Real Analysis Exchange
Vol. 18(2), 1992/93, pp. 343–351

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NORMS AND DERIVATIVES

1. Introduction

The main purpose of this note is to investigate the equation

$$(*) \qquad f^2 + g^2 = h^2$$

where f, g, and h are derivatives. In particular, we shall look for conditions under which $(f^2 + g^2)^{1/2}$, the Euclidean norm of the pair (f, g), is again a derivative. It turns out that the corresponding results hold also for some other norms. In Theorems 3.7, 3.8, and 6.3 we investigate n-tuples of derivatives, where n is any integer greater than 1.

Setting $f(x) = \sin \frac{1}{x}$, $g(x) = \cos \frac{1}{x}$ for $x \neq 0$ and $f(0) = g(0) = 0$ we have derivatives for which $f^2 + g^2$ is not the square of any derivative. On the other hand we would like to find nontrivial examples of triples (f, g, h) of derivatives fulfilling $(*)$; this can be done with the help of Proposition 3.6 and Theorems 3.7 and 3.8. Theorem 6.2 gives information about g and h under some assumptions about f provided that $(*)$ holds. Theorems 6.3, 6.8, and 6.9 point in the opposite direction; if f and g fulfill certain conditions, there is a derivative h for which $(*)$ holds. Examples 2 and 3 indicate that it would not be easy to weaken the assumptions in Theorems 6.8 and 6.9.

2. Notation

The word function means a mapping to the real line $\mathbb{R}$. By C, L, C_{ap}, D we understand the systems of all continuous functions, Lebesgue functions, approximately continuous functions and derivatives on the interval $I = [0, 1]$, respectively. Symbols like $\int_a^b f$, $\int_Q f$ denote the corresponding Lebesgue integrals. If Y is any system of functions, then bY $[Y^+]$ stands for the system of all elements of Y that are bounded [nonnegative].

Received by the editors April 24, 1992

Throughout this paper, p is an element of $(1,\infty)$ and $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. Further we set $S_p = \{f \in D;\ |f|^p \in D\}$,

$$T_p = \{f \in D;\ \limsup \frac{1}{y-x} \int_x^y |f|^p < \infty\ (y \to x, y \in I) \text{ for each } x \in I\}.$$

It is well-known that $bC_{ap} = bL$, $L \subset D \cap C_{ap}$ and that a function f integrable on I is in D if and only if $\frac{1}{y-x}\int_x^y f \to f(x)$ $(y \to x, y \in I)$ for each $x \in I$.

For any system $Y \subset D$ let $M(Y) = \{g \in D;\ fg \in D \text{ for each } f \in Y\}$.

3. Elementary and Known Results

Lemma 3.1 . *Let $f, g \in C_{ap}$, $|g| \leqq f \in D$. Then $g \in L$.*(See [M1], 1.8.)

Lemma 3.2 *L is a vector space. If $f \in L$, then $|f| \in L$.* (This is well-known.)

Proposition 3.3 *S_p is a vector space. If $f \in S_p$, then f, $|f|^p \in L$.*

PROOF. Let $f \in S_p$. It follows from Lemma 4.4 in [MW] that f, $|f|^p \in L$. If also $g \in S_p$, then $f + g \in L$ and $|f+g|^p \leqq 2^p(|f|^p + |g|^p) \in L$. By Lemma 3.1 we have $|f+g|^p \in L$ so that $f + g \in S_p$.

Proposition 3.4 *Let $q \in (1,\infty)$, $\frac{1}{p} + \frac{1}{q} = 1$. Let $f \in S_p$, $g \in T_q$. Then $fg \in D$.*

PROOF. Let $x \in I$. By Proposition 3.3 we have $f - f(x) \in S_p$ whence $\frac{1}{y-x}\int_x^y |f - f(x)|^p \to 0$. Thus $|\frac{1}{y-x}\int_x^y (f - f(x))g| \leqq (\frac{1}{y-x}\int_x^y |f - f(x)|^p)^{1/p} \cdot (\frac{1}{y-x}\int_x^y |g|^q)^{1/q} \to 0$ $(y \to x, y \in I)$. Hence $\frac{1}{y-x}\int_x^y fg = \frac{1}{y-x}\int_x^y (f - f(x))g + f(x)\frac{1}{y-x}\int_x^y g \to f(x)g(x)$ $(y \to x, y \in I)$.

