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ON BOREL SETS WITH SMALL COVER: A PROBLEM OF M. LACZKOVICH

The following problem has been raised by M. Laczkovich [1]: is it true, that a Borel set $H \subset \mathbb{R}$ can not be covered by zero measure $\mathcal{F}_\sigma$ set (the small cover referred to in the title) if and only if H is residual in a closed set metrically dense in itself (i.e. every portion of the closed set has positive measure)?

In this paper we answer the above question in the affirmative, moreover we show that the equivalence of the two properties in question holds also for analytic sets. It remains open, how far the equivalence could be extended to higher projective classes. As for measurable sets, the next remark was communicated by J. Mycielski and R. Laver:

There exists a set $X \subset [0,1]$ of measure 0 such that X cannot be covered by an $\mathcal{F}_\sigma$ of measure 0 and X is not residual in any perfect set.

The same counterexample was found by one of the other referees as well. Moreover, as it was pointed out by him, assuming the Axiom of Constructibility it follows that there are projective classes for which this equivalence fails. Thus the question to ask is whether it is consistent with ZFC (for instance, assuming the Axiom of Projective Determinacy) that every projective set satisfies this equivalence, or it is provable in ZFC that a projective counterexample exists.

We carry out the proof for subsets of [0,1], though our theorem holds for polish (i.e. separable complete metric) spaces replacing $[0,1]$, and for finite continuous Borel measures replacing the Lebesgue measure, as well. The proof itself easily extends to the case when the underlying space is a closed subspace of the irrational numbers. Any zero dimensional polish space is homeomorphic to such a space, on the other hand, given a finite Borel measure on any polish space, one can easily find a zero dimensional $\mathcal{G}_\delta$ subspace of full measure, and hence the generalization readily follows.

Definitions and notation. The closure and the interior of H is denoted by $\overline{H}$, *int* H respectively, and λ denotes the Lebesgue measure.

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to be metrically dense in itself, or briefly, P is an md set. If P is a closed set and $I \cap P$ is a portion, then $\overline{I} \cap P$ is called a closed portion of P.

Let $\mathcal{C}$ denote the family of sets of real numbers that can be covered by a zero measure $\mathcal{F}_\sigma$ set. It is immediate, that $\mathcal{C}$ is a sub-σ-ideal in the system $\mathcal{Z}$ of sets of measure zero as well as in the system $\mathcal{I}$ of sets of first category: $\mathcal{C} \subset \mathcal{Z} \cap \mathcal{I}$.

Sets with $H \notin \mathcal{C}$ will be called noncoverable, or (NC) sets and we refer to this relation as the (NC) property.

A set H is said to have the (RR) (relative residual) property, if there exists a closed md set P such that $H \cap P$ is a residual subset relative in P.

We use the Souslin operation to represent analytic sets in the standard way (see [2], or [4]). $\mathcal{N}$ denotes the set $\{s\}$ of all infinite sequences of natural numbers. p, q, r denote multi-indices, i.e. finite sequences of natural numbers. If $p = (n_1, ..., n_k)$, $q = (n_1, ..., n_k, n_{k+1}, ..., n_l)$, $r = (m_1, ..., m_j)$, we write $|p| = k$, $p|q$, $pr = (n_1, ..., n_k, m_1, ..., m_j)$. For $s = (n_1, ..., n_k, ...) \in \mathcal{N}$ we put $s|k = (n_1, ..., n_k)$, $s^k = (n_{k+1}, n_{k+2}, ...)$, $rs = (m_1, ..., m_j, n_1, ...)$.

For every interval $I = (x - h, x + h)$ and $t > 0$, denote $tI = (x - th, x + th)$.

Our main result in this paper is the following

Theorem 1 *Let $H \subset [0,1]$ be an analytic set. If $H \notin \mathcal{C}$, then there exist a closed md set B and a $\mathcal{G}_\delta$ set $C \subset (H \cap B)$ such that C is everywhere dense in B. In particular, every analytic (NC) set contains a $\mathcal{G}_\delta$ (NC) set.*

Remarks. (i) It should be noted, that the implication (RR) $\Longrightarrow$ (NC) is trivial for any set H. Indeed, if P is md and F is a closed set with $\lambda(F) = 0$, then $F \cap P$ must be nowhere dense in P, and hence any set $H \in \mathcal{C}$ is of first category relative in P. Thus, having (NC) $\Longrightarrow$ (RR) by the Theorem, we obtain that if $H \subset [0,1]$ is an analytic set, then properties (NC) and (RR) on H are indeed equivalent to each other.

