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Density and $\mathcal{I}$-density continuous homeomorphisms

1. Preliminaries

Let $\mathcal{H}_\mathcal{N}$ and $\mathcal{H}_\mathcal{I}$ stand for the increasing homeomorphisms that are density and $\mathcal{I}$-density continuous and let $\mathcal{H}_\mathcal{N}^{-1}$ and $\mathcal{H}_\mathcal{I}^{-1}$ denote the classes of inverses of functions from $\mathcal{H}_\mathcal{N}$ and $\mathcal{H}_\mathcal{I}$, respectively; i.e., classes of increasing homeomorphisms that preserve density and $\mathcal{I}$-density points. In the paper we prove that classes $\mathcal{H}_\mathcal{N}$, $\mathcal{H}_\mathcal{I}$ and $\mathcal{H}_\mathcal{I}^{-1}$ are closed under the addition operation. A similar result for the class $\mathcal{H}_\mathcal{N}^{-1}$ has been proved by Niewiarowski [7]. The theorem that the class $\mathcal{H}_\mathcal{I}^{-1}$ is closed under the addition operation is also contained in the paper of Aversa and Wilczyński [1, Theorem 4]. (See also [11, Theorem 25].) However, their proof contains an essential gap. (The gap will be discussed in the last paragraph of the paper.)

This paper contains also the examples showing that none of the above theorems is correct if we admit the possibility that one of the homeomorphism is increasing, and the second one is decreasing, even in the case when their sum is still a homeomorphism.

The notation used throughout this paper is standard. In particular, $\mathbb{R}$ stands for the set of real numbers and $\mathbb{N} = \{1,2,3,\ldots\}$. For $A, B \subset \mathbb{R}$ and $d \in \mathbb{R}$ the complement of A is denoted by A^c, while $B-d = \{x-d \in \mathbb{R} \colon x \in B\}$ and $dB = \{dx \in \mathbb{R} \colon x \in B\}$. The symbols $\mathcal{L}$ and $\mathcal{B}$ stand for the families of subsets of $\mathbb{R}$ which are Lebesgue measurable and have the Baire property, respectively. $\mathcal{N}$ and $\mathcal{I}$ denote the ideals of Lebesgue measure zero and first category subsets of $\mathbb{R}$. If $A \in \mathcal{L}$, its Lebesgue measure is denoted by $\mathrm{m}(A)$.

To define the density topology $\mathcal{T}_\mathcal{N}$ and the $\mathcal{I}$-density topology $\mathcal{T}_\mathcal{I}$ we need the following notions of density and $\mathcal{I}$-density points [8, 11].

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Let $A \in \mathcal{L}$. A number x, not necessarily in A, is a *density point* of A if

$$\lim_{h \to 0+} \frac{\mathrm{m}(A \cap (x-h, x+h))}{2h} = 1.$$

The set of all density points of $A \in \mathcal{L}$ we denote by $\Phi_{\mathcal{N}}(A)$. The family of sets

$$\mathcal{T}_{\mathcal{N}} = \{A \in \mathcal{L} \colon A \subset \Phi_{\mathcal{N}}(A)\}$$

forms a topology on $\mathbb{R}$ [6, 8] and is called the *density topology* on $\mathbb{R}$.

We say that 0 is an *$\mathcal{I}$-density point* of a set $A \in \mathcal{B}$ [11, Theorem 1] (see also [10, Corollary 1] and [9]) if for every increasing sequence $\{t_k\}_{k\in\mathbb{N}}$ of positive numbers diverging to infinity there exists a subsequence $\{t_{k_i}\}_{i\in\mathbb{N}}$ such that

$$\lim_{i\to\infty} \chi_{t_{k_i} A \cap (-1,1)} = \chi_{(-1,1)} \ \mathcal{I}\text{-a.e.}$$

It is worth noticing that the above condition is equivalent to the fact that the set $\liminf_{i\to\infty} t_{k_i} A = \bigcup_{j\in\mathbb{N}} \bigcap_{i\geq j} t_{k_i} A$ is residual in $(-1,1)$. We say that a point a is an $\mathcal{I}$-density point of $A \in \mathcal{B}$ if 0 is an $\mathcal{I}$-density point of $A - a$. The set of all $\mathcal{I}$-density points of $A \in \mathcal{L}$ we denote by $\Phi_{\mathcal{I}}(A)$. The family of sets

$$\mathcal{T}_{\mathcal{I}} = \{A \in \mathcal{B} \colon A \subset \Phi_{\mathcal{I}}(A)\}$$

forms a topology on $\mathbb{R}$ [9, 11] called the *$\mathcal{I}$-density topology* on $\mathbb{R}$.

We also use the following notions dual to the density definitions given above. We say that x is a *dispersion* (*$\mathcal{I}$-dispersion*) point of A if x is a density ($\mathcal{I}$-density) point of A^c. In particular, 0 is an $\mathcal{I}$-dispersion point of B if for every increasing sequence $\{t_k\}_{k\in\mathbb{N}}$ of positive numbers diverging to infinity there exists a subsequence $\{t_{k_i}\}_{i\in\mathbb{N}}$ such that

$$(1) \qquad (-1,1) \cap \bigcap_{j\in\mathbb{N}} \bigcup_{i\geq j} t_{k_i} B = (-1,1) \cap \limsup_{i\to\infty} (t_{k_i} B) \in \mathcal{I}$$

and 0 is a dispersion point of B if

$$(2) \qquad \limsup_{h\to 0+} \frac{\mathrm{m}(B \cap (-h,h))}{2h} = 0.$$

A function $f \colon \mathbb{R} \to \mathbb{R}$ is *density continuous* (*$\mathcal{I}$-density continuous*) if it is continuous with respect to the density ($\mathcal{I}$-density) topology on the domain and the range. A homeomorphism $h \colon \mathbb{R} \to \mathbb{R}$ preserves density ($\mathcal{I}$-density) points if h^{-1} is density ($\mathcal{I}$-density) continuous.

All the continuity and density definitions given above can be restated in more-or-less obvious ways in one-sided versions. For technical reasons it is often more convenient to work with one-sided density or continuity. For example, to show that a point a is an $\mathcal{I}$-density point of a set A, it is often easier to establish that it is both a left and right $\mathcal{I}$-density point. Such simple technical extensions to the definitions will be used without further comment.

We say that a set $\bigcup_{n\in\mathbb{N}}(a_n, b_n)$ is a *right interval set* if $b_{n+1} < a_n < b_n$ for $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = 0$.

In what follows we will need the following facts.