Definition 1 *Let $Y \subset D$. We say that Y has property V_p if $(|\alpha|^p + |\beta|^p)^{1/p} \in Y$, whenever $\alpha, \beta \in Y$.*

Proposition 3.5 *Let Y have property V_p. Let $n \in \{2,3,\ldots\}$ and let $\alpha_1, \ldots, \alpha_n \in Y$. Then $(\sum_{j=1}^n |\alpha_j|^p)^{1/p} \in Y$.*

(The proof is left to the reader).

Proposition 3.6 *Let $Y \in \{C, bC_{ap}, S_p, L\}$. Then Y has property V_p.*

PROOF. It is obvious that C and bC_{ap} have property V_p. Now let $\alpha, \beta \in L$, $\gamma = (|\alpha|^p + |\beta|^p)^{1/p}$. Then $\gamma \in C_{ap}$, $|\gamma| \leqq |\alpha| + |\beta|$ so that, by Lemmas 3.2 and 3.1, $\gamma \in L$. If $\alpha, \beta \in S_p$, then, by Proposition 3.3, $\gamma^p \in L$ whence $\gamma \in S_p$.

Theorem 3.7 *Let Y have property V_p. Let $n \in \{2,3,\ldots\}$, $\alpha_1,\ldots,\alpha_n \in Y$, $\psi \in M(Y)$. Set $f_j = \psi\alpha_j$ $(j = 1,\ldots,n)$, $h = \psi \cdot (\sum |\alpha_j|^p)^{1/p}$. Then $f_j, h \in D$ and $\sum |f_j|^p = |h|^p$.*

(The proof is left to the reader).

Remark 1 *The systems $M(C)$ and $M(bC_{ap})$ have been characterized in [M3] (Theorems 7 and 12). According to 4.2 in [M1] we have $M(L) = bD$. Proposition 3.4 says that $T_q \subset M(S_p)$; it will be proved elsewhere that $M(S_p) = T_q$.*

Remark 2 *The next theorem shows that if $Y = S_p$, then the assumption $\alpha_j \in Y$ can be replaced by the weaker assumption that $\alpha_j \in D$ and $\sum |\alpha_j|^p \in D$.*

Theorem 3.8 *Let $n \in \{2,3,\ldots\}$. Let $\alpha_1,\ldots,\alpha_n \in D$, $\sum |\alpha_j|^p \in D$. Let $\psi \in T_q$, where $\frac{1}{p} + \frac{1}{q} = 1$. Set $f_j = \psi\alpha_j$ $(j = 1,\ldots,n)$, $h = \psi \cdot (\sum |\alpha_j|^p)^{1/p}$. Then $f_j, h \in D$ and $\sum |f_j|^p = |h|^p$.*

PROOF. Choose an $s \in (1,p)$. Then $\sum |a_j|^p \in Q_s$, where Q_s is as in Section 5.4 of [MW] (we take $r = q_1 = 1$ there), so that we can apply Theorem 5.5 of [MW] (where we take $p_i = p$ and s instead of p) and we get $\alpha_j \in L$. By Lemma 3.1 we have $|\alpha_j|^p \in L$ so that $\alpha_j \in S_p$ $(j = 1,\ldots,n)$. Now we apply Proposition 3.4, Proposition 3.6, and Theorem 3.7.

4. Conventions

For any (Lebesgue) measurable set $Q \subset \mathbb{R}$ let $|Q|$ be its measure. If Q is such a set and $b \in \mathbb{R}$, we write $d(Q,b) = \lim |Q \cap (b-x, b+x)|/2x$ $(x \to 0+)$, provided that this limit exists.

Throughout the rest of this note we write $M = M(bC_{ap})$ $(= M(bL))$.

For $v, w \in \mathbb{R}^2$ let $v \cdot w$ be their scalar product.

If $z = (x,y) \in \mathbb{R}^2$, then $\|z\|_p$ means $(|x|^p + |y|^p)^{1/p}$.

In Lemma 5.2 and Proposition 6.1 we suppose that $\|\cdot\|$ is a norm in $\mathbb{R}^2$ and that θ is a mapping of $\mathbb{R}^2$ to $\mathbb{R}^2$ with the following properties:

(i) For each $(x,y) \in \mathbb{R}^2$ we have $|y| \leqq \|(x,y)\| = \|(x,-y)\|$.

