathrm{cl}\,(f(U))$ connected. Then $f(C(f))$ is dense in $f(X)$.*

PROOF. Consider $gr(f) = \{(x, f(x)) \mid x \in X\} \subset X \times Y$ and put $M = \mathrm{cl}\,(gr(f))$. Observe that $gr(f)$ is an ambivalent set in $X \times Y$, i.e. both of type

$\mathbf{F}_\sigma$ and $\mathbf{G}_\delta$. Indeed, $gr(f)$ is an $\mathbf{F}_\sigma$-set since f is piecewise continuous and since any continuous function has a closed graph. (Compare [6].) Secondly, $gr(f)$ is $\mathbf{G}_\delta$ whenever f is a Baire-one function; in our case we have simply $gr(f) = \bigcap_{m=1}^{\infty}\{(x,y) \mid x \notin F_m \text{ or } \sigma(f(x),y) < 1/m\}$ with nondecreasing F_m from Definition 1.iv). Since M is complete, we see that $gr(f)$ is residual and $M \setminus gr(f)$ nowhere dense in M. Consequently, cl (int$_M$ $gr(f)$) $= M$ and it suffices to prove that $x \in C(f)$ whenever $(x, f(x)) \in$ int$_M$ $gr(f)$. Otherwise, we find $\delta > 0$ and sets $x \in U_{n+1} \subset U_n \subset B(x, 1/n)$ such that $B(f(x),\delta)$ is compact and any cl $f(U_n)$ is connected and intersecting $Y \setminus B(f(x),\delta)$. It follows from the proof of Lemma 6.1.25 in [5] that for each $n \geq 1$ the connected component of $f(x)$ in cl $f(U_n) \cap B(f(x),\delta)$ is a continuum meeting $Y \setminus U(f(x),\delta)$. Hence, $C = \bigcap_{n=1}^{\infty}$ cl $f(U_n) \cap B(f(x),\delta)$ is a continuum containing $f(x)$ and intersecting $Y \setminus U(f(x),\delta)$. We choose $n > 1/\delta$ such that $\{(x',y') \in M \mid \rho(x,x') + \sigma(y', f(x)) \leq 1/n\} \subset gr(f)$. Obviously, we find $y' \in C$ with $\rho(f(x), y') = 1/2n$. Now, for any $m \geq 1$ there is an $x' \in U_m$ such that $\sigma(f(x'), y') < 1/m$ and hence $((x,y'), gr(f)) \leq 2/m$. This shows that $(x,y') \in M \setminus gr(f)$. This is a contradiction, finishing the proof.

If we drop the assumption of "almost connected" images of small neighborhoods, $i_{\{0\}}$ is a trivial counterexample. However, we construct a more complicated one which is related to questions considered in [6]. A function $f : X \to Y$ is said there to be a first level Borel isomorphism provided f is a bijection and f as well as f^{-1} are first level Borel functions. The question appears whether such an f is a homeomorphism between residual subsets of X and Y. In [6] an affirmative answer was given for X and Y being locally Euclidean of dimension $n \geq 1$. The following example shows that even for $X = Y \subset [0,1]$ compact the situation can be bad.

Let $C \subset [0,1]$ be the classical (1/3)-Cantor set and put $X = C \cup [2/3, 1]$. (Obviously, X is not locally Euclidean.) Define $g : X \to X$ by

$$g(x) = \begin{cases} x & x \notin C \\ \frac{x}{3} & x \in C \cap [0, \frac{1}{3}] \\ x - \frac{4}{9} & x \in C \cap [\frac{6}{9}, \frac{7}{9}] \\ 1 - 3(1-x) & x \in C \cap [\frac{8}{9}, 1] \end{cases} .$$

Then $g(X \setminus C) = X \setminus C$; $g(C) = C$ and g is injective and continuous on both C and $X \setminus C$. Since $X \setminus C$ and C are $\mathbf{F}_\sigma$-sets in X, we infer that g is a first level Borel function. Because X is compact, g maps $\mathbf{F}_\sigma$-sets onto $\mathbf{F}_\sigma$-sets and is even a first level Borel isomorphism. Now, let $D \subset X$ be dense and such that $g_{|D}$ is continuous. We show that $g(D) \cap [2/9, 3/9] = \emptyset$. Indeed, we have cl $(D \setminus C) \supset [2/3, 1]$ and therefore, $g(x) \neq x = \lim_{y \to x, y \in D} g(x)$ whenever $x \in C \cap [2/3, 1]$. This shows $D \cap C \cap [2/3, 1] = \emptyset$ and $g(D) \subset g(X \setminus (C \cap [2/3, 1])) \subset X \setminus [2/9, 3/9]$.

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