Proposition 1.1. *Let $h\colon \mathbb{R} \to \mathbb{R}$ be an increasing homeomorphism with the property that $h(0) = 0$. Then*

(i) h is right density ($\mathcal{I}$-density) continuous at 0 if, and only if, 0 is not a dispersion ($\mathcal{I}$-dispersion) point of $h(D)$ for every closed set $D \subset [0,\infty)$ such that 0 is not a dispersion ($\mathcal{I}$-dispersion) point of D;

(ii) h preserves right $\mathcal{I}$-density points at 0 if, and only if, for every right interval set E for which 0 is a right $\mathcal{I}$-density point, 0 is a right $\mathcal{I}$-density point of a right interval set $h(E)$.

PROOF. (i) follows easily from [2, Theorem 3] in density case and from [1, Theorem 3] in $\mathcal{I}$-density case. For (ii) see [1, Theorem 3]. □

Proposition 1.2. *0 is a right $\mathcal{I}$-density point of a right interval set E if, and only if, for every increasing sequence $\{t_k\}_{k\in\mathbb{N}}$ of positive numbers diverging to infinity and every nonempty interval $(A,B) \subset (0,1)$ there exists a nonempty subinterval $J \subset (A,B)$ and a subsequence $\{t_{k_i}\}_{i\in\mathbb{N}}$ such that for every $i \in \mathbb{N}$*

$$J \subset t_{k_i}E.$$

PROOF. See [4, Lemma 6.1(iii)]. □

Proposition 1.3. *Let $P \subset (0,1]$ be closed and nowhere dense and let $\{d_k\}_{k\in\mathbb{N}}$ be a sequence of positive numbers such that $\lim_{k\to\infty} d_{k+1}/d_k = 0$. Then there is an open set $V \supset \bigcup_{k\in\mathbb{N}} d_kP$ such that 0 is an $\mathcal{I}$-dispersion point of V.*

PROOF. See [4, Lemma 2.4]. In fact, Proposition 1.3 says that 0 is a deep-$\mathcal{I}$-dispersion point of $\bigcup_{k\in\mathbb{N}} d_kP$. □

Proposition 1.4. *Let $a > 0$ and let $\{d_k\}_{k\in\mathbb{N}}$, $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ be sequences of positive numbers such that $a < a_k < b_k$ for every $k \in \mathbb{N}$ and*

$$\lim_{k\to\infty} d_k = \lim_{k\to\infty} [b_k - a_k] = 0.$$

Then, 0 *is an* $\mathcal{I}$*-dispersion point of*

$$\bigcup_{k\in\mathbb{N}} d_k(a_k, b_k).$$

PROOF. It follows immediately from [11, Theorem 2]. See also [10, Theorem 1]. □

Proposition 1.5. *Let* $h\colon \mathbb{R} \to \mathbb{R}$ *be a homeomorphism. If* h *and* h^{-1} *satisfy a local Lipschitz condition then* h *and* h^{-1} *preserve density and* $\mathcal{I}$*-density points and are density and* $\mathcal{I}$*-density continuous.*

PROOF. For the density case see [3, Lemma 1]. (Compare also [2, Corollary 2].) The $\mathcal{I}$-density case can be found in [1, Corollary 1], [11, Theorem 26] or [4, Theorem 5.8]. □

2. Density continuous homeomorphisms

In this section we prove that the sum of two increasing density continuous homeomorphisms is density continuous.

Theorem 2.1. *If* $f, g \in \mathcal{H}_{\mathcal{N}}$*, then* $f + g \in \mathcal{H}_{\mathcal{N}}$.

PROOF. Let $f, g \in \mathcal{H}_{\mathcal{N}}$ and let $a \in \mathbb{R}$. It is enough to prove that $f+g$ is right density continuous at a, as the left-hand side argument is similar. Without loss of generality we may assume that $a = f(a) = g(a) = 0$. Let $D \subset [0,\infty)$ be a closed set for which 0 is not a dispersion point of D. By Proposition 1.1(i), it is enough to prove that 0 is not a dispersion point of $(f+g)(D)$.

Let $D_f = \{x \in D\colon g(x) \le f(x)\}$ and $D_g = \{x \in D\colon f(x) \le g(x)\}$. Then 0 is not a dispersion point of either D_f or D_g. Assume that 0 is not a dispersion point of D_f. We may assume, without loss of generality, that $D = D_f$. Thus,

$$g(x) \le f(x) \quad \text{for every} \quad x \in D.$$

But $f \in \mathcal{H}_{\mathcal{N}}$ and 0 is not a dispersion point of D. So, 0 is not a dispersion point of $f(D) \subset [0,\infty)$ and, by (2), there exist $\varepsilon > 0$ and a decreasing sequence $\{h_n\}_{n\in\mathbb{N}}$ of positive numbers converging to 0 such that

$$\limsup_{n\to\infty} \frac{\mathrm{m}(f(D) \cap (0, h_n))}{h_n} = \varepsilon.$$

Let $h_n' = \sup f(D) \cap (0, h_n] \in (0, h_n]$, $t_n = f^{-1}(h_n') \in D$ and define $p_n = (f+g)(t_n) \le 2f(t_n)$. Then, $\lim_{n\to\infty} p_n = 0$ and

$$
\begin{aligned}
\limsup_{n\to\infty} \frac{\mathrm{m}((f+g)(D)\cap(0,p_n))}{p_n} &= \limsup_{n\to\infty} \frac{\mathrm{m}((f+g)(D\cap(0,t_n)))}{p_n} \\
&\ge \limsup_{n\to\infty} \frac{\mathrm{m}(f(D\cap(0,t_n)))}{2f(t_n)} \\
&= \frac{1}{2}\limsup_{n\to\infty} \frac{\mathrm{m}(f(D)\cap(0,h_n'))}{h_n'} \\
&= \frac{1}{2}\limsup_{n\to\infty} \frac{\mathrm{m}(f(D)\cap(0,h_n))}{h_n'} \\
&\ge \frac{1}{2}\limsup_{n\to\infty} \frac{\mathrm{m}(f(D)\cap(0,h_n))}{h_n} \\
&= \frac{\varepsilon}{2} > 0,
\end{aligned} \tag{3}
$$

where the numerator part of inequality (3) holds, because $\mathrm{m}((f+g)(A)) \ge \mathrm{m}(f(A))$ for every $A \in \mathcal{L}$. Thus, by (2), 0 is not a dispersion point of $(f+g)(D)$. This finishes the proof of Theorem 2.1. □

Corollary 2.2. *If $f, g \in \mathcal{H}_{\mathcal{N}}$, then $f\,g \in \mathcal{H}_{\mathcal{N}}$.*

PROOF. By Proposition 1.5, functions exp and ln are density continuous. We show that $f\,g$ is density continuous at $a \in \mathbb{R}$. Translating and restricting f and g to an open neighborhood of a, if necessary, we may assume that f and g are positive. Then, $\ln f, \ln g \in \mathcal{H}_{\mathcal{N}}$, as composition of density continuous increasing homeomorphisms is a density continuous increasing homeomorphism. Thus, by Theorem 2.1, density continuous is also

$$f\,g = \exp(\ln f + \ln g).$$ □

Let us also notice that in fact we proved the following result, which is a little bit stronger that Theorem 2.1.