(ii) If $z = (x,y) \in \mathbb{R}^2$, $x \neq 0$ and $\theta(z) = (A,B)$, then $A \neq 0$ and $|B| \cdot \|z\| < z \cdot \theta(z)$.

(iii) If $v, z \in \mathbb{R}^2$ and $\|v\| = \|z\|$, then $v \cdot \theta(z) \leqq z \cdot \theta(z)$.

5. Preliminary Results

Proposition 5.1 *For each $z = (x, y) \in \mathbb{R}^2$ set $\|z\| = \|z\|_p$ and set $\theta(z) = (|x|^{p-1} sgn\, x, |y|^{p-1} sgn\, y)$. Then the conditions (i)-(iii) are fulfilled.*

PROOF. (i) is obvious. Let $z = (x, y) \in \mathbb{R}^2$, $x \neq 0$, $\theta(z) = (A, B)$. Clearly $A \neq 0$, $z \cdot \theta(z) = \|z\|^p = \|z\| \cdot \|z\|^{p-1}$ and $\|z\|^{p-1} = (|x|^p + |y|^p)^{(p-1)/p} > |y|^{p-1} = |B|$. Now let $v, z \in \mathbb{R}^2$, $\|v\| = \|z\|$. Define q by $\frac{1}{p} + \frac{1}{q} = 1$. Then $p = q(p-1) = 1 + p/q$, $v \cdot \theta(z) \leqq \|v\| \cdot \|\theta(z)\|_q = \|z\| \cdot (|x|^{(p-1)q} + |y|^{(p-1)q})^{1/q} = \|z\| \cdot (|x|^p + |y|^p)^{1/q} = \|z\|^{1+p/q} = \|z\|^p = z \cdot \theta(z)$.

Lemma 5.2 *Let $z \in \mathbb{R}^2 \setminus \{(0,0)\}$. Set $K = z \cdot \theta(z)/\|z\|$. Then $v \cdot \theta(z) \leqq K\|v\|$ for each $v \in \mathbb{R}^2$.*

PROOF. Let $v \in \mathbb{R}^2 \setminus \{(0,0)\}$. Set $w = v\|z\|/\|v\|$. Then $\|w\| = \|z\|$ so that, by (iii), $v \cdot \theta(z) = w \cdot \theta(z)\|v\|/\|z\| \leqq z \cdot \theta(z)\|v\|/\|z\| = K\|v\|$.

6. Main Results

Proposition 6.1 *Let $f \in M$, $g \in D \setminus M$, $f > 0$. Suppose the conditions (i) - (iii) are fulfilled. Then $\|(f, g)\| \notin D$.*

PROOF. Set $h = \|(f, g)\|$ and suppose that $h \in D$. Then all the functions f, g, h are Lebesgue integrable. By Theorem 12 in [M3] there is a $b \in I$ and a measurable set $Q \subset I$ such that $d(Q, b) = 0$ and that the relation $\frac{1}{x} \int_{Q \cap (b-x, b+x)} g \to 0$ $(x \to 0+)$ fails. We may suppose that $b = 0$ and that $\limsup \frac{1}{x} \int_{Q \cap (0,x)} g$ $(x \to 0+)$ is positive; call it u. Applying again Theorem 12 in [M3] we get

$$\frac{1}{x} \int_{Q \cap (0,x)} f \to 0 \ (x \to 0+). \tag{1}$$

Set $z = (f(0), g(0)), \theta(z) = (A, B)$, $K = z \cdot \theta(z)/\|z\|$. By Lemma 5.2 we have

$$Af + Bg \leqq Kh. \tag{2}$$

Hence

$$A \int_Z f + B \int_Z g \leqq K \int_Z h \tag{3}$$

for each measurable set $Z \subset I$. There are $x_1, x_2, \ldots \in (0, 1)$ such that $x_n \to 0$ and that, setting $Q_n = Q \cap (0, x_n)$ and $u_n = \frac{1}{x_n} \int_{Q_n} g$, we have $u_n \to u$ and

