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STOCHASTIC INTEGRALS OF ITÔ AND HENSTOCK

1. Introduction

The Itô integral [4, 5] is well-known. It has been actively studied in recent years and applied successfully to solving stochastic differential equations. The technique used is measure-theoretic. On the other hand, the Henstock integral [2, 7, 10] uses Riemann sums in its definition and is able to achieve such generality that it is known to include Wiener and Feynman integration. Stochastic integrals using Henstock's theory have been attempted by McShane [9], T. W. Lee [8] and most recently by Henstock [3]. In this note, we shall show that it also includes the Itô integral. This is achieved by combining the ideas of Henstock [3] using Riemann sums and of McShane [9] using belated divisions. Furthermore, using the stochastic integral of Henstock we obtain Itô's formula.

2. The Stochastic integral of Itô

We give a brief description of the Itô integral, which is essential for the understanding of the next section. We follow mainly Ikeda and Watanabe [4]. A good reference on stochastic integrals for analysts is Kopp [6]. Let W denote the set of all real-valued continuous functions on [0,1] with a metric ρ given by

$$\rho(w_1, w_2) = \sup\{|w_1(t) - w_2(t)|;\ 0 \le t \le 1\}.$$

The class of all Borel cylinder sets B in W, denoted by $\mathcal{C}$, is a collection of all the sets B in W of the form

$$B = \{w; (w(t_1), w(t_2), \cdots, w(t_n)) \in E\}$$

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where $0 \leq t_1 < t_2 < \cdots < t_n \leq 1$ and E is a Borel set in $\mathbb{R}^n$ (n is not fixed). The Borel σ-field of $\mathcal{C}$ is denoted by $\mathcal{B}(W)$, i.e., it is the smallest σ-field which contains $\mathcal{C}$. Finally, let P_W be the Wiener measure defined on $(W, \mathcal{B}(W))$. Then the triple $(W, \mathcal{B}(W), P_W)$ is a probability space with $P_W(W) = 1$. We remark that it is possible to develop the Wiener integral [11] using Henstock's general theory of division spaces [2, 7, 10], in which the Borel cylinder sets are taken as intervals in the division space. For details, see [1, 10].

Next, let $L_2 = L_2(W, \mathcal{B}(W), P_W)$ be the space of all random variables q (real-valued $\mathcal{B}(W)$-measurable functions on W) such that

$$||q||_{L_2}^2 = \int_W |q(w)|^2 dP_W(w) < +\infty.$$

Since $\mathcal{B}(W)$ is separable, so is L_2. That is, there is a countable dense set $\{q_1, q_2, \cdots\}$ in L_2. This fact will be used later in Section 2.

Let $X = \{X(t,w)\}_{0 \leq t \leq 1}$ be a Brownian motion (or Wiener process) so that $X(t,w) = w(t)$ for $w \in W$, $t \in [0,1]$, and $X(t,w)$ is adapted to $\{\mathcal{B}_t; 0 \leq t \leq 1\}$. That is to say, $X(t,w)$ is $\mathcal{B}_t$-measurable for each $t \in [0,1]$, where $\mathcal{B}_t = \sigma\{X(s,w); s \leq t\}$ is the smallest σ-field generated by $\{X(s,w); s \leq t\}$. Here $X(t,w)$ is called a canonical Brownian motion of $(W, \mathcal{B}(W), P_W; \{\mathcal{B}_t; 0 \leq t \leq 1\})$. We denote by $\mathcal{L}_2$ the space of all measurable processes $\{\varphi(t,w)\}_{0 \leq t \leq 1}$ defined on $(W, \mathcal{B}(W), P_W)$ (φ is a measurable function on $[0,1] \times W$), adapted to $\{\mathcal{B}_t\}$ such that

$$||\varphi||_{\mathcal{L}_2}^2 = \int_0^1 [\int_W |\varphi(t,w)|^2 dP_W(w)] dt < +\infty.$$

For convenience, we write

$$E(Q(w)) = \int_W Q(w) dP_W(w),$$

where E is called the expectation of a random variable Q with respect to P_W, and

$$||\varphi||_{\mathcal{L}_2}^2 = \int_0^1 E|\varphi(t,w)|^2 dt.$$

We may regard a process as a family of random variables. We can construct a dense set in $\mathcal{L}_2$ as follows. Let $\mathcal{L}_0$ be the set of all step processes $\varphi(t,w)$ satisfying the following conditions :

(i) there is $M > 0$ such that $|\varphi(t,w)| \leq M$ for $t \in [0,1]$, $w \in W$;

(ii) there are a finite sequence of points $t_0 = 0 < t_1 < t_2 < \cdots < t_n < t_{n+1} = 1$ and a finite sequence of random variables $f_i(w)$, $i = 0, 1, 2, \cdots, n$, such that $\varphi(0, w) = f_0(w)$ and

$$\varphi(t, w) = \sum_{i=0}^{n} f_i(w)\chi_{(t_i, t_{i+1}]}(t) \text{ for } t \in (0, 1]$$

where χ denotes the characteristic function of $(t_i, t_{i+1}]$ and $f_i(w)$ is measurable with respect to $\mathcal{B}_{t_i}$ for $i = 0, 1, \cdots, n$.