(ii) By (i) it is obvious, that $\mathcal{C}$ is a proper subfamily in $\mathcal{Z} \cap \mathcal{I}$ which indicates the "smallness" of its elements.

(iii) By Theorem 13.4 in [3], $A \subset \mathbb{R}$ is a set of first category if and only if there exists a homeomorphism h of the real line onto itself such that $h(A) \in \mathcal{C}$. This is obviously equivalent to the following statement: $A \subset \mathbb{R}$ is of second category if and only if $h(A)$ is (NC) for every homeomorphism $h : \mathbb{R} \to \mathbb{R}$. Hence, making use of our result, the following corollary is immediate.

Corollary 1 *An analytic set $A \subset \mathbb{R}$ is of second category if and only if $h(A)$ has the (RR) property for every homeomorphism $h : \mathbb{R} \to \mathbb{R}$.*

Lemma 1 *Let the sets F_n be (relative) open in $\overline{F_n}$. Then $F = \cap F_n$ is a $\mathcal{G}_\delta$ set in $P = \cap \overline{F_n}$. In particular, if F is dense in P, then it is a dense $\mathcal{G}_\delta$ set in P.*

PROOF. We have $F_n = G_n \cap \overline{F_n}$ for some proper open sets G_n.

$$F = \bigcap_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty}(G_n \cap \overline{F_n}) = (\bigcap_{n=1}^{\infty} G_n) \cap (\bigcap_{n=1}^{\infty} \overline{F_n}) = P \cap \bigcap_{n=1}^{\infty} G_n,$$

showing that F is indeed a $\mathcal{G}_\delta$ subset.

Lemma 2 *For an arbitrary* $X \subset [0,1]$ *denote*

$$X^* = X \setminus \bigcup \{I : \text{ the portion } \; I \cap X \in \mathcal{C}\}.$$

Then
(i) $X^* = X \setminus \bigcup\{I : I$ *is a rational interval and* $I \cap X \in \mathcal{C}\}$*;*
(ii) $X \setminus X^* \in \mathcal{C}$*,* X^* *is relative closed in* X*;* $X \in \mathcal{C} \iff X^* = \emptyset$*;*
(iii) if $X \notin \mathcal{C}$*, then for every portion* $I \cap X^* \notin \mathcal{C}$*; in particular,* $\overline{X^*}$ *is md or empty;*
(iv) if $X = \bigcup_{j=1}^{\infty} X_j$ *then* $\bigcup_{j=1}^{\infty} X_j^* \subset X^* \subset \overline{X^*} \subset \overline{\bigcup_{j=1}^{\infty} X_j^*}$.

PROOF. All the statements are trivial. Let us verify $X^* \subset \overline{\bigcup_{j=1}^{\infty} X_j^*}$ only. Suppose $x \notin \overline{\bigcup_{j=1}^{\infty} X_j^*}$. Then there exists a neighborhood $x \in I$ such that $I \cap (\overline{\bigcup_{j=1}^{\infty} X_j^*}) = \emptyset$, and hence $I \cap X_j^* = \emptyset$, $j = 1, \ldots$. That is, $I \cap X_j \in \mathcal{C}$, $j = 1, \ldots$. Thus $I \cap X = \bigcup_{j=1}^{\infty}(I \cap X_j) \in \mathcal{C}$ proving $x \notin X^*$.