Corollary 2.3. *Let f and g be increasing homeomorphisms such that $f(a) = g(a)$ for some $a \in \mathbb{R}$. If $g(x) \le f(x)$ for every $x \ge a$ and f is right density continuous at a then $f+g$ is also right density continuous at a.*

3. $\mathcal{I}$-density continuous homeomorphisms

The purpose of this section is to prove that the sum of two increasing $\mathcal{I}$-density continuous homeomorphisms is $\mathcal{I}$-density continuous. For this we need the following lemmas.

Lemma 3.1. *Let $D \in \mathcal{B}$ be such that 0 is not a right $\mathcal{I}$-dispersion point of D. Then there exists an increasing sequence $\{t_k\}_{k\in\mathbb{N}}$ of positive numbers diverging to infinity and a nonempty interval $(a,b) \subset (0,1)$ such that*

$$\left(\liminf_{k\to\infty} t_k D\right) \text{ is dense in } (a,b).$$

PROOF. Since 0 is not a right $\mathcal{I}$-dispersion point of D then, by (1), there exists an increasing sequence $\{s_n\}_{n\in\mathbb{N}}$ of positive numbers diverging to infinity such that for every its subsequence $\{s_{n_k}\}_{k\in\mathbb{N}}$

$$\left(\limsup_{k\to\infty} s_{n_k} D\right) \cap (0,1) \notin \mathcal{I}. \tag{4}$$

Let $(p_k, q_k) \subset (0,1)$ be a sequence of all nonempty intervals with rational endpoints. Let us construct, by induction on k, sequences $\{s_n^k\}_{n\in\mathbb{N}}$ such that $\{s_n^0\}_{n\in\mathbb{N}} = \{s_n\}_{n\in\mathbb{N}}$ and $\{s_n^k\}_{n\in\mathbb{N}}$ is a subsequence of $\{s_n^{k-1}\}_{n\in\mathbb{N}}$ such that

(5)
$$\text{either} \quad \left(\limsup_{n\to\infty} s_n^k D\right) \cap (p_k, q_k) = \emptyset \quad \text{or} \quad \left(\liminf_{n\to\infty} s_n^k D\right) \cap [p_k, q_k] \neq \emptyset.$$

Put $t_k = s_k^k$. Then, by (4), $(\limsup_{k\to\infty} t_k D) \cap (0,1) \notin \mathcal{I}$; i.e., there exists a nonempty interval $(a,b) \subset (0,1)$ such that

$$\left(\limsup_{k\to\infty} t_k D\right) \text{ is dense in } (a,b).$$

But this, together with (5), guarantees that then also

$$\left(\liminf_{k\to\infty} t_k D\right) \text{ is dense in } (a,b).$$

This finishes the proof of Lemma 3.1. □

Lemma 3.2. *Let $h\colon \mathbb{R} \to \mathbb{R}$ be an increasing $\mathcal{I}$-density continuous homeomorphism such that $h(0) = 0$ and let $\{t_k\}_{k\in\mathbb{N}}$ be an increasing sequence of positive numbers diverging to infinity. Then for every nontrivial interval $[a,b] \subset (0,1)$ there exists a nonempty interval $(c,d) \subset (a,b)$ and a subsequence $\{t_{k_i}\}_{i\in\mathbb{N}}$ of $\{t_k\}_{k\in\mathbb{N}}$ such that the limit*

$$\lim_{i\to\infty} \frac{h(c/t_{k_i})}{h(d/t_{k_i})}$$

exists and is positive.

PROOF. By way of contradiction assume that it cannot be done; i.e., that

$$\limsup_{k\to\infty} \frac{h(c/t_k)}{h(d/t_k)} = 0 \quad \text{for every} \quad a \le c < d \le b. \tag{6}$$

We will show that this contradicts $\mathcal{I}$-density continuity of h.

So, let $\{q_k : k \in \mathbb{N}\}$ be an enumeration of $Q = [a,b] \cap \mathbb{Q}$ and for each $i \in \mathbb{N}$ let $d_1, \dots, d_i$ be an increasing enumeration of $q_1, \dots, q_i$. Choose $\{t_{k_i}\}_{i\in\mathbb{N}}$ such that

$$\frac{h(b/t_{k_{i+1}})}{h(a/t_{k_i})} \le \frac{h(d_{j+1}/t_{k_i})}{h(d_j/t_{k_i})} \le \frac{1}{i} \quad \text{for every} \quad j < i, \ i \in \mathbb{N}. \tag{7}$$

This can be done by (6). Let

$$U_i = \bigcup_{j \le i} h(d_j/t_{k_i}) \left(1 - \frac{1}{i}, 1 + \frac{1}{i}\right).$$

and put $U = \bigcup_{i\in\mathbb{N}} U_i$. Then, by (7) and Proposition 1.4, 0 is an $\mathcal{I}$-dispersion point of U. But 0 is not an $\mathcal{I}$-dispersion point of $h^{-1}(U)$, since for any subsequence $\{t_m\}_{m\in\mathbb{N}}$ of $\{t_{k_i}\}_{i\in\mathbb{N}}$ the open set $\bigcup_{m\ge m_0} t_m h^{-1}(U) \supset Q$ is dense in (a,b) for every $m_0 \in \mathbb{N}$, and so,

$$(-1,1) \cap \limsup_{m\to\infty} (t_m h^{-1}(U)) \notin \mathcal{I}.$$

This finishes the proof of Lemma 3.2. □

Lemma 3.3. *Let $a < b$, $H_k \colon [a,b] \to \mathbb{R}$ be a sequence of increasing homeomorphisms and let us assume that there exists a dense subset Q of $[a,b]$ containing a and b such that the limit $H(q) = \lim_{k\to\infty} H_k(q)$ exists for every $q \in Q$. If $H(Q)$ is dense in $[H(a), H(b)]$ and $H(x) = \inf H(Q \cap [x,\infty))$ for every $x \in [a,b]$, then H_k converges uniformly to H.*

PROOF. First notice that the function $H(q) = \lim_{k\to\infty} H_k(q)$ on Q is nondecreasing, so indeed $H(q) = \inf H(Q \cap [q,\infty))$ for every $q \in Q$.