$u_n > 0$ for each n. Define t_n by

$$A\int_{Q_n} f + B\int_{Q_n} g = t_n \int_{Q_n} h. \tag{4}$$

Since $g \leqq h$, we have $|t_n|\int_{Q_n} g \leqq |t_n|\int_{Q_n} h \leqq |A|\int_{Q_n} f + |B|\int_{Q_n} g$, $|t_n|u_n \leqq |A|\frac{1}{x_n}\int_{Q_n} f + |B|u_n$; by (1) we get $\limsup |t_n| \leqq |B|$. Hence we may suppose that the sequence $\langle t_n\rangle$ converges to some t. Then $t \leqq |B|$. By (ii) we have $|B| < K$ so that $t < K$. There is an m such that $t_n < K$ for each $n > m$. From (4) and (3) with $Z = (0, x_n)\setminus Q_n$ we get $A\int_0^{x_n} f + B\int_0^{x_n} g \leqq K\int_0^{x_n} h - (K - t_n)\int_{Q_n} h$ whence $K\int_0^{x_n} h \geqq A\int_0^{x_n} f + B\int_0^{x_n} g + (K - t_n)\int_{Q_n} g$ $(n > m)$. We see that $u < \infty$. Since $z \cdot \theta(z) = K\|z\|$ and $\|z\| = h(0)$, we have

$$Kh(0) \geqq Af(0) + Bg(0) + (K - t)u = K\|z\| + (K - t)u > Kh(0);$$

a contradiction.

Theorem 6.2 *Let $f \in M$, $g, h \in D$, $f > 0$ and let $f^p + |g|^p = |h|^p$. Then $g, h \in M$.*

PROOF. We may suppose that $h > 0$. Then $h = \|(f, g)\|_p$. By Proposition 5.1 and Proposition 6.1 we have $g \in M$. Since $h \leqq f + |g|$, it follows easily from [M3] (see condition (ii) in Theorem 12) that $h \in M$.

Remark 3 Let $1 = x_0 > x_1 > \cdots$, $x_n \to 0$, $x_n/x_{n+1} \to 1$. Set $d_n = x_{n-1} - x_n$. Let $x_n < y_n < z_n < x_{n-1}$, $z_n - y_n < d_n/n$. Let l be a nonnegative function on I such that $l(0) = 1$, l is continuous on $(0,1]$, $l = 0$ on $(0,1]\setminus\bigcup(y_n, z_n)$ and $\int_{y_n}^{z_n} l = d_n$ $(n = 1, 2, \ldots)$. It is easy to see that $l \in D$. Now let $v_n \in (x_n, y_n)$, $w_n \in (z_n, x_{n-1})$, $w_n - v_n < d_n/n$. There is a function ω such that $0 \leqq \omega \leqq 1$ on I, ω is continuous on $(0,1]$, $\omega = 1$ on $\bigcup(y_n, z_n)$ and $\omega = 0$ on $I\setminus\bigcup(v_n, w_n)$. It is obvious that $\lim \text{ ap } l(x) = 0$ $(x \to 0+)$, $\omega \in bC_{ap}$ and $l\omega = l$ on $(0,1]$. Since $(l\omega)(0) = 0$, we have $l\omega \notin D$ so that $l \notin M$.

Now it is clear that there are $l, \mu \in D^+\setminus M$ continuous on $(0,1]$ such that $l(0) = \mu(0) = 1$, $\lim \text{ ap } l(x) = \lim \text{ ap } \mu(x) = 0$ $(x \to 0+)$ and $l\mu = 0$ on $(0,1]$.

Remark 4 *Example 1 shows that the requirement "$f > 0$" in Theorem 6.2 cannot be replaced by "$f > 0$ on $(0,1]$".*

Example 1 Let l be as in Remark 1. Set $f(x) = x$ $(x \in I)$, $g = 1 + l$, $h = (f^2 + g^2)^{1/2}$, $\varphi = f/g$. Then $\varphi \in C$ and $h = g(\varphi^2 + 1)^{1/2}$. Since $D^+ \subset M(C)$ (see, e.g., Theorem 7 in [M3]), we have $h \in D$. Clearly $f \in M$, $f^2 + g^2 = h^2$, $g \in D\setminus M$.