Lemma 3 *Let* A_n*,* $n = 1, 2, \ldots$ *be a sequence of perfect sets, denote* $U = \bigcup_{n=1}^{\infty} A_n$*,* $F = \overline{U}$*. Let* $\varepsilon > 0$*,* $\eta > 0$*, and finitely many open intervals* $J_1, J_2, \ldots, J_N$ *be given. Then there exist pairwise disjoint closed sets* $C_k \subset A_k$ *such that*

(i) $C = \bigcup_{k=1}^{\infty} C_k$ *is a dense open set in* $B = \overline{\bigcup_{k=1}^{\infty} C_k} = \overline{C}$*;*

(ii) $\lambda(B \cap J_k) \geq (1 - \varepsilon)\lambda(F \cap J_k)$ $(k = 1, \ldots, N)$*;*

(iii) if $C_k \neq \emptyset$*,* C_k *consists of finitely many closed portions of* A_k*, such that the measure of each portion is at most* η *and the end points of the underlying interval are not isolated points of the portion ;*

(iv) if A_k *is an md set for* $k = 1, \ldots,$ *then* B *is md as well.*

PROOF. The sets C_k will be defined by induction. We put $C_1 = A_1$. Note that, if $I \cap D$ is a portion of a perfect set D, then $\overline{I \cap D} = \overline{I'} \cap D$ is a closed portion of D (not necessarily equal to $\overline{I} \cap D$) such that the end points of I' are never isolated points of $\overline{I'} \cap D$. Choose now an integer $K > \frac{1}{\eta}$, let the closed intervals I_i^1 be defined by $I_i^1 \cap F = \overline{(\frac{i-1}{K}, \frac{i}{K}) \cap F}$, and put

$$F_1 = \bigcup_{i=1}^{N} I_i^1 \cap F.$$

Thus F_1 consists of nonoverlapping closed portions of F, and each one of these portions is a perfect set.

Suppose that the closed sets F_j and $C_j \subset (A_j \cap F_j)$ have been defined for $j = 1, ..., n$ and for $2 \leq j \leq n$ they satisfy the following properties:

$$(1) \qquad F_j \cap C_{j-1} = \emptyset, \quad F_j \subset F_{j-1},$$

$$(2) \qquad \lambda((F_j \cup C_{j-1}) \cap J_k) \geq (1 - \frac{\varepsilon}{2^{j-1}})\lambda(F_{j-1} \cap J_k) \quad (k = 1, ..., N),$$

$$(3) \qquad F_j = \bigcup_{i=1}^{\nu_j} (I_i^j \cap F_1),$$

where I_i^j are nonoverlapping closed intervals for every fixed j such that the endpoints of I_i^j are not isolated points of the closed portion $I_i^j \cap F_1$. The intervals I_i^j are called the main intervals of F_j. Now the induction proceeds as follows. Consider $F_n = \bigcup_{i=1}^{\nu_n}(I_i^n \cap F_1)$. Suppose first, that $I_i^n \cap C_n \neq \emptyset$ for some i, and denote shortly $P = I_i^n \cap F_1$. Then $P \setminus C_n$ is open in P, thus $P \setminus C_n = \bigcup_{j=1}^{\infty}(P \cap I_j)$, where $I_j \subset I_i^n$ are the disjoint contiguous intervals of a suitable (relative in I_i^n) open set $G = \bigcup_{j=1}^{\infty} I_j$. For a real number $m > 0$ denote

$$L_m = L_m(P) = (\bigcup_{j \leq m} [I_j \setminus (1 - \frac{1}{m})I_j)] \cup (\bigcup_{j > m} I_j).$$

Since $\lambda(L_m) \to 0$ for $m \to \infty$, we can choose m so large, that the estimates

$$(4) \qquad \lambda((F_n \setminus \bigcup_P L_m(P)) \cap J_k) \geq (1 - \frac{\varepsilon}{2^n})\lambda(F_n \cap J_k) \quad (k = 1, ..., N)$$

all hold true. Let $M = \{j : 1 \leq j \leq m, \ ((1 - \frac{1}{m})I_j) \cap P \neq \emptyset\}$ and for $j \in M$ let the closed interval $I_j(P)$ be defined by $I_j(P) \cap P = \overline{((1 - \frac{1}{m})I_j) \cap P}$. By the remark above, the endpoints of $I_j(P)$ are not isolated points of $I_j(P) \cap F_1$. F_{n+1} within P is defined by $F_{n+1} \cap P = (\bigcup_{j \in M} I_j(P)) \cap F_n = (\bigcup_{j \in M} I_j(P)) \cap F_1$. Thus the intervals $I_j(P)$, $j \in M$ are main intervals for F_{n+1}. If, on the other hand $I_i^n \cap C_n = \emptyset$ for some i, then I_i^n will be preserved as a main interval for F_{n+1} as well. That is