Let us fix $\varepsilon > 0$. For $x \in [a,b]$ choose distinct $q_1, q_2 \in Q$, $q_1 \le x \le q_2$, such that $q_1 < x < q_2$ for $x \in (a,b)$ and

$$|H(q_2) - H(q_1)| < \varepsilon/5.$$

Let $N_x \in \mathbb{N}$ be such that

$$|H(q_i) - H_n(q_i)| < \varepsilon/5$$

for every $n > N_x$ and $i = 1, 2$. Put $U_x = (q_1, q_2)$ for $x \in (a, b)$, $U_x = [q_1, q_2)$ for $x = a$ and $U_x = (q_1, q_2]$ for $x = b$. Thus, U_x is an open neighborhood of x in $[a, b]$ and, for every $y \in U_x$ and $n > N_x$,

$$H(q_1) \le H(y) \le H(q_2) \quad \text{and} \quad H_n(q_1) \le H_n(y) \le H_n(q_2),$$

so that

$$\begin{aligned} |H(y) - H_n(y)| &\le |H(y) - H(q_2)| + |H(q_2) - H_n(q_2)| + |H_n(q_2) - H_n(y)| \\ &< |H(q_1) - H(q_2)| + \varepsilon/5 + |H_n(q_2) - H_n(q_1)| \\ &< \varepsilon/5 + \varepsilon/5 + \\ &\quad |H_n(q_2) - H(q_2)| + |H(q_2) - H(q_1)| + |H(q_1) - H_n(q_1)| \\ &< \varepsilon/5 + \varepsilon/5 + \varepsilon/5 + \varepsilon/5 + \varepsilon/5 = \varepsilon. \end{aligned}$$

Choose a finite subcover $\{U_{x_1}, \dots, U_{x_k}\}$ of the open cover $\mathcal{U} = \{U_x\}_{x \in [a,b]}$ of $[a, b]$ and put $N = \sup\{N_{x_1}, \dots, N_{x_k}\}$. Then we obtain

$$|H(y) - H_n(y)| < \varepsilon$$

for every $y \in [a, b]$ and $n > N$. This finishes the proof of Lemma 3.3. □

Lemma 3.4. *Let $h\colon \mathbb{R} \to \mathbb{R}$ be an increasing $\mathcal{I}$-density continuous homeomorphism such that $h(0) = 0$ and let $[a, b] \subset (0, 1)$ be a nontrivial interval. If $\{s_k\}_{k \in \mathbb{N}}$ and $\{t_k\}_{k \in \mathbb{N}}$ are increasing sequences of positive numbers diverging to infinity such that $H_k(x) = s_k h(x/t_k) \in [0, 1]$ for every $x \in [a, b]$, then there exist a nonempty interval $(c, d) \subset (a, b)$ and a subsequence $\{H_{k_i}\}_{i \in \mathbb{N}}$ of $\{H_k\}_{k \in \mathbb{N}}$ such that the sequence $H_{k_i}|_{[c,d]}$ converges uniformly to a function $H\colon [c, d] \to [0, 1]$.*

Moreover, if $\liminf_k H_k(a) > 0$ *then we can assume that the function H is one-to-one.*

Proof. First notice that functions H_k are increasing.

Let $Q = \{q_i\colon i \in \mathbb{N}\}$ be a dense subset of $[a, b]$ containing a and b. The functions $H_k|_Q$ are elements of a compact metric space $[0, 1]^Q$. So, there exists a subsequence $\{H_{k_i}\}_{i \in \mathbb{N}}$ of $\{H_k\}_{k \in \mathbb{N}}$ that converges in $[0, 1]^Q$; i.e., such that for every $j \in \mathbb{N}$ there exists $H(q_j) \in [0, 1]$ with the property that

$$\lim_{i \to \infty} H_{k_i}(q_j) = H(q_j).$$

If $H(q_r) = 0$ for some $q_r \in (a, b)$ then, by Lemma 3.3, interval $[c, d] = [a, q_r]$ and the function $H(x) = 0$ for every $x \in [c, d]$ work. So, decreasing $[a, b]$, if necessary, we can assume that $H(a) > 0$. This is also the case when $\liminf_k H_k(a) > 0$.

We prove that

$$P = \mathrm{cl}\,(H(Q \cap [a', b'])) \subset (0, 1] \tag{8}$$

is not nowhere dense for every nonempty interval $(a', b') \subset (a, b)$ such that $a', b' \in Q$. Notice that this will finish the proof, because it implies existence of a nontrivial interval $[c, d] \subset [a, b]$, $c, d \in Q$, such that $H(Q \cap [c, d])$ is dense in $[H(c), H(d)]$. So, Lemma 3.3 gives us the desired uniform convergence. Moreover, condition (8) guarantees also that H will be one-to-one on $[c, d]$.

By way of contradiction let us assume that condition (8) fails; i.e., that P is nowhere dense for some nonempty interval $(a', b') \subset (a, b)$ such that $a', b' \in Q$. Choosing a subsequence, if necessary, we can assume that

$$\lim_{i \to \infty} s_{k_{i+1}}^{-1} / s_{k_i}^{-1} = 0.$$

Then, by Proposition 1.3, there exists an open set $W \supset \bigcup_{i \in \mathbb{N}} s_{k_i}^{-1} P$ such that 0 is an $\mathcal{I}$-dispersion point of W. We will construct a set V such that 0 is not an $\mathcal{I}$-dispersion point of V, while $h(V) \subset W$; i.e., $h(0) = 0$ is an $\mathcal{I}$-dispersion point of $h(V)$. This will contradict the assumption that h is $\mathcal{I}$-density continuous.