Remark 5 *If $f, g, h \in D$, $f^2 + g^2 = h^2$ and if* $\liminf \text{ ap } h(y) > 0$ $(y \to x, y \in I)$ *for each $x \in I$, then, by Proposition 4.6 of [MW], there are α, β, and ψ (namely $\psi = h$) such that*

$$f = \alpha\psi, \; g = \beta\psi, \; \alpha, \beta \in C_{ap}. \tag{5}$$

Examples 1 and 2 in [MW] (sections 5.12-5.13) indicate the importance of the corresponding assumptions. If we set $\alpha(x) = x \sin \frac{1}{x}$, $\beta(x) = x \cos \frac{1}{x}$, $\psi(x) = \frac{1}{x}$ $(x \in (0,1])$, $\alpha(0) = \beta(0) = \psi(0) = 0$, $f = \alpha\psi$, $g = \beta\psi$, $h = (f^2 + g^2)^{1/2}$, we see that the assumption $\liminf \text{ ap } h(y) > 0$ $(y \to x, y \in I)$ for each $x \in I$ does not imply that $h > 0$ on I (even if (5) holds and $f, g \in bD$).

Now we would like to find conditions for f and g that would allow us to deduce from (5) the existence of an $h \in D$ fulfilling $f^2 + g^2 = h^2$.

Theorem 6.3 *Let $n \in \{2, 3, \ldots\}$, $f_1, \ldots, f_n \in M$, $\alpha_1, \ldots, \alpha_n \in C_{ap}$. Let $\sum \alpha_j^2 > 0$ on I. Let ψ be a function on I such that $f_j = \alpha_j \psi$ $(j = 1, \ldots, n)$. Then there is an $h \in M$ such that $\sum |f_j|^p = |h|^p$.*

PROOF. Set $\gamma = (\sum |\alpha_j|^p)^{1/p}$, $\beta_j = |\alpha_j/\gamma|^{p-1} \cdot \text{sgn } \alpha_j$, $h = \sum \beta_j f_j$. Then $\beta_j \in bC_{ap}$ so that $h \in D$; clearly $h \in M$, $h = \psi \sum \beta_j \alpha_j = \psi \sum |\alpha_j|^p / \gamma^{p-1} = \psi\gamma$ whence $|h|^p = \gamma^p |\psi|^p = \sum |\alpha_j \psi|^p = \sum |f_j|^p$.

Remark 6 *The reader may compare this theorem with Proposition 5.10 and Theorem 5.11 in [MW].*

Our next goal is Theorem 6.8 which is a modification of Theorem 6.3 with $n = 2$. In Theorem 6.8 we still assume that $f_1 \in M$, but the requirement $f_2 \in M$ is replaced by other conditions. We need a few lemmas.

Lemma 6.4 *Let $f \in M$, $g \in D$, $f^2 + g^2 > 0$, $g \geqq -|f|$. Let α, β, ψ be functions on I such that (5) holds. Let $a, b \in I$, $a < b$ and let $\psi > 0$ on (a, b). Then $\psi(a) > 0$.*

PROOF. Clearly $\psi(a) \neq 0$. We distinguish two cases.

(i) $f(a) \neq 0$. Set $\gamma = (\alpha^2 + \beta^2)^{1/2}$, $\varphi = \psi\alpha^2/\gamma$. Since $\varphi \geqq 0$ on (a, b) and $\varphi = f\alpha/\gamma \in D$, we have $\varphi(a) \geqq 0$, $\psi(a) > 0$.

(ii) $f(a) = 0$. Then $\alpha(a) = 0$, $\beta(a) \neq 0$, $g(a) \geqq -|f(a)| = 0$. Since $\psi > 0$ on (a, b), we have $\beta \geqq -|\alpha|$ on (a, b), $\beta(a) \geqq -|\alpha(a)| = 0$, $\beta(a) > 0$, $\psi(a) > 0$.

Lemma 6.5 *Let $f, g, \alpha, \beta, \psi$ be as before. Then sgn ψ is constant.*

PROOF. Set $\sigma = \operatorname{sgn} \psi$. Clearly $\psi = (\alpha f + \beta g)/(\alpha^2 + \beta^2)$, $\sigma = \psi/|\psi|$. Then σ is a Baire one function. Let G be the set of all points where σ is continuous (with respect to I). It is easy to see that G is open in I and that σ is constant on each component of G. Let $F = I \setminus G$. Suppose that $F \neq \emptyset$. Then there is a $b \in F$ such that σ is continuous at b with respect to F. There is an open interval J such that $b \in J$ and that σ is constant on $F \cap J$. By Lemma 6.4 σ is constant on the closure of each component of G. Hence σ is constant on $J \cap I \subset G$; a contradiction. Thus $F = \emptyset$, $G = I$ and σ is constant on I.