$$F_{n+1} = \bigcup_i \{I_i^n \cap F_1 : \ I_i^n \cap C_n = \emptyset\} \cup \bigcup_{j,P} \{I_j(P) \cap F_1 : \ P \cap C_n \neq \emptyset, \ j \in M\}.$$

Thus $C_n \cap F_{n+1} = \emptyset$, showing (1) for $j = n+1$, and since $(F_n \setminus \bigcup_P L_m(P)) \subset (C_n \cup F_{n+1})$, we have

$$\lambda((C_n \cup F_{n+1}) \cap J_k) \geq (1 - \frac{\varepsilon}{2^n})\lambda(F_n \cap J_k)$$

for $k = 1, ..., N$, which is (2) for $j = n+1$. (3) is obvious by the definition. Now we put

$$C_{n+1} = \bigcup_{i=1}^{\nu_{n+1}} \overline{A_{n+1} \cap int\, I_i^{n+1}} \subset A_{n+1} \cap F_{n+1}. \tag{5}$$

(Note that $C_{n+1} = A_{n+1} \cap F_{n+1} \setminus$ finite set.) Thus the sets C_k, F_k satisfy (1), (2) and (3) by induction. In particular, C_j are pairwise disjoint, $(C_1 \cup C_2 \cup ... \cup C_{j-1}) \cap F_j = \emptyset$, and $C_k \subset F_j$ for any $k \geq j$. Thus $\overline{\bigcup_{l=j}^{\infty} C_l} \cap (C_{j-1} \cup ... \cup C_1) = \emptyset$. This shows that C_k is open in B, and hence C is indeed a dense open set in B. In order to prove (ii), we show first that

$$B = \overline{C} = C \cup (\bigcap_{j=1}^{\infty} F_j \setminus Q), \tag{6}$$

where Q, consisting of some isolated and one sided limit points of $\bigcap F_n$, is a countable set. Let first $x \in \overline{C} \setminus C$, then any neighborhood I of x meets infinitely many C_j sets, thus I meets infinitely many, and hence by (1) all the F_j sets, therefore $x \in \bigcap_{j=1}^{\infty} F_j$. On the other hand, let $x \in \bigcap_{j=1}^{\infty} F_j$ such that x is a two sided limit point of $\bigcap_{j=1}^{\infty} F_j$ and let I be a neighborhood of x. We choose for every n a main interval I^n of F_n such that $I^n \supset I^{n+1}$ and $x \in (I^n \cap F_n) = (I^n \cap F_1)$. Suppose $\lambda(I^n)$ does not tend to 0. Let T denote the interior of the nondegenerate closed interval $\bigcap_{n=1}^{\infty} I^n$, then x being a two sided limit point, we have $\emptyset \neq (T \cap F_1) \subset (I^n \cap F_n)$ for every n. Since U is dense in F_1, there is an index n such that $A_n \cap T \cap F_1 \neq \emptyset$, say $y \in A_n \cap T \cap F_1$. This implies both $y \in (A_n \cap int\, I^n) \subset C_n$ and $y \in F_{n+1}$, contradicting $F_{n+1} \cap C_n = \emptyset$. Thus we have $\lambda(I^n) \to 0$, and we can choose n such that $I^n \subset I$ and $\lambda(I^{n+1}) < \lambda(I^n)$. If the main interval I^n is not preserved, then by the definition of F_{n+1}, we must have $C_n \cap I^n \neq \emptyset$. Hence $I \cap C \neq \emptyset$, i.e. $x \in B$ and (6) is proved. Now the statement (ii) of the lemma follows easily. Adding $\lambda(C_{j-2} \cap J_k)$ to both sides of (2) we obtain