So, let us choose a countable base $\{I_i\}_{i \in \mathbb{N}}$ of $[a', b']$ and for every $i, j \in \mathbb{N}$, $j \leq i$, choose $q_{i,j}, q'_{i,j} \in Q$ such that $q_{i,j} < q'_{i,j}$, $[q_{i,j}, q'_{i,j}] \subset I_j$, and

$$H([q_{i,j}, q'_{i,j}]) \subset s_{k_i} W. \tag{9}$$

This can be done, since $P \subset s_{k_i} W$, so the distance d_i between P and the complement of $s_{k_i} W$ is positive and any interval $[q_{i,j}, q'_{i,j}]$ for which $H(q'_{i,j}) - H(q_{i,j}) < d_i$ satisfies condition (9). In addition, choosing subsequence of $\{k_i\}_{i \in \mathbb{N}}$, if necessary, we can also assume that for every $i, j \in \mathbb{N}$, $j \leq i$, $H_{k_i}(q_{i,j})$ and $H_{k_i}(q'_{i,j})$ are closer to $H(q_{i,j})$ then d_i. This means that

$$H_{k_i}((q_{i,j}, q'_{i,j})) \subset s_{k_i} W \quad \text{for every } i, j \in \mathbb{N},\ j \leq i. \tag{10}$$

Let $V_i = \bigcup_{j \leq i} (q_{i,j}, q'_{i,j})$ and

$$V = \bigcup_{i \in \mathbb{N}} \frac{1}{t_{k_i}} V_i.$$

Then, by (10),

$$h\left(\frac{1}{t_{k_i}} V_i\right) = \frac{1}{s_{k_i}} \left[s_{k_i} h\left(\frac{1}{t_{k_i}} V_i\right)\right] = \frac{1}{s_{k_i}} H_{k_i}(V_i) \subset \frac{1}{s_{k_i}} [s_{k_i} W] = W$$

for every $i \in \mathbb{N}$ and, indeed, $h(V) \subset W$.

On the other hand, 0 is not $\mathcal{I}$-dispersion point of V, since for any subsequence $\{t_{k_{i_p}}\}_{p\in\mathbb{N}}$ of $\{t_{k_i}\}_{i\in\mathbb{N}}$ the set $\bigcup_{p\geq p_0} t_{k_{i_p}}V \supset \bigcup_{p\geq p_0} V_{i_p}$ is open and dense in (a', b') for every $p_0 \in \mathbb{N}$, and so,

$$(-1,1)\cap \limsup_{p\to\infty}(t_{k_{i_p}}V) \notin \mathcal{I}.$$

This finishes the proof of Lemma 3.4. □

Theorem 3.5. *If $f, g \in \mathcal{H}_\mathcal{I}$, then $f + g \in \mathcal{H}_\mathcal{I}$.*

PROOF. Let $f, g \in \mathcal{H}_\mathcal{I}$ and let $a \in \mathbb{R}$. It is enough to prove that $f + g$ is right $\mathcal{I}$-density continuous at a. Without loss of generality we may assume that $a = f(a) = g(a) = 0$. Let $D_0 \subset [0,\infty)$ be a closed set for which 0 is not an $\mathcal{I}$-dispersion point of D_0. By Proposition 1.1(i), it is enough to prove that 0 is not an $\mathcal{I}$-dispersion point of $(f+g)(D_0)$.

Let $D_f = \{x \in D_0 \colon g(x) \leq f(x)\}$ and $D_g = \{x \in D_0 \colon f(x) \leq g(x)\}$. Then 0 is not an $\mathcal{I}$-dispersion point of either D_f or D_g. Assume that 0 is not an $\mathcal{I}$-dispersion point of D_f. We may assume, without loss of generality, that $D_0 = D_f$; i.e., that

$$g(x) \leq f(x) \quad \text{for every} \quad x \in D_0. \tag{11}$$

Let D be the interior of D_0. Since 0 is an $\mathcal{I}$-dispersion point of the closed nowhere dense set $D_0 \setminus D$, 0 is not an $\mathcal{I}$-dispersion point of D. Then, by Lemma 3.1, there is an increasing sequence $\{t_k\}_{k\in\mathbb{N}}$ of positive numbers diverging to infinity and a nontrivial interval $[a,b] \subset (0,1)$ such that

$$Q = \liminf_{k\to\infty} t_k D \cap (a,b) \quad \text{is dense in } (a,b). \tag{12}$$

We may also easily assume that $b \in \liminf_{k\to\infty} t_k D$; i.e., that $b/t_k \in D \subset D_0$ for almost all $k \in \mathbb{N}$. This, together with (11), implies that

$$g(b/t_k) \leq f(b/t_k) \quad \text{for almost all } k \in \mathbb{N}. \tag{13}$$

Now, by Lemma 3.2 used for the function f, the sequence $\{t_k\}_{k\in\mathbb{N}}$ and the interval $[a,b]$, we may find a subsequence $\{t_{k_i}\}_{i\in\mathbb{N}}$ of $\{t_k\}_{k\in\mathbb{N}}$, and a nonempty interval $(c,d) \subset (a,b)$ such that $\lim_{i\to\infty} f(c/t_{k_i})/f(d/t_{k_i}) > 0$. Without loss of generality we may assume that $\{t_{k_i}\}_{i\in\mathbb{N}} = \{t_k\}_{k\in\mathbb{N}}$ and $[c,d] = [a,b]$; i.e., that

$$\lim_{k\to\infty} \frac{f(a/t_k)}{f(b/t_k)} > 0. \tag{14}$$

Let $s_k = 1/(f+g)(b/t_k)$, $F_k(x) = s_k f(x/t_k)$ and $G_k(x) = s_k g(x/t_k)$ for $x \in [0,1]$. Then,

$$(15) \qquad (F_k + G_k)(x) = s_k(f+g)(x/t_k) \in [0,1] \text{ for } x \in [a,b],\ k \in \mathbb{N},$$

$s_k(f+g)(b/t_k) = (F_k+G_k)(b) = 1$ and, by conditions (13) and (14),

$$(16) \qquad \begin{aligned} \liminf_{k\to\infty} F_k(a) &= \liminf_{k\to\infty} s_k f(a/t_k) \\ &= \liminf_{k\to\infty} \frac{f(a/t_k)}{(f+g)(b/t_k)} \\ &\geq \liminf_{k\to\infty} \frac{f(a/t_k)}{2f(b/t_k)} \\ &> 0. \end{aligned}$$

By Lemma 3.4 used twice, we can find a nonempty interval $(c,d) \subset (a,b)$ and a sequence $\{k_i\}_{i\in\mathbb{N}}$ of natural numbers such that $\{F_{k_i}|_{[c,d]}\}_{i\in\mathbb{N}}$ converges uniformly to some F and $\{G_{k_i}|_{[c,d]}\}_{i\in\mathbb{N}}$ converges uniformly to a function G. Moreover, by (16), we can also assume that F and $F+G$ are increasing homeomorphisms on $[c,d]$. Without loss of generality we may assume that $[c,d]=[a,b]$.