Lemma 6.6 *Let $A = \{(0,y); y \in (-\infty, 0]\}$, $G = \mathbb{R}^2 \setminus A$. Define a function F on G setting $F(x,y) = ((|x|^p + |y|^p)^{1/p} - y)/x$ ($x \neq 0$), $F(0,y) = 0$ ($y > 0$). Then F is continuous.*

PROOF. It is obvious that F is continuous at each point (x,y), where $x \neq 0$. Now let y_0, $\varepsilon \in (0,\infty)$. Let $y > y_0/2$, $|x| < y_0(\varepsilon p)^{1/(p-1)}/2$. Set $t = \varepsilon|x|/y$. Then $|x|^{p-1} < y^{p-1}\varepsilon p$, $|x|^p < y^p t p$, $|x|^p + y^p < y^p(1+tp) \leqq y^p(1+t)^p = (y+\varepsilon|x|)^p$, $(|x|^p + y^p)^{1/p} - y < \varepsilon|x|$, $|F(x,y)| < \varepsilon$. This proves the continuity of F at $(0, y_0)$.

Lemma 6.7 *Let G, F be as before. Let $k \in (0,\infty)$, $H = \{(x,y) \in G;\ y \geqq -k|x|\}$. Then $|F| \leqq 2k+1$ on H.*

PROOF. If $x, y \in (0,\infty)$, then $x^p + y^p < (x+y)^p$ so that $0 < F(x,y) < 1$. Now let $x > 0$, $-kx \leqq y \leqq 0$. Then $(x^p + |y|^p)^{1/p} - y \leqq x + |y| + |y| \leqq (2k+1)x$ whence $0 < F(x,y) \leqq 2k+1$. Clearly $F(-x,y) = -F(x,y)$.

Theorem 6.8 *Let $f \in M$, $g \in D$, $f^2 + g^2 > 0$. Let α, β, ψ be functions such that (5) holds. Let $k \in (0,\infty)$, $g \geqq -k|f|$. Then $(|f|^p + |g|^p)^{1/p} \in D$.*

PROOF. Let G, H, F be as before. By Lemma 6.5 we may suppose that $\psi > 0$. Then $(\alpha(t), \beta(t)) \in H$ for each $t \in I$. Clearly $y + xF(x,y) = (|x|^p + |y|^p)^{1/p}$ for each $(x,y) \in G$. Hence $(|f|^p + |g|^p)^{1/p} = g + fF(f,g)$ on I. It is easy to see that $F(f,g) = F(\alpha,\beta)$; by Lemmas 6.6 and 6.7 we have $F(\alpha,\beta) \in bC_{ap}$. Thus $fF(f,g) \in D$ which proves our assertion.

Remark 7 Taking $f(x) = \sin\frac{1}{x}$ ($x \in (0,1]$), $f(0) = 0$, $g = \psi = f$ and $\alpha = \beta = k = 1$ we see that the assumption "$f^2+g^2 > 0$" in Theorem 6.8 cannot be dropped. If, however, $f \in M^+$, $g \in D$, $\beta \in C_{ap}$, β is bounded below and $g = \beta f$, then, clearly, $(f^p + |g|^p)^{1/p} = g + f\varphi$, where $\varphi = (1+|\beta|^p)^{1/p} - \beta \in bC_{ap}$ so that $(f^p + |g|^p)^{1/p} \in D$. Thus we may ask whether "$f^2 + g^2 > 0$" can be replaced by "$f \geqq 0$". The next theorem gives a positive answer to this question.