Then we can prove that $\mathcal{L}_0$ is dense in $\mathcal{L}_2$. More precisely, for every $\varphi \in \mathcal{L}_2$ there is a sequence $\{\varphi_1, \varphi_2, \cdots\}$ in $\mathcal{L}_0$ such that

$$\|\varphi_m - \varphi\|_{\mathcal{L}_2} \to 0 \text{ as } m \to \infty.$$

If $\varphi \in \mathcal{L}_0$ is a step process as given in (ii) above, then we define the Itô integral of φ to be

$$I(\varphi)(w) = \sum_{i=0}^{n} f_i(w)[w(t_{i+1}) - w(t_i)].$$

Note that $I(\varphi) \in L_2$. In general, whenever $\varphi \in \mathcal{L}_2$ there is a sequence $\{\varphi_1, \varphi_2, \cdots\}$ in $\mathcal{L}_0$ such that $\|\varphi_m - \varphi\|_{\mathcal{L}_2} \to 0$ as $m \to \infty$. Then we define the Itô integral $I(\varphi)$ of φ to be

$$I(\varphi) = \lim_{m \to \infty} I(\varphi_m) \text{ in } L_2,$$

that is,

$$\int_W |I(\varphi_m) - I(\varphi)|^2 dP_W \to 0 \text{ as } m \to \infty.$$

We can prove that the Itô integral $I(\varphi)$ is uniquely determined in L_2.

3. The stochastic integral of Henstock

A full cover Δ is a family of interval-point pairs $([u, v], \xi)$ such that $\xi \in [u, v] \subset (\xi - \delta(\xi), \xi + \delta(\xi))$ for some given $\delta(\xi) > 0$. We shall define a belated full cover Δ_b as follows. Fix a sequence of pairwise disjoint measurable subsets $M_1, M_2, \cdots$ of $[0, 1]$ whose union is $[0, 1]$. Let $t_k = inf\{t; t \in M_k\}$. Define $\delta(\xi) > 0$ so that $t_k < \xi - \delta(\xi)$ whenever $\xi \in M_k \setminus \{t_k\}$. Obviously, the family Δ of all interval-point pairs $([u, v], \xi)$ satisfying $\xi \in [u, v] \subset (\xi - \delta(\xi), \xi + \delta(\xi))$ forms a full cover. Then a belated full cover Δ_b is a collection of all interval-point pairs $([u, v], y(\xi))$ such that $([u, v], \xi) \in \Delta$ and $y(\xi)$ is defined as follows:

(i) when $\xi \in M_k$ and $\xi = t_k$ (note that t_k may not belong to M_k), put $y(\xi) = 0$,

(ii) when $\xi \in M_k \setminus \{t_k\}$ and $([u, v], \xi) \in \Delta$, put $y(\xi) \in M_k \cap [t_k, u]$.

Note that $y(\xi)$ depends on $[u,v]$ and ξ when $\xi \in M_k \setminus \{t_k\}$. Since there exists a division $D = \{([u,v],\xi)\}$ of $[0,1]$ from Δ, there also exists a division (a belated division) $D = \{([u,v],y(\xi))\}$ of $[0,1]$ from Δ_b. The above belated full cover Δ_b is said to depend on $\{M_k\}$ and $\delta(\xi)$.

Given $\{M_k\}$ with $M_1, M_2, \cdots$ pairwise disjoint and $\cup_{k=1}^{\infty} M_k = [0,1]$, another sequence $\{M_k^*\}$ is called a split of $\{M_k\}$ if for each k, $M_k^* \subset M_i$ for some i, $M_1^*, M_2^*, \cdots$ are pairwise disjoint and $\cup_{k=1}^{\infty} M_k^* = [0,1]$. A measurable process $\{\varphi(t,w)\}_{0\le t\le 1}$ is said to be Henstock belatedly integrable or HB integrable to $I(\varphi) \in L_2$ if for every $\varepsilon > 0$ there is a sequence of pairwise disjoint measurable subsets $M_1, M_2, \cdots$ of $[0,1]$ with $\cup_{k=1}^{\infty} M_k = [0,1]$ such that for every split $\{M_k^*\}$ of $\{M_k\}$ there exist $\delta(\xi) > 0$ and a belated full cover Δ_b depending on $\{M_k^*\}$ and $\delta(\xi)$, and for any $D = \{([u,v],y(\xi))\}$ from Δ_b we have $E|(D)\sum \varphi(y(\xi),w)(w(v)-w(u)) - I(\varphi)|^2 < \varepsilon$, where $(D)\sum$ denotes the sum over all interval-point pairs $([u,v],y(\xi))$ in D, and E the expectation with respect to P_W. We write

$$I(\varphi) = (HB)\int_0^1 \varphi(t,w)dX(t,w)$$

and call $I(\varphi)$ the Henstock belated integral of φ.