$$\lambda((F_j \cup C_{j-1} \cup C_{j-2}) \cap J_k) \geq (1 - \frac{\varepsilon}{2^{j-1}})\lambda((F_{j-1} \cup C_{j-2}) \cap J_k)$$

$$\geq (1 - \frac{\varepsilon}{2^{j-1}})(1 - \frac{\varepsilon}{2^{j-2}})\lambda(F_{j-2} \cap J_k),$$

thus applying (2) repeatedly we get

(7)

$$\lambda((F_j \cup C_{j-1} \cup \dots \cup C_1) \cap J_k) \geq \prod_{l=1}^{j-1}(1 - \frac{\varepsilon}{2^l})\lambda(F_1 \cap J_k) \geq (1-\varepsilon)\lambda(F_1 \cap J_k).$$

Applying (6) and (7)

$$\begin{aligned} \lambda(B \cap J_k) &= \lambda(C \cap J_k) + \lambda(\bigcap_{n=1}^{\infty}(F_n \cap J_k)) \\ &= \lim_{n=\infty}(\lambda((C_1 \cup \dots \cup C_{n-1}) \cap J_k) + \lambda(F_n \cap J_k)) \\ &= \lim_{n=\infty}\lambda([(C_1 \cup \dots \cup C_{n-1}) \cup F_n] \cap J_k) \geq (1-\varepsilon)\lambda(F_1 \cap J_k), \end{aligned}$$

and (*ii*) is verified. Finally, by the choice of the intervals I_i^1 the length of any main interval is obviously at most η, thus (*iii*) is clear by (5). (*iv*) is obvious by (*iii*) and the non-isolated end point property of the main intervals. Hence the proof is complete.

Lemma 4 *Let $H = \bigcup_s \bigcap_n H_{s|n}$ be an analytic (NC) set in [0,1] represented by a monotone Souslin scheme $\{H_p\}$ of closed sets. Then there exists an analytic (NC) set $A \subset H$, and monotone Souslin scheme $\{A_p\}$ of closed sets such that $A = \bigcup_s \bigcap_n A_{s|n}$ and*

(*i*) *for every p, A_p is an neighborhood set or empty;*

(*ii*) $\overline{\bigcup_{j=1}^{\infty} A_{pj}} = A_p$ *for every p. In particular, if $A_p \neq \emptyset$, then for some suitable j also $A_{pj} \neq \emptyset$.*

PROOF. For every fixed index p denote

$$L_p = \bigcup_s \bigcap_{n=1}^{\infty} H_{p(s|n)},\ A_p = \overline{L_p^*},\ A = \bigcup_s \bigcap_{n=1}^{\infty} A_{s|n}.$$

Let $L = \bigcup_s \bigcap_n L_{s|n}$. We show first $L = H$. Since $L_p \subset H_p$ for every p, $L \subset H$ is obvious. Let $x \in H$. Then there exists s such that $x \in H_{s|k}$ for every k. This means, that for a fixed k, $x \in H_{s|(k+j)}$ for every j. Denote $\sigma = s^k$, then $s|(k+j) = (s|k)(\sigma|j)$ and we obtain $x \in H_{(s|k)(\sigma|j)}$, $j = 1, \dots$. Hence $x \in L_{s|k}$ for every k, and $L = H$ follows, indeed. Since $L_p^* \subset L_p \subset H_p$, we have $\overline{L_p^*} \subset \overline{L_p} \subset H_p$, and hence $A \subset H$. Notice that

$$\begin{aligned} \bigcup_{j=1}^{\infty} L_{pj} &= \bigcup_{j=1}^{\infty}\bigcup_s\left(\bigcap_{n=1}^{\infty} H_{pj(s|n)}\right) = \bigcup_{j=1}^{\infty}\bigcup_s\left(\bigcap_{n=2}^{\infty} H_{p((js)|n)}\right) \\ &= \bigcup_s \bigcap_{n=2}^{\infty} H_{p(s|n)} = L_p, \end{aligned}$$

and hence by Lemma 2 (*iv*) we obtain

$$(8) \qquad A_p = \overline{L_p^*} \subset \overline{\bigcup_{j=1}^{\infty} L_{pj}^*} = \overline{\bigcup_{j=1}^{\infty} \overline{L_{pj}^*}} = \overline{\bigcup_{j=1}^{\infty} A_{pj}}.$$

Thus (*ii*) follows, and (*i*) is clear by Lemma 2 (*iii*).