Theorem 6.9 *Let $f \in M^+$, $g \in D$. Let α, β, ψ be functions such that* (5) *holds. Let $k \in (0, \infty)$, $g \geqq -kf$. Then $(f^p + |g|^p)^{1/p} \in D$.*

PROOF. Set $h = (f^p + |g|^p)^{1/p}$. If $f(x) = g(x) = 0$, set $\varphi(x) = 0$; otherwise set $\varphi(x) = F(f(x), g(x))$, where F is as in Lemma 6.6. Clearly $h = g + f\varphi$ and, by Lemma 6.7, $|\varphi| \leqq 2k + 1$ on I. A simple computation shows that $g = (|k\alpha + \beta| - k|\alpha|)|\psi|$. Thus we may suppose that $\alpha, \psi \geqq 0$. Set $W = \{f > 0\}$. On W we have $\psi > 0$ and $\varphi = F(\alpha, \beta)$ so that φ is approximately continuous at each point of W. It follows easily from Proposition 11 in [M3] (see condition (iv)) that $\frac{1}{y-x}\int_x^y f\varphi \to f(x)\varphi(x)$ $(y \to x, y \in I)$ for each $x \in W$. If $f(x) = 0$, then the inequalities $g \leqq |g| \leqq h \leqq |g| + f \leqq g + (2k+1)f$ imply that $\frac{1}{y-x}\int_x^y h \to g(x) = h(x)$. Thus $h \in D$.

Remark 8 Let $f = 1$, $g \in C_{ap} \cap D$, $\int_I |g| = \infty$. Set $\alpha = \psi = 1$, $\beta = g$. Then $f \in M$, $f^2 + g^2 > 0$, (5) holds, but, obviously, $(f^2 + g^2)^{1/2} \notin D$. We see that the requirement "$g \geqq -k|f|$" in Theorem 6.8 or Theorem 6.9 cannot be dropped. This example raises naturally the question whether "$g \geqq -k|f|$" cannot be replaced by "$\int_I |g| < \infty$". Example 2 shows that this is not possible; the corresponding function g is the difference of two nonnegative derivatives. (Not every Lebesgue integrable derivative can be expressed in this way.) A more complicated example (not given here) shows that not even "$|g| \in D$" can replace "$g \geqq -k|f|$" in Theorem 6.8 or Theorem 6.9.

Example 3 in [MW] (section 5.14) shows that the requirement "$f \in M$" in Theorem 6.8 cannot be replaced by "$f \in D^+$". Our Example 3 shows the same thing in a simpler way. Theorem 6.11 shows that the assumption "$f \in M$" is very essential. This theorem follows easily from Proposition 6.10 that is stated without proof.

Example 2 Let l, μ be as in Remark 1, section 15. Set $f = 1$, $g = l - \mu$, $h = (f^2 + g^2)^{1/2}$, $\alpha = \psi = 1$, $\beta = g$. Then $f \in M$ and (5) holds. However, $h \geqq |g| = l + \mu$ on $(0, 1]$ whence $\liminf \frac{1}{x}\int_0^x h \geqq l(0) + \mu(0) = 2 > 1 = h(0)$. Thus $h \notin D$.

Example 3 Let l, μ be as before. Set $f = 1 + l$, $g = 1 + \mu$, $h = (f^2 + g^2)^{1/2}$. Then $f, g \in D$, $g/f \in C_{ap}$. Since $l^2 + \mu^2 = (l + \mu)^2$ on $(0, 1]$, we have there $h^2 > 1 + 2(l + \mu) + l^2 + \mu^2 = (1 + l + \mu)^2$ so that $\liminf \frac{1}{x}\int_0^x h \geqq 1 + l(0) + \mu(0) = 3 > 2\sqrt{2} = h(0)$. Thus $h \notin D$. It is obvious that $g > -|f|$ and that (5) holds with $\alpha = 1$, $\beta = g/f$ and $\psi = f$.

Proposition 6.10 *Let $f \in D \setminus M$ and let $\varepsilon \in (0, 1)$. Then there is a $\beta \in C_{ap}$ such that $|\beta - 1| < \varepsilon$ on I, $\beta f \in D$ and $(f^2 + (\beta f)^2)^{1/2} \notin D$.*

Theorem 6.11 *Let $f \in D \setminus M$ and let $f > 0$ on I. Then there is a $k \in (0, \infty)$ and functions g, α, β, ψ such that* (5) *holds, $g \in D$, $g \geqq -k|f|$ and $(f^2 + g^2)^{1/2} \notin D$.*

PROOF. Let $\varepsilon = \frac{1}{2}$ and let β be as in Proposition 6.10. Now it suffices to take $\alpha = k = 1$, $\psi = f$, and $g = \beta f$.

Remark 9 *Some of the results of this note with $p = 2$ have been stated without proof in [M2].*

References

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