We shall verify briefly that the Henstock belated integral $I(\varphi)$ as defined above is unique. Suppose there are $I_1(\varphi)$ and $I_2(\varphi)$ satisfying the above conditions with $\{M_{1k}\}$ and $\{M_{2k}\}$ respectively. Then consider $M_{1i} \cap M_{2k}$ for all i,k and label it $\{M_k^*\}$. Note that $\{M_k^*\}$ is a split of both $\{M_{1k}\}$ and $\{M_{2k}\}$. Hence by definition there exist $\delta(\xi) > 0$ and a belated full cover Δ_b depending on $\{M_k^*\}$ and $\delta(\xi)$ such that for any $D = \{([u,v],y(\xi))\}$ from Δ_b we have

$$\begin{aligned} E|I_1(\varphi) - I_2(\varphi)|^2 &\le 2E|I_1(\varphi) - (D)\sum \varphi(y(\xi),w)(w(v)-w(u))|^2 \\ &\quad + 2E|(D)\sum \varphi(y(\xi),w)(w(v)-w(u)) - I_2(\varphi)|^2 \\ &< 4\varepsilon. \end{aligned}$$

That is, $I_1(\varphi) = I_2(\varphi)$ in L_2.

The idea of a split is due to Chew Tuan Seng. We remark that it helps to prove the uniqueness of the HB integral easily and it is not used in the proof later. In fact, we shall show in Section 4 that it can be dropped.

Theorem 1 *If $\varphi \in \mathcal{L}_2$ then for every $\varepsilon > 0$ there exists a belated full cover Δ_b such that for any division $D = \{([u,v],y(\xi))\}$ from Δ_b we have*

$$\int_0^1 E|(D)\sum \varphi(y(\xi),w)\chi_{(u,v]}(t) - \varphi(t,w)|^2 dt < \varepsilon,$$

where $(D)\sum$ sums over all interval-point pairs $([u,v],y(\xi))$ in D.

PROOF. First, fix a countable dense set $\{q_1, q_2, \cdots\}$ in L_2. Let $0 < \varepsilon < 1/4$ and define $O_k(\varepsilon) = \{q \in L_2;\ \|q - q_k\|_{L_2} < \varepsilon/2\}$, for $k = 1, 2, \cdots$. Since $\{q_1, q_2, \cdots\}$ is dense in L_2, we have $\bigcup_{k=1}^{\infty} O_k(\varepsilon) = L_2$. Next, define

$$\begin{aligned} M_1 &= \{t \in [0,1];\ \varphi(t,\cdot) \in O_1(\varepsilon)\}, \\ M_k &= \{t \in [0,1];\ \varphi(t,\cdot) \in O_k(\varepsilon) \setminus \bigcup_{j=1}^{k-1} O_j(\varepsilon)\}, \end{aligned}$$

$k = 2, 3, \cdots$. Note that $\bigcup_{k=1}^{\infty} M_k = [0,1]$. Let $\eta = \sum_{k=1}^{\infty} \eta_k$ with $\eta_k < \varepsilon 2^{-k-3}(\|q_k\|_{L_2} + \varepsilon)^{-2}$. Using $\{M_k\}$ and $\{\eta_k\}$ we can choose a sequence of open sets $G_1, G_2, \cdots$ such that $G_k \supset M_k$ and $|G_k \setminus M_k| < \eta_k$ for each k, and choose $\delta(\xi)$ such that $(\xi - \delta(\xi), \xi + \delta(\xi)) \subset G_k$ whenever $\xi \in M_k$ for some k. Furthermore, write $t_k = inf\{t;\ t \in M_k\}$ and put $t_k < \xi - \delta(\xi)$ whenever $\xi \in M_k \setminus \{t_k\}$ and also $\delta(\xi) \le \eta_k/2$ when $\xi = t_k \in M_k$. Consequently, we define a belated full cover Δ_b depending on $\{M_k\}$ and $\delta(\xi)$ with $y(\xi)$ defined as usual, i.e., $y(\xi) = 0$ when $\xi = t_k \in M_k$ and $y(\xi) \in M_k \cap [t_k, u]$ when $\xi \in M_k \setminus \{t_k\}$ and $\xi \in [u,v] \subset (\xi - \delta(\xi), \xi + \delta(\xi))$.

Then take any division $D = \{([u,v], y(\xi))\}$ of $[0,1]$ from Δ_b and split D into D_1 and D_2 in which D_1 consists of $([u,v], y(\xi))$ such that $\xi = t_k \in M_k$ for some k and D_2 consists of $([u,v], y(\xi))$ such that $\xi \in M_k \setminus \{t_k\}$. For brevity, we denote by M_ξ the set M_k when $\xi \in M_k$. Then we have

$$\begin{aligned} &\int_0^1 E|(D)\sum \varphi(y(\xi), w)\chi_{(u,v]}(t) - \varphi(t,w)|^2 dt \\ \le\ & (D_1)\sum \int_{[u,v]} E|\varphi(0,w) - \varphi(t,w)|^2 dt \\ & + (D_2)\sum \int_{[u,v]\cap M_\xi} E|\varphi(y(\xi),w) - \varphi(t,w)|^2 dt \\ & + 2(D_2)\sum \int_{[u,v]\setminus M_\xi} E|\varphi(y(\xi),w)|^2 dt \\ & + 2(D_2)\int_{[u,v]\setminus M_\xi} E|\varphi(t,w)|^2 dt \\ =\ & R_1 + R_2 + R_3 + R_4. \end{aligned}$$