Let $(A,B) = ((F+G)(a),(F+G)(b)) \subset (0,1]$. By (12), the set $(F+G)(Q)$ is dense in (A,B). But if $q \in Q$ then, by (12), $q/t_k \in D$ for almost all $k \in \mathbb{N}$. So, for every sequence $\{k_i\}_{i\in\mathbb{N}}$ of natural numbers and every $j \in \mathbb{N}$,

$$(F+G)(q) = \lim_{i\to\infty} s_{k_i}(f+g)(q/t_{k_i}) \in cl\left(\bigcup_{i\geq j} s_{k_i}(f+g)(D)\right)$$

which implies that the set $\bigcup_{i\geq j} s_{k_i}(f+g)(D)$ is dense in (A,B). Thus, the G_δ set

$$(-1,1)\cap \limsup_{i\to\infty} s_{k_i}(f+g)(D) = (-1,1)\cap \bigcap_{j\in\mathbb{N}}\bigcup_{i\geq j} s_{k_i}(f+g)(D) \notin \mathcal{I};$$

i.e., 0 is not an $\mathcal{I}$-dispersion point of $(f+g)(D)$. This finishes the proof of Theorem 3.5. □

Corollary 3.6. *If $f,g \in \mathcal{H}_\mathcal{I}$, then $fg \in \mathcal{H}_\mathcal{I}$.*

PROOF. Same as for Corollary 2.2. □

Let us also notice that the previous proof works also for the following result, that is a little bit stronger that Theorem 3.5.

Corollary 3.7. *Let f and g be increasing homeomorphisms such that $f(a) = g(a)$ for some $a \in \mathbb{R}$. If $g(x) \leq f(x)$ for every $x \geq a$ and f is right $\mathcal{I}$-density continuous at a then $f+g$ is also right $\mathcal{I}$-density continuous at a.*

4. Homeomorphisms that preserve $\mathcal{I}$-density points

In this section we prove the theorem that $f+g$ preserves $\mathcal{I}$-density points provided f and g are increasing homeomorphisms preserving $\mathcal{I}$-density points. For this we need the following lemma analogous to Lemma 3.4.

Lemma 4.1. *Let $h\colon \mathbb{R} \to \mathbb{R}$ be an increasing homeomorphism preserving $\mathcal{I}$-density points such that $h(0) = 0$ and let $\{s_k\}_{k\in\mathbb{N}}$ and $\{t_k\}_{k\in\mathbb{N}}$ be the increasing sequences of positive numbers diverging to infinity such that $H_k(x) = s_k h(x/t_k) \in [0,1]$ for every $x \in [0,1]$. Then for every nontrivial interval $[a,b] \subset (0,1)$ there exists a nonempty interval $(c,d) \subset (a,b)$ and a subsequence $\{H_{k_i}\}_{i\in\mathbb{N}}$ of $\{H_k\}_{k\in\mathbb{N}}$ such that the sequence $H_{k_i}|_{[c,d]}$ converges uniformly to a function $H\colon [c,d] \to [0,1]$.*

Moreover, if $\limsup_k(H_k(b) - H_k(a)) > 0$ then we can assume that the function H is one-to-one.

PROOF. Let $Q = \{q_i : i \in \mathbb{N}\}$ be a dense subset of $[a,b]$ containing a and b. Functions $H_k|_Q$ are elements of a compact metric space $[0,1]^Q$. So, there exists an increasing sequence $\{k_i\}_{i\in\mathbb{N}}$ of natural numbers such that $H_{k_i}|_Q$ converges in $[0,1]^Q$; i.e., that for every $j \in \mathbb{N}$ there exists $H(q_j) \in [0,1]$ such that

$$\lim_{i\to\infty} H_{k_i}(q_j) = H(q_j).$$

Moreover, if $\limsup_k(H_k(b) - H_k(a)) > 0$ then we can also assume that

$$H(a) < H(b).$$

If $H(a) = H(b)$ then, by Lemma 3.3, interval $[c,d] = [a,b]$ and the function $H = H(a)\chi_{[c,d]}$ work. So, we can assume that $H(a) < H(b)$.

By Lemma 3.3 in order to prove the first part of Lemma 4.1 it is enough to show that $H(Q)$ is dense in $[H(a), H(b)] \subset [0,1]$. So, by way of contradiction, assume that $H(Q)$ is not dense in $[H(a), H(b)]$. Then, there exists a nonempty interval $(A,B) \subset [H(a), H(b)]$ such that $H(Q) \cap [A,B] = \emptyset$, and we can find $a_i, b_i \in Q$, $0 < b_i - a_i < 1/i$, such that $H(a_i) < A < B < H(b_i)$ for every $i \in \mathbb{N}$. Now, taking subsequence of $\{k_i\}_{i\in\mathbb{N}}$, if necessary, we can conclude that

$$s_{k_i} h(a_i/t_{k_i}) = H_{k_i}(a_i) < A < B < H_{k_i}(b_i) = s_{k_i} h(b_i/t_{k_i})$$

for every $i \in \mathbb{N}$.

Let $U = \bigcup_{i\in\mathbb{N}} t_{k_i}^{-1}(a_i, b_i)$. Then, by Proposition 1.4, 0 is an $\mathcal{I}$-dispersion point of U. But,

$$[A,B] \subset (H_{k_i}(a_i), H_{k_i}(b_i)) = s_{k_i} h\left(t_{k_i}^{-1}(a_i, b_i)\right) \subset s_{k_i} h(U)$$

for every $i \in \mathbb{N}$. So, by (1), 0 is not $\mathcal{I}$-dispersion point of $h(U)$. This contradicts the assumption that h preserves $\mathcal{I}$-density points.

To prove the additional condition let us assume, by way of contradiction, that H is not one-to-one on any nonempty interval $(c,d) \subset (a,b)$. Then, the set

$$U = \bigcup\{(c,d) \subset (a,b)\colon H(c) = H(d)\}$$

is dense in (a,b) and the set $H(U)$ is countable. In particular, the set $P = [a,b] \setminus U$ is nowhere dense in $[a,b]$, while $H(P)$ is dense in $[H(a), H(b)]$. We will show that this implies that h does not preserve right $\mathcal{I}$-density at 0.

Choosing a subsequence of $\{k_i\}_{i\in\mathbb{N}}$, if necessary, we may assume that

$$\lim_{i\to\infty} t_{k_{i+1}}^{-1}/t_{k_i}^{-1} = 0.$$

Then, by Proposition 1.3, there exists an open set $V \supset \bigcup_{i\in\mathbb{N}} t_{k_i}^{-1}P$ such that 0 is an $\mathcal{I}$-dispersion point of V. We will show that 0 is not an $\mathcal{I}$-dispersion point of the open set $h(V)$.

