Real Analysis Exchange
Vol. 18(2), 1992/93, pp. 339–342

Elzbieta Wagner-Bojakowska, Institute of Mathematics, University of Łódź, 90-238 Łódź, Poland.

SOME REMARKS ON DENSITY TOPOLOGIES ON THE PLANE

The aim of this note is to prove that the topological spaces $(\mathbb{R}^2, d^2)$ and $(\mathbb{R}^2, d \times d)$ are not homeomorphic.

Let $\mathbb{N}$ denote the set of positive integers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{R}$ the real line, $\mathbb{R}^2$ the plane, $\mathcal{L}^1, \mathcal{L}^2$ the families of Lebesgue measurable sets on the real line and on the plane, respectively.

If $A \in \mathcal{L}^i$, then $m_i(A)$ denotes the Lebesgue measure of A, $i = 1, 2$.

Let $A \in \mathcal{L}^1$, $x \in \mathbb{R}$. The density of A at x is defined as follows:

$$d(A, x) = \lim_{h \to 0+} \frac{m_1(A \cap (x - h, x + h))}{2h}.$$

If $d(A, x) = 1$, then we say that x is a density point of A. The set of all density points of A is denoted by $d(A)$.

The family of sets $d = \{A \in \mathcal{L}^1 : A \subset d(A)\}$ forms a topology called density topology (see [4]). In the analogous way we define the density topology d^2 on the plane, using in the definition of the density of A at a point (x, y) the square $(x - h, x + h) \times (y - h, y + h)$.

Let $d \times d$ denote the product of two density topologies.

If τ is a topology, then by $\mathcal{B}(\tau)$, $\mathcal{G}_\delta(\tau)$, $\mathcal{F}_\sigma(\tau)$ we denote the families of Borel sets, $\mathcal{G}_\delta$ sets and $\mathcal{F}_\sigma$ sets with respect to the topology τ, respectively.

Observe first that most of the topological properties (for terminology see [3], Chapter 1) of the spaces $(\mathbb{R}^2, d^2)$ and $(\mathbb{R}^2, d \times d)$ are the same. It is easy to see that these topological spaces are not separable because countable sets are closed in both of them.

From Theorem 2 and 3 in [2] and from Theorem 2.3.11 in [1] it follows that the topological spaces $(\mathbb{R}^2, d \times d)$ and $(\mathbb{R}^2, d^2)$ are completely regular but not normal. Consequently, they are not Lindelöf spaces (see Th. 3.8.2 in [1]).

Theorem 1 *The spread, the weight and the Lindelöf-degree of* $(\mathbb{R}^2, d \times d)$ *and* $(\mathbb{R}^2, d^2)$ *are equal to* $2^{\aleph_0}$.

Received by the editors January 24, 1992

PROOF. Let C be the Cantor set of Lebesgue measure zero. It is easy to see that $C \times \{0\}$ is a closed discrete subspace of cardinality $2^{\aleph_0}$ of both of the spaces $(\mathbb{R}^2, d^2)$ and $(\mathbb{R}^2, d \times d)$. Consequently the spread of these spaces is equal to $2^{\aleph_0}$.

From Theorem 4.10 in [5] and from the table of invariants of operations in [1] it follows that the weights of both topological spaces are equal to $2^{\aleph_0}$.

Theorem 2.1 (b) in [3] implies that the Lindelöf-degrees of $(\mathbb{R}^2, d^2)$ and $(\mathbb{R}^2, d \times d)$ are less than or equal to $2^{\aleph_0}$. On the other hand, the family of sets $\{U_x, x \in C\}$, where C is the Cantor set of Lebesgue measure zero on the x-axis and

$$U_x = \mathbb{R} \times [(\infty, 0) \cup (0, \infty)] \cup [(\mathbb{R} \backslash C) \cup \{x\}] \times \mathbb{R},$$

is a $d \times d$ - and d^2 - open cover of $\mathbb{R}^2$ which has no subcover of cardinality less than $2^{\aleph_0}$.