Without loss of generality we can assume $E|\varphi(0,w)|^2 < \infty$ throughout this paper. Since $(D_1)\sum |v - u| < \sum_{k=1}^{\infty} \eta_k = \eta$ and $\varphi \in \mathcal{L}_2$, we can choose η_k and in fact η sufficiently small so that $R_1 < \varepsilon/4$. When $y(\xi)$, t belong to the same M_k for some k, we obtain

$$\begin{aligned} \|\varphi(y(\xi),\cdot) - \varphi(t,\cdot)\|_{L_2} &\le \|\varphi(y(\xi),\cdot) - q_k\|_{L_2} + \|\varphi(t,\cdot) - q_k\|_{L_2} \\ &< \varepsilon. \end{aligned}$$

Therefore $R_2 < \varepsilon^2 \leq \varepsilon/4$. Note that when $\xi \in M_k$ for some k, we have

$$E|\varphi(y(\xi), w)|^2 \leq (\|q_k\|_{L_2} + \varepsilon)^2.$$

It follows that

$$\begin{aligned} R_3 &\leq 2\sum_{k=1}^{\infty}(\|q_k\|_{L_2} + \varepsilon)^2|G_k \setminus M_k| \\ &< 2\sum_{k=1}^{\infty}(\|q_k\|_{L_2} + \varepsilon)^2\eta_k < \varepsilon/4. \end{aligned}$$

Finally, note that $\sum_{k=1}^{\infty} |G_k \setminus M_k| < \eta$ and $\varphi \in \mathcal{L}_2$. Hence again for choosing sufficiently small η we have $R_4 < \varepsilon/4$. Consequently, we obtain

$$R_1 + R_2 + R_3 + R_4 < \varepsilon.$$

The proof is complete.

We remark that Theorem 1 also holds true with Δ_b replaced by a full cover Δ. However belated divisions are required later when considering the Itô integral, hence Theorem 1 was stated in terms of a belated full cover Δ_b.

Theorem 2 *If $\varphi \in \mathcal{L}_2$, then φ is HB integrable and*

$$(HB)\int_0^1 \varphi(t, w)dX(t, w) = (I)\int_0^1 \varphi(t, w)dX(t, w)$$

where the right-hand side above denotes the Itô integral.

PROOF. Let $I(\varphi) \in L_2$ be the Itô integral of $\varphi \in \mathcal{L}_2$. For $\varepsilon > 0$, let Δ_b be a belated full cover depending on $\{M_k\}$ and $\delta(\xi)$ as defined in the proof of Theorem 1. Take a division $D = \{([u, v], y(\xi))\}$ from Δ_b and write

$$\varphi_D(t, w) = \begin{cases} \varphi(0, w) & \text{when } t = 0 \\ \varphi(y(\xi), w) & \text{when } t \in (u, v], \end{cases}$$

for each $([u, v], y(\xi))$ from D. Note that $\varphi_D(t, w)$ is a step process and therefore its Itô integral exists. Then by Theorem 1 and the Itô isometry [4; p.48, 6; p.15] that

$$E|(I)\int_0^1 \varphi(t, w)dX(t, w)|^2 = \int_0^1 E|\varphi(t, w)|^2 dt$$

for $\varphi \in \mathcal{L}_2$, we have

$$\begin{aligned} E|(D)\sum \varphi(y(\xi),w)(w(v)-w(u)) - I(\varphi)|^2 &= E|(I)\int_0^1 [\varphi_D(t,w) \\ &\qquad -\varphi(t,w)]dX(t,w)|^2 \\ &= \int_0^1 E|\varphi_D(t,w)-\varphi(t,w)|^2 dt \\ &< \varepsilon. \end{aligned}$$

To prove the existence of the HB integral, let $\{M_k^*\}$ be a split of $\{M_k\}$. The above proof still goes through with $\{M_k\}$ replaced by $\{M_k^*\}$. Hence if the Itô integral exists, so does the HB integral with the same value.