So, let $\{k_p\}_{p\in\mathbb{N}}$ be an arbitrary subsequence of $\{k_i\}_{i\in\mathbb{N}}$. Then, for every $x \in P$,

$$H(x) = \lim_{p\to\infty} H_{k_p}(x) = \lim_{p\to\infty} s_{k_p} h(x/t_{k_p}) \in \mathrm{cl}\left(\bigcup_{r\geq p} s_{k_r} h(V)\right)$$

which implies that the set $\bigcup_{r\geq p} s_{k_r} h(V)$ is dense in $[H(a), H(b)]$ for every $p \in \mathbb{N}$. Thus, the G_δ set

$$(0,1) \cap \limsup_{p\to\infty} s_{k_p} h(V) = (0,1) \cap \bigcap_{r\in\mathbb{N}} \bigcup_{p\geq r} s_{k_p} h(V) \notin \mathcal{I},$$

because it is dense in $(H(a), H(b)) \neq \emptyset$. Now, by (1), 0 is not an $\mathcal{I}$-dispersion point of $h(V)$. This finishes the proof of Lemma 4.1. □

Theorem 4.2. *If $f, g \in \mathcal{H}_{\mathcal{I}}^{-1}$, then $f + g \in \mathcal{H}_{\mathcal{I}}^{-1}$.*

PROOF. Let $f, g \in \mathcal{H}_{\mathcal{I}}^{-1}$ and let $a \in \mathbb{R}$. It is enough to prove that $f + g$ preserves right $\mathcal{I}$-density at a. Without loss of generality we may assume that $a = f(a) = g(a) = 0$. Let E be a right interval set such that 0 is a right $\mathcal{I}$-density point of E, let $\{s_k\}_{k\in\mathbb{N}}$ be an increasing sequence of positive numbers diverging to infinity and let $0 < A < B < 1$. By Propositions 1.1(ii) and 1.2, it is enough to prove that there exist a subsequence $\{s_{k_i}\}_{i\in\mathbb{N}}$ of $\{s_k\}_{k\in\mathbb{N}}$ and a nonempty open interval $J \subset (A, B)$ such that

$$J \subset s_{k_i}(f+g)(E)$$

for every $i \in \mathbb{N}$.

Let us define

$$t_k = 1/(f+g)^{-1}(B/s_k), \quad a_k = 1/(f+g)^{-1}(A/s_k),$$
$$F_k(x) = s_k f(x/t_k), \quad G_k(x) = s_k g(x/t_k)$$

and

$$H_k(x) = (F_k + G_k)(x) = s_k(f+g)(x/t_k).$$

Then, $t_k < a_k$, $A = s_k(f+g)(1/a_k)$ and $B = s_k(f+g)(1/t_k)$. In particular,

$$H_k\left(\left[\frac{t_k}{a_k}, 1\right]\right) = s_k(f+g)\left(\frac{1}{t_k}\left[\frac{t_k}{a_k}, 1\right]\right) = [A, B].$$

Let $\{k_i\}_{i\in\mathbb{N}}$ be a sequence of natural numbers such that the following limits exist

$$a = \lim_{i\to\infty} \frac{t_{k_i}}{a_{k_i}} \in [0,1],$$
$$F(a) = \lim_{i\to\infty} F_{k_i}(a), \quad G(a) = \lim_{i\to\infty} G_{k_i}(a).$$

We will show that

$$(F+G)(a) = A. \tag{17}$$

By way of contradiction, let us assume that it is not the case. We will assume that

$$s_{k_i}(f+g)(1/a_{k_i}) = A < (F+G)(a) = \lim_{i\to\infty} s_{k_i}(f+g)(a/t_{k_i}).$$

The other inequality is similar. Let $A < C < (F+G)(a)$. Then,

$$s_{k_i}(f+g)(1/a_{k_i}) = A < C < s_{k_i}(f+g)(a/t_{k_i})$$

for almost all $i \in \mathbb{N}$. Assume that it is true for all $i \in \mathbb{N}$. Then,

$$\frac{f(1/a_{k_i}) + g(1/a_{k_i})}{f(a/t_{k_i}) + g(a/t_{k_i})} = \frac{s_{k_i}(f+g)(1/a_{k_i})}{s_{k_i}(f+g)(a/t_{k_i})} < \frac{A}{C} < 1.$$

Hence, for every $i \in \mathbb{N}$, either $\frac{f(1/a_{k_i})}{f(a/t_{k_i}))} \le \frac{A}{C}$ or $\frac{g(1/a_{k_i})}{g(a/t_{k_i}))} \le \frac{A}{C}$. Without loss of generality, passing to a subsequence, if necessary, we can assume that for all $n \in \mathbb{N}$

$$\frac{f\left(\frac{1}{t_{k_i}}\frac{t_{k_i}}{a_{k_i}}\right)}{f\left(\frac{1}{t_{k_i}}a\right)} = \frac{f(1/a_{k_i})}{f(a/t_{k_i})} \le \frac{A}{C} < 1.$$

Let $u_{k_i} = f\left(\frac{1}{t_{k_i}}a\right)$. Then

$$u_{k_i}^{-1} f\left(\frac{1}{t_{k_i}} \frac{t_{k_i}}{a_{k_i}}\right) \leq \frac{A}{C} < 1 = u_{k_i}^{-1} f\left(\frac{1}{t_{k_i}} a\right);$$

i.e.,

$$\left(\frac{A}{C}, 1\right) \subset u_{k_i}^{-1} f\left(\frac{1}{t_{k_i}} \left(\frac{t_{k_i}}{a_{k_i}}, a\right)\right) \tag{18}$$

for every $i \in \mathbb{N}$. But, choosing subsequence, if necessary, we can assume that $\lim_{i\to\infty} t_{k_{i+1}}^{-1}/t_{k_i}^{-1} = 0$ and hence, by Proposition 1.4, 0 is an $\mathcal{I}$-dispersion point of

$$D = \bigcup_{i\in\mathbb{N}} \frac{1}{t_{k_i}} \left(\frac{t_{k_i}}{a_{k_i}}, a\right).$$

On the other hand, by (18),

$$\left(\frac{A}{C}, 1\right) \subset u_{k_i}^{-1} f(D)$$

for every $i \in \mathbb{N}$; i.e., 0 is not an $\mathcal{I}$-dispersion point of $f(D)$. This contradicts the assumption that f preserves $\mathcal{I}$-density points. Condition (17) is proved.