The cellularities of $(\mathbb{R}^2, d \times d)$ and $(\mathbb{R}^2, d^2)$ are equal to $\aleph_0$ because of C.C.C. It is easy to see that only finite sets are compact in both the spaces.

Theorem 2 *The densities, the tightness, the π-weights and the characters of $(\mathbb{R}^2, d^2)$ and $(\mathbb{R}^2, d \times d)$ are greater than $\aleph_0$ but not greater than $2^{\aleph_0}$.*

PROOF. Since countable sets are closed with respect to both the topologies, therefore the densities of those two topological spaces are greater than $\aleph_0$. Also, from Theorem 2.1 (b) in [3] it follows that the density is not greater than the weight for every topological space. Consequently, the densities of $(\mathbb{R}^2, d^2)$ and $(\mathbb{R}^2, d \times d)$ are not greater than $2^{\aleph_0}$.

From Theorem 2.1 (a) in [3] it follows that the π-weight of each of the topological spaces is greater than $\aleph_0$ but not greater than $2^{\aleph_0}$.

Theorem 2.1 (e) in [3] implies that the character is not greater than the weight for every topological space. Consequently, the characters of $(\mathbb{R}^2, d^2)$ and $(\mathbb{R}^2, d \times d)$ are not greater than $2^{\aleph_0}$. On the other hand, from Theorem 4.11 in [5] it follows that the characters of $(\mathbb{R}, d)$ and $(\mathbb{R}^2, d^2)$ are greater than $\aleph_0$. By the table of invariants of operations in [1], the character of $(\mathbb{R}^2, d \times d)$ is greater than $\aleph_0$, too.

It is easy to see ([3], Th. 2.1 (f)) that the tightness of a topological space is not greater than the cardinality of this space. On the other hand, countable sets are closed in $(\mathbb{R}^2, d^2)$ and $(\mathbb{R}^2, d \times d)$. Consequently, the tightness of each of the considered spaces is greater than $\aleph_0$ but not greater than $2^{\aleph_0}$.

If we suppose Martin's Axiom or the continuum hypothesis, then all cardinal functions from the last theorem are equal to $2^{\aleph_0}$ (compare [5], Th. 4.12).

If $E \subset \mathbb{R}$, $a \in \mathbb{R}$, then we put $E - a = \{x - a, x \in E\}$.

Let $A = \{(x,y) \in (0,1) \times (0,1) : y - x \in \mathbb{Q}\}$. We have

$$A = \bigcup_{w \in \mathbb{Q}} ([(0,1) \times (0,1)] \cap \{(x,y) in \mathbb{R}^2 : y - x = w\}),$$

so, A is a set of type $\mathcal{F}_\sigma$ with respect to the Euclidean topology on the plane and also with respect to the topology $d \times d$.

Theorem 3 *The set A is not of type $\mathcal{G}_\delta$ with respect to the topology $d \times d$.*

PROOF. For every $H \subset \mathbb{R}^2$ we shall denote $W(H) = \{y - x : (x,y) \in H\}$. We shall prove that if U is a $\mathcal{G}_\delta$ set in the $d \times d$ topology containing the set A, then $W(U)$ is uncountable. Since $W(A) \subset \mathbb{Q}$, this will imply that A is not a $\mathcal{G}_\sigma$ set in the $d \times d$ topology.

Let $A \subset U = \bigcap_{n=1}^{\infty} G_n$, where each G_n is a $d \times d$ - open set. Suppose that $W(U)$ is countable, and let $W(U) = \{w_n\}_{n \in \mathbb{N}}$. We shall construct a sequence of non-empty compact sets $F_0 \supset F_1 \supset \cdots$ such that $F_n \subset G_n$ and $w_n \notin W(F_n)$ for every $n = 1, 2, \ldots$.