Theorem 3 *Let $\varphi \in \mathcal{L}_2$. If φ is HB integrable to $I(\varphi)$ then the Itô integral exists and we have*

$$E|I(\varphi)|^2 = \int_0^1 E|\varphi(t,w)|^2 dt.$$

PROOF. Since φ is HB integrable, by definition there is a sequence of belated divisions $D_n = \{([u_n,v_n],y(\xi_n))\}$ such that

$$\lim_{n\to\infty} E|(D_n)\sum \varphi(y(\xi_n),w)(w(v_n)-w(u_n)) - I(\varphi)|^2 = 0.$$

Alternatively, we write

$$\lim_{n\to\infty} E|(HB)\int_0^1 \varphi_{D_n}(t,w)dX(t,w) - I(\varphi)|^2 = 0.$$

It is clear that for step processes φ_D the HB integral and the Itô integral coincide and the Itô isometry holds. So we have

$$\begin{aligned} E|(HB)\int_0^1 \varphi_D(t,w)dX(t,w)|^2 &= E|(I)\int_0^1 \varphi_D(t,w)dX(t,w)|^2 \\ &= \int_0^1 E|\varphi_D(t,w)|^2 dt. \end{aligned}$$

Hence it follows from Theorem 1 that

$$\begin{aligned} E|I(\varphi)|^2 &= \lim_{n\to\infty} E|(HB)\int_0^1 \varphi_{D_n}(t,w)dX(t,w)|^2 \\ &= \lim_{n\to\infty}\int_0^1 E|\varphi_{D_n}(t,w)|^2 dt \\ &= \int_0^1 E|\varphi(t,w)|^2 dt. \end{aligned}$$

Using Theorem 3, we see that if $\varphi \in \mathcal{L}_2$ and φ is HB integrable to $I(\varphi)$ then $I(\varphi)$ is also the Itô integral of φ. Hence the HB integral has provided a Riemann-type definition for the Itô integral.

4. The HB integral

We shall show that the use of splits in the definition of the HB integral can be dispensed with.

Theorem 4 *Let φ be a L_2-valued measurable process. Then φ is HB integrable to $I(\varphi)$ if and only if for every $\varepsilon > 0$ there exists a belated full cover Δ_b depending on $\{M_k\}$ and $\delta(\xi)$, where $\{M_k\}$ is a sequence of pairwise disjoint measurable subsets of $[0,1]$ with $\cup_{k=1}^{\infty} M_k = [0,1]$,*

$$\lambda_k = \sup\{E|\varphi(t,w)|^2;\ t \in M_k\} < \infty$$

for $k = 1, 2, \cdots$, and $\delta(\xi) > 0$, such that for any $D = \{([u,v], y(\xi))\}$ from Δ_b we have

$$E|(D)\sum \varphi(y(\xi), w)(w(v) - w(u)) - I(\varphi)|^2 < \varepsilon,$$

where $(D)\sum$ denotes the sum over interval-point pairs $([u,v], y(\xi))$ in D.

PROOF. From the proof of Theorem 1, we see that if φ is HB integrable then we can choose $\{M_k\}$ so that λ_k is finite for each k. Indeed, for $t \in M_k$ we have $\|\varphi(t,\cdot)\|_{L_2} < \varepsilon/2 + \|q_k\|_{L_2}$ or $E|\varphi(t,w)|^2 < (\varepsilon/2 + \|q_k\|_{L_2})^2$. That is, λ_k is finite.

Conversely, suppose the condition is satisfied. Take a split $\{M_k^*\}$ of $\{M_k\}$. As usual, let $t_k^* = \inf\{t;\ t \in M_k^*\}$ and define $\delta^*(\xi) > 0$ so that $t_k^* < \xi - \delta^*(\xi)$ whenever $\xi \in M_k^* \setminus \{t_k^*\}$. We may assume $\delta^*(\xi) \le \delta(\xi)$. Again, Δ^* is a full cover using $\delta^*(\xi)$ and Δ_b^* a belated full cover depending on $\{M_k^*\}$ and $\delta^*(\xi)$. Now take a belated division $D = \{([u,v], y^*(\xi))\}$ from Δ_b^*. Note that $([u,v], y^*(\xi)) = ([u,v], y(\xi)) \in \Delta_b$ whenever $\xi \ne t_k^*$ for all k. Let D_1 be the partial division of D in which $\xi = t_k^* \in M_k^*$ for some k. Obviously, $y^*(t_k^*) = 0$. Replace D_1 by D_2 in which $([u,v], y(t_k^*)) \in \Delta_b$. Then the division $(D\backslash D_1) \cup D_2$, denoted by D_3, comes from Δ_b with $(D)\sum = (D_3)\sum + (D_1)\sum - (D_2)\sum$ and again by the Itô isometry we obtain

$$\begin{aligned}
E\left|(D)\sum\varphi(y^*(\xi),w)(w(v)-w(u))-I(\varphi)\right|^2 \\
\le\ & 3E\left|(D_3)\sum\varphi(y(\xi),w)(w(v)-w(u))-I(\varphi)\right|^2 \\
&+3E\left|(D_1)\sum\varphi(0,w)(w(v)-w(u))\right|^2 \\
&+3E\left|(D_2)\sum\varphi(y(\xi),w)(w(v)-w(u))\right|^2 \\
\le\ & 3\varepsilon+3E|(D_1)\sum\int_u^v\varphi(0,w)dX(t,w)|^2 \\
&+3E|(D_2)\sum\int_u^v\varphi(y(\xi),w)dX(t,w)|^2 \\
\le\ & 3\varepsilon+3(D_1)\sum\int_u^v E|\varphi(0,w)|^2dt \\
&+3(D_2)\sum\int_u^v E|\varphi(y(\xi),w)|^2dt.
\end{aligned}$$