Now notice that condition (17) implies, in particular, that $a < 1$, since $\lim_{i\to\infty}(F_{k_i} + G_{k_i})(1) = B > A = \lim_{i\to\infty}(F_{k_i} + G_{k_i})(a)$.

Using Lemma 4.1 twice for functions F_{k_i} and G_{k_i}, passing to a subsequence, if necessary, we can find a nontrivial interval $[c, d] \subset (a, 1)$ such that $\{F_{k_i}|_{[c,d]}\}_{i\in\mathbb{N}}$ converges uniformly to some function F and that $\{G_{k_i}|_{[c,d]}\}_{i\in\mathbb{N}}$ converges uniformly to a function G. Let us notice also that condition (17) implies that either $\limsup_{k\to\infty}(F_k(1) - F_k(a)) > 0$ or $\limsup_{k\to\infty}(G_k(1) - G_k(a)) > 0$, since $\limsup_{k\to\infty}(F_k(1) - F_k(a)) + (G_k(1) - G_k(a)) = H(1) - H(a) = B - A > 0$. Thus, we can also assume that the function $H = F + G$ is a homeomorphism on $[c, d]$.

By Proposition 1.2, choosing a subsequence of $\{k_i\}_{i\in\mathbb{N}}$ and a subinterval of (c, d), if necessary, we may also assume that

$$(c, d) \subset t_{k_i} E \quad \text{for every } i \in \mathbb{N},$$

which implies that

$$\begin{aligned}
((F_{k_i} + G_{k_i})(c), (F_{k_i} + G_{k_i})(d)) &= (F_{k_i} + G_{k_i})((c, d)) \\
&\subset (F_{k_i} + G_{k_i})(t_{k_i} E) \\
&= s_{k_i}(f + g)\left(\frac{1}{t_{k_i}}(t_{k_i} E)\right) \\
&= s_{k_i}(f + g)(E).
\end{aligned}$$

Now, if $c < c' < d' < d$ then

$$A \leq (F+G)(c) < (F+G)(c') < (F+G)(d') < (F+G)(d) \leq B$$

so, $J = ((F+G)(c'), (F+G)(d')) \subset (A, B)$ and

$$J \subset ((F_{k_i} + G_{k_i})(c), (F_{k_i} + G_{k_i})(d)) \subset s_{k_i}(f+g)(E)$$

for i's large enough, since $\{F_{k_i} + G_{k_i}\}$ converges to $F+G$. Thus, we may assume that

$$J \subset s_{k_i}(f+g)(E) \cap (A, B)$$

for every $i \in \mathbb{N}$. This finishes the proof of Theorem 4.2. □

Corollary 4.3. *If $f, g \in \mathcal{H}_{\mathcal{I}}^{-1}$, then $fg \in \mathcal{H}_{\mathcal{I}}^{-1}$.*

PROOF. Same as for Corollary 2.2. □

5. Discussion and examples

We start this section with an explicit statement of the density analog of Theorem 4.2, that has been proved by Niewiarowski [7].

Theorem 5.1. *If $f, g \in \mathcal{H}_{\mathcal{N}}^{-1}$, then $f + g \in \mathcal{H}_{\mathcal{N}}^{-1}$.*

We are going to present examples showing that none of the Theorems 2.1, 3.5, 4.2 or 5.1 remains valid if we admit that one of the homeomorphisms is decreasing, even when their sum is an increasing homeomorphism. Moreover, the decreasing function can be defined by $g(x) = -x$.

Example 1. *There exists $h \in \mathcal{H}_{\mathcal{I}} \cap \mathcal{H}_{\mathcal{I}}^{-1} \cap \mathcal{H}_{\mathcal{N}}^{-1} \cap C^\infty$, $h: \mathbb{R} \to \mathbb{R}$, such that $f: \mathbb{R} \to \mathbb{R}$, $f(x) = h(x) - x$, is strictly increasing but $f \notin \mathcal{H}_{\mathcal{I}} \cup \mathcal{H}_{\mathcal{I}}^{-1} \cup \mathcal{H}_{\mathcal{N}}^{-1}$.*

PROOF. Let

$$f(x) = \begin{cases} e^{-x^{-2}} & x > 0 \\ 0 & x = 0 \\ -e^{-x^{-2}} & x < 0 \end{cases}$$

It is known that f is C^∞, which is not $\mathcal{I}$-density continuous and does not preserve $\mathcal{I}$-density points [5, Example 10]. (See also [4, Example 5.7].) It also follows easily from Bruckner [2, Theorem 7] that f does not preserve density points.

Define $h(x) = f(x) + x$. Then h is an increasing, C^∞ homeomorphism such that h and h^{-1} satisfy a local Lipschitz conditions. Thus, by Proposition 1.5, h and h^{-1} are density and $\mathcal{I}$-density continuous. □

Example 2. *There exists $h \in \mathcal{H}_{\mathcal{N}} \cap C^{\infty}$, $h:\mathbb{R} \to \mathbb{R}$, such that $f:\mathbb{R} \to \mathbb{R}$, $f(x) = h(x) - x$, is strictly increasing but $f \notin \mathcal{H}_{\mathcal{N}}$.*

PROOF. In [3, Example 1] it is constructed a nondecreasing function $f \in C^{\infty}$ which is not density continuous. In fact, f is strictly increasing except for the intervals forming some right interval set. It is not difficult to modify this function to be strictly increasing C^{∞} and not density continuous. Then function $h(x) = f(x) + x$ works. □

Let us finish this paper with the remark that Theorem 4.2 is also contained in Aversa and Wilczyński [1, Theorem 4].[1] However, their proof, considerably shorter that the one presented in this paper, contains an essential gap. In their proof Aversa and Wilczyński show that for every homeomorphisms f and g preserving $\mathcal{I}$-density points, any sequence $\{n_k\}$ of natural numbers and any open set A for which 0 is an $\mathcal{I}$-dispersion point and a nonempty interval $J \subset (0,1)$ there exists a nonempty interval $I \subset (0,1)$ and subsequence $\{n_{k_i}\}$ such that $f(I) \cup g(I) \subset J$, and

$$f(I) \cap n_{k_i} f(A) = \emptyset = g(I) \cap n_{k_i} g(A).$$

From this they conclude that $(f+g)(I) \cap n_{k_i}(f+g)(A) = \emptyset$. This evidently might be false. To see this you can take, for example, $f(x) = x^3$, $g(x) = x$, $n_{k_i} = 8$, $I = (1/3, 1/2)$ and $A = ([1/24, 1/16] \cup [1/6, 1/4])^c$.

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