It remains to verify, that A is indeed an (NC) set. Let $x \in H \setminus A$, or, by $L = H$, $x \in L \setminus A$. Then there exists an infinite sequence s such that $x \in L_{s|n}$ for every n, and there must exist a j such that $x \notin A_{s|j}$, i.e. $x \in (L_{s|j} \setminus A_{s|j}) \subset (L_{s|j} \setminus L_{s|j}^*)$. This shows

$$(H \setminus A) \subset \bigcup_p (L_p \setminus L_p^*) \in \mathcal{C}$$

by Lemma 2, and hence A is (NC) as stated.

PROOF. (of the Theorem) Let H be an analytic (NC) set. Referring to Lemma 4 we take an analytic (NC) subset $A = \bigcup_s \bigcap_n A_{s|n} \subset H$ such that the Souslin scheme $\{A_p\}$ satisfies the properties (*i*) and (*ii*) of Lemma 4. Since A is (NC), we may assume without loss of generality, that A_p, still satisfying (*i*) and (*ii*) of Lemma 4 are perfect sets (replacing A_p by its perfect kernel in the Souslin operation, we only lose countably many points of A). Using induction we define a new Souslin scheme $\{C_p\}$ by applying Lemma 3 on the systems $\{A_p,\ |p| = j\}$, $j = 1, \ldots$ as follows. Put $B_0 = \overline{\bigcup_{k=1}^{\infty} A_k}$, and let $C_1, C_2, \ldots$ defined by Lemma 3 applied on the sets $A_1, A_2, \ldots$ with $\eta = 1$, $\varepsilon = \frac{1}{4}$, and $J_1 = (0,1)$. Suppose that for $1 \le j \le n$ and $|p| = j$ the closed sets $C_p \subset A_p$ and $B_j = \overline{\bigcup_{|p|=j} C_p}$ have been defined with the following properties.

(i) If C_p is nonempty, it consists of a finite number of closed portions of A_p called the main portions of C_p:

$$(9) \qquad C_p = \bigcup_{j=1}^{\nu_p} I_j(p) \cap A_p, \quad \lambda(I_j(p)) \le \frac{1}{|p|},$$

where the end points of the closed intervals $I_j(p)$ are not isolated points of the main portion $I_j(p) \cap A_p$;

(10) (ii) C_p are pairwise disjoint for fixed j, $|p| = j$, and $C_q \subset C_p$ if $p|q$;

(iii) if J is a rational interval, and the denominators of the end points are at most j, then

$$(11) \qquad \lambda(B_j \cap J) \ge (1 - \frac{1}{4^j})\lambda(B_{j-1} \cap J).$$

We consider now the closed sets $D_{pj} = C_p \cap A_{pj}$ for every $|p| = n$ and $j = 1, 2, \dots$ such that $int\, I_k(p) \cap A_{pj} \neq \emptyset$ for some k. Since $\bigcup_{j=1}^{\infty} A_{pj}$ is dense in A_p and C_p consists of portions of A_p, also the set $\bigcup_{j=1}^{\infty} D_{pj}$ is everywhere dense in C_p. Therefore $\overline{\bigcup_{p,j} D_{pj}} = B_n$. Note that

$$D_{pj} \cap I_k(p) = I_k(p) \cap C_p \cap A_{pj} = I_k(p) \cap A_p \cap A_{pj} = I_k(p) \cap A_{pj}$$

are perfect sets. Let $\eta = \frac{1}{|p|+1}$, $\varepsilon = \frac{1}{4^n+1}$ and let $J_1, \dots, J_N$ be the system of rational intervals such that the end points have denominators at most $n+1$. Now we apply Lemma 3 with these parameters for the perfect sets $D_{pj} \cap I_k(p)$ for $k = 1, \dots, \nu_p$ one after each other and denote the union of all the resulting sets by C_{pj}. Properties (9), (10), (11) are immediate by Lemma 3. Since we apply Lemma