In fact, the last two terms are

$$3(D_1)\sum E|\varphi(0,w)|^2(v-u)+3(D_2)\sum E|(y(\xi),w)|^2(v-u).$$

Write $\eta_k^*=2\delta(t_k^*)$. It is easy to see that by choosing η_k^* so that

$$3\sum_{k=1}^{\infty}E|\varphi(0,w)|^2\eta_k^* \ <\ \varepsilon$$

we have

$$3(D_1)\sum\int_u^v E|\varphi(0,w)|^2dt \ <\ \varepsilon.$$

Further, write $\eta_k=\sum\eta_i^*$ in which the sum is over all $t_i^*\in M_k$. Choose η_i^* and consequently η_k so that $\sum_{k=1}^{\infty}\lambda_k\eta_k\le\varepsilon/3$. Note that when $t_i^*\in M_k$, we have $y(t_i^*)\in M_k$ and $E|\varphi(y(\xi),w)|^2\le\lambda_k$. Then we obtain

$$3(D_2)\sum\int_u^v E|\varphi(y(\xi),w)|^2dt \ <\ \varepsilon.$$

Hence the proof is complete.

In view of Theorem 4, the condition there can be used as an alternative definition of the HB integral, and furthermore the integral so-defined is uniquely determined.

5. Itô's formula

This is one of the most important tools in the study of stochastic integrals. We shall show that it can also be verified using the HB integral.

Theorem 5 *Suppose that*

(i) $F(y,w)$, $-\infty < y < \infty$, is a process defined on $(W, \mathcal{B}(W), P_W; \{\mathcal{B}_t; 0 \le t \le 1\})$ and the map $y \to F(y,w)$ is continuous with probability one;

(ii) $F'_y(y,w)$ and $F''_y(y,w)$ are bounded and both $y \to F'_y(y,w)$, $y \to F''_y(y,w)$ are continuous with probability one;

(iii) Both $F'_y(X(t,w),w)\}_{0\le t\le 1}$ and $\{F''_y(X(t,w),w)\}_{0\le t\le 1}$ are adapted to $\{\mathcal{B}_t;\ 0 \le t \le 1\}$ where $X(t,w)$ is the canonical Brownian motion of $(W, \mathcal{B}(W), P_W; \{\mathcal{B}_t;\ 0 \le t \le 1\})$.

Then for every $T \in [0,1]$ we have

$$F(X(T,w),w) - F(X(0,w),w) = \int_0^T F'_y(X(s,w),w)dX(s,w) + \frac{1}{2}\int_0^T F''_y(X(s,w),w)ds.$$

To prove Theorem 5 which is known as Itô's formula, we need the following two lemmas.

Lemma 6 *Let $\varphi = \{\varphi(t,w)\}_{0\le t\le 1}$ be defined on $(W, \mathcal{B}(W), P_W)$ and the map $t \to \varphi(t,\cdot)$ from $[0,1]$ into L_2 be continuous. Then Theorem 4 holds with $D = \{([u,v], y(\xi))\}$ replaced by $D_1 = \{([u,v], u)\}$.*

PROOF. Suppose φ is HB integrable to $I(\varphi)$, i.e.

$$I(\varphi) = (HB)\int_0^1 \varphi(t,w)dX(t,w).$$

Then for every $\varepsilon > 0$ there exists a belated full cover Δ_b depending on $\{M_k\}$ and $\delta(\xi)$ such that the conditions in Theorem 4 are satisfied. Now take a division $D = \{([u,v], y(\xi))\}$ from Δ_b, replace $y(\xi)$ in D by u, and denote the new division by $D_1 = \{([u,v], u)\}$. Then using the Itô isometry we have

$$E|(D_1)\sum \varphi(u,w)(w(v) - w(u)) - I(\varphi)|^2 = \int_0^1 E|\varphi_{D_1}(t,w) - \varphi(t,w)|^2 dt$$

where φ_{D_1} is defined accordingly as in the proof of Theorem 2. We can choose $\delta(\xi) > 0$ such that

$$\|\varphi(t,w) - \varphi(\xi,w)\|^2_{L_2} < \varepsilon/4 \text{ whenever } |t-\xi| < \delta(\xi).$$

Consequently,

$$\begin{aligned}\int_0^1 E|\varphi_{D_1}(t,w)-\varphi(t,w)|^2dt &= (D_1)\sum\int_u^v E|\varphi(u,w)-\varphi(t,w)|^2dt \\ &\le 2(D_1)\sum\int_u^v E|\varphi(u,w)-\varphi(\xi,w)|^2dt \\ &\quad +2(D_1)\sum\int_u^v E|\varphi(\xi,w)-\varphi(t,w)|^2dt \\ &< \varepsilon(D_1)\sum|v-u| = \varepsilon.\end{aligned}$$

Hence the inequality in Theorem 4 holds with D replaced by D_1.