We put $F_0 = [0,1] \times [0,1]$. Let $n \geq 0$ and suppose that $F_n = A_n \times B_n$ has been defined such that A_n, B_n are compact subsets of $\mathbb{R}$ of positive measure. Let $f(t) = m_1(A_n \cap (B_n - t))$, $t \in \mathbb{R}$. It is well known that f is a continuous function of t. Since $f(t) > 0$ for some t (for example, if a and b are density points of A_n and B_n, respectively, then $f(b-a) > 0$), we can select a $t \in \mathbb{Q}$ such that $t \neq w_{n+1}$ and $f(t) > 0$. Let x be a density point of $A_n \cap (B_n - t)$. Then $(x, x+t) \in A \subset G_{n+1}$ and hence there are d - open sets $E, F \subset \mathbb{R}$ such that $(x, x+t) \in E \times F \subset G_{n+1}$. Then x is a density point of both of the sets A_n and E and $x + t$ is a density point of both of the sets B_n and F. Let $0 < \delta < |w_{n+1} - t|/2$, and let

$$A_{n+1} \subset A_n \cap E \cap (x - \delta, x + \delta),$$

$$B_{n+1} \subset B_n \cap F \cap (x + t - \delta, x + t + \delta)$$

be closed sets of positive measure. Putting $F_{n+1} = A_{n+1} \times B_{n+1}$, we have $F_{n+1} \subset F_n \cap G_{n+1}$ and $w_{n+1} \notin W(F_{n+1})$, since $(x,y) \in F_{n+1}$ implies $|y - x - t| < 2\delta$ and $|w_{n+1} - t| > 2\delta$.

In this way we have constructed the sets F_n for every $n = 0, 1, \ldots$. Then $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$; let (x,y) be a point of this intersection. Then $(x,y) \in \bigcap_{n=1}^{\infty} G_n = U$ and hence $y - x \in W(U)$. On the other hand, $y - x \in W(F_n)$ for every n, and thus $y - x \neq w_n$, $(n = 1, 2, \ldots)$, which is a contradiction.

Remark 1 *A more elaborate version of this proof gives that if U is a $\mathcal{G}_\delta$ set in the $d \times d$ topology containing the set A, then $W(U)$ contains a closed uncountable set and hence its cardinality is continuum.*

Corollary 1 *The topological spaces* $(\mathbb{R}^2, d \times d)$ *and* $(\mathbb{R}^2, d^2)$ *are not homeomorphic.*

PROOF. Observe that $\mathcal{F}_\sigma(d^2) = \mathcal{L}^2$. The inclusions

$$\mathcal{F}_\sigma(d^2) \subset \mathcal{B}(d^2) \subset \mathcal{L}^2$$

are obvious. If $B \in \mathcal{L}^2$, then $B = D \cup E$ where D is of type $\mathcal{F}_\sigma$ with respect to the Euclidean topology on the plane, and $m_2(E) = 0$. Thus $D \in \mathcal{F}_\sigma(d^2)$ and E is d^2-closed. Consequently, $B \in \mathcal{F}_\sigma(d^2)$ and $\mathcal{L}^2 = \mathcal{F}_\sigma(d^2) = \mathcal{G}_\delta(d^2)$.

Suppose now that there exists a homeomorphism $H : (\mathbb{R}^2, d\times d) \to (\mathbb{R}^2, d^2)$. The set A from the last theorem is of type $\mathcal{F}_\sigma$ with respect to the topology $d \times d$, so, $H(A)$ is of type $\mathcal{F}_\sigma$ with respect to the topology d^2. But $\mathcal{F}_\sigma(d^2) = \mathcal{G}_\sigma(d^2)$. Consequently, $H(A) \in \mathcal{G}_\delta(d^2)$ and $A = H^{-1}(H(A)) \in \mathcal{G}_\delta(d\times d)$, which contradicts Theorem 3.

References

[1] R. Engelking, *General topology*, PWN, Warszawa, 1977.

[2] C. Goffman, C. Neugebauer, T. Nishiura, *Density topology and approximate continuity*, Duke Math. J. **28** (1961), 497–505.

[3] I. Juhász, *Cardinal functions in topology - ten years later*, Mathematisch Centrum, Amsterdam, 1980.

[4] J. C. Oxtoby, *Measure and category*, Springer-Verlag, New York, 1971.

[5] F. D. Tall, *The density topology*, Pacific J. Math. **62** (1976), 275–284.