Conversely, suppose the inequality in Theorem 4 holds with D replaced by D_1 and $I(\varphi)$ by $I_1(\varphi)$. Since the map $t \to \varphi(t,\cdot)$ is continuous in L_2, we have as $t' \to t$

$$|\|\varphi(t',\cdot)\|_{L2} - \|\varphi(t,\cdot)\|_{L_2}| \le \|\varphi(t',\cdot)-\varphi(t,\cdot)\|_{L_2} \to 0.$$

It follows that

$$\int_0^1 E|\varphi(t,w)|^2dt < \infty,$$

and $\varphi \in \mathcal{L}_2$. Then φ is HB integrable. Suppose

$$(HB)\int_0^1 \varphi(t,w)dX(t,w) = I_2(\varphi).$$

By going through the same argument as the uniqueness proof of the HB integral, we can show that $I_1(\varphi) = I_2(\varphi)$. Hence the proof is complete.

Lemma 7 *Let $\varphi = \{\varphi(t,w)\}_{0\le t\le 1}$ be defined on $(W, \mathcal{B}(W), P_W)$ and the map $t \to \varphi(X(t,w),w)$ be continuous with probability one and adapted to $\{\mathcal{B}_t; 0 \le t \le 1\}$. Further, let φ be a bounded process. Then for every $\varepsilon > 0$ there exists a full cover Δ such that for any $D = \{([u,v],\xi)\}$ from Δ, we replace ξ in D by any $\xi_w^* \in [u,v]$, denote the new division by D^*, and if*

$$\sigma = (D^*)\sum\varphi(w(\xi_w^*),w)(w(v)-w(u))^2 \in L_2$$

then we have

$$E\left|\sigma - \int_0^1 \varphi(X(s,w),w)ds\right|^2 < \varepsilon.$$

PROOF. For brevity, write $\Delta t = v - u$ and $\Delta w = w(v) - w(u)$. Then we have for $D^* = \{([u,v],\xi_w^*)\}$ and $D_1 = \{([u,v],u)\}$ in which D_1 is obtained from D^* by replacing $\xi_w^* \in [u,v]$ with u

$$\begin{aligned} & E\left|(D^*)\sum \varphi(w(\xi_w^*),w)|\Delta w|^2 - \int_0^1 \varphi(X(s,w),w)ds\right|^2 \\ \leq\ & 3E\left|(D^*)\sum \varphi(w(\xi_w^*),w)|\Delta w|^2 - (D_1)\sum \varphi(w(u),w)|\Delta w|^2\right|^2 \\ & +3E\left|(D_1)\sum \varphi(w(u),w)(|\Delta w|^2 - \Delta t)\right|^2 \\ & +3E\left|(D_1)\sum \varphi(w(u),w)\Delta t - \int_0^1 \varphi(X(s,w),w)ds\right|^2 \\ =\ & I_1 + I_2 + I_3. \end{aligned}$$

We shall show that I_1, I_2 and I_3 are small for suitably chosen D^*.

Simple calculation in probability theory (see, for example, [6; p.14]) shows that

$$E|\Delta w|^2 = \Delta t \quad and \quad E|\Delta w|^4 = 3|\Delta t|^2,$$

which in turn imply

$$E\left||\Delta w|^2 - \Delta t\right|^2 = 2|\Delta t|^2.$$

Since φ is bounded, we can assume

$$|\varphi(y,w)| \leq C_0 \quad for \quad all \quad y,w.$$

It follows that

$$\begin{aligned} I_2 &\leq 3C_0^2(D_1)\sum E\left||\Delta w|^2 - \Delta t\right|^2 \\ &\leq 6C_0^2(D_1)\sum |\Delta t|^2 \\ &\leq 6C_0^2(\max_{D_1} \Delta t)(D_1)\sum \Delta t \\ &\leq 6C_0^2 \max_{D_1} \Delta t. \end{aligned}$$

Hence given $\varepsilon > 0$ we can choose a full cover Δ with $2\delta(\xi) < \varepsilon/18C_0^2$ for each ξ such that for any D from Δ with D_1 defined as above, i.e., replace $\xi \in [u,v]$ in D by u, we have

$$I_2 < \varepsilon/3.$$

Further we define

$$\varphi_{D_1}(w(t),w) = \begin{cases} \varphi(w(0),w) & when \quad t = 0, \\ \varphi(w(u),w) & when \quad t \in (u,v], \end{cases}$$

for each $([u,v],u)$ from D_1. Put

$$M(u,v) = \sup\{|\varphi(w(s),w) - \varphi(w(t),w)|;\ s,t \in [u,v]\}.$$

For convenience, we sometimes write M_i, Δw_i, Δt_i for $M(u,v)$, Δw, Δt respectively where i runs from 1 to n. It follows that

$$\begin{aligned}
I_3 &= 3E\left|\int_0^1 \{\varphi_{D_1}(X(s,w),w) - \varphi(X(s,w),w)\}ds\right|^2 \\
&\le 3E\left|(D_1)\sum M(u,v)\Delta t\right|^2 \\
&= 3E\left|\sum_{i=1}^n M_i\Delta t_i\right|^2 \\
&= 3E\left|\sum_{i=1}^n M_i^2|\Delta t_i|^2 + 2\sum_{i<j} M_iM_j\Delta t_i\Delta t_j\right| \\
&\le 12C_0^2\sum_{i=1}^n |\Delta t_i|^2 + 2\sum_{i<j}(EM_i^2)^{1/2}(EM_j^2)^{1/2}\Delta t_i\Delta t_j \\
&\le 12C_0^2 \max_i \Delta t_i + 2(\max_i EM_i^2)\sum_{i<j}\Delta t_i\Delta t_j \\
&\le 12C_0^2 \max_i \Delta t_i + \max_i EM_i^2.
\end{aligned}$$

Next, we shall estimate $\max EM_i^2$ or $\max EM(u,v)^2$. Since $t \to \varphi(w(t),w)$ is continuous with probability one, for any $\xi \in [0,1]$ and for any $\eta_1,\eta_2 > 0$ there exists $h_0 = h_0(\xi,\eta_1,\eta_2) > 0$ such that whenever $h < h_0$

$$P_W\{\sup_{|s-\xi|\le h} |\varphi(w(s),w) - \varphi(w(\xi),w)| \ \ge\ \eta_1\} \ <\ \eta_2.$$

We may assume

$$\eta_1 + 4C_0^2\eta_2 \ <\ \frac{\varepsilon}{6} \quad \textit{and} \quad \delta(\xi) \ <\ \min\{h_0, \frac{\varepsilon}{144C_0^2}\}.$$

Then it follows that

$$\max_{D_1} EM(u,v)^2 \ \le\ \eta_1^2 + 4C_0^2\eta_2.$$

Consequently,

$$I_3 \ <\ \frac{\varepsilon}{6} + \frac{\varepsilon}{6} \ =\ \frac{\varepsilon}{3}.$$

Following the same argument as in the proof of $I_3 < \varepsilon/3$, we obtain

$$I_1 < \varepsilon/3.$$

Hence the proof is complete.

Proof of Theorem 5. Take a division $D = \{([u,v],\xi)\}$ of $[0,T]$. Write D_1 when ξ in D is replaced by u, and write D_2 when ξ in D is replaced by a given $\xi_w^* \in [u,v]$, where ξ_w^* is to be determined later. We can write with probability one

$$\begin{aligned} F(w(T),w) - F(w(0),w) &= (D)\sum\{F(w(v),w) - F(w(u),w)\} \\ &= (D_1)\sum F_y'(w(u),w)(w(v)-w(u)) \\ &\quad +\frac{1}{2}(D_2)\sum F_y''(w(\xi_w^*),w)(w(v)-w(u))^2 \end{aligned}$$

where ξ_w^* denotes some suitable value in $[u,v]$ such that the above equality holds. Note that ξ_w^* depending on w may not be a random variable. However the above equality shows that the sum $(D_2)\sum$ does belong to L_2.

Since F_y' is bounded, the conditions in Lemma 6 are satisfied with $\varphi(t,w) = F_y'(w(t),w)$. Given $\varepsilon > 0$, in view of Lemma 6 there exists a belated full cover Δ_b depending on $\{M_k\}$ and $\delta(\xi)$ and satisfying the conditions in Theorem 4 such that for any $D = \{([u,v],y(\xi))\}$ from Δ_b we write $D_1 = \{([u,v],u)\}$ and obtain

$$E|(D_1)\sum F_y'(w(u),w)(w(v)-w(u)) - \int_0^T F_y'(X(s,w),w)dX(s,w)|^2 < \varepsilon.$$

Next, in view of Lemma 7 there exists a full cover Δ_1 such that for any $D = \{([u,v],\xi)\}$ from Δ_1 we write $D_2 = \{([u,v],\xi_w^*)\}$ and obtain

$$E\left|(D_2)\sum F_y''(w(\xi_w^*),w)(w(v)-w(u))^2 - \int_0^1 F_y''(X(s,w),w)ds\right|^2 < \varepsilon.$$

We may assume Δ_b and Δ_1 above share the same $\delta(\xi) > 0$. Hence combining the above inequalities we obtain

$$\begin{aligned} E|F(w(T),w) &- F(w(0),w) - \int_0^T F_y'(X(s,w),w)dX(s,w) \\ &- \frac{1}{2}\int_0^T F_y''(X(s,w),w)ds|^2 < 3\varepsilon. \end{aligned}$$

Since ε is arbitrary, the proof is complete.

It is instructional to go through the above proof again for the special case when $F(y,w) = \frac{1}{2}y^2$ and $X(t,w)$ is a standard Brownian motion. It can be seen more clearly there how the HB integral is used to prove results. In that case, Itô's formula becomes

$$\frac{1}{2}B(t)^2 = \int_0^t B(s)dB(s) + \frac{1}{2}t,$$

where $B(t)$ denotes standard Brownian motion.

We remark that the boundedness condition of F'_y and F''_y in Theorem 5 can be removed by means of the usual localization technique (see [4; Theorem 2.5.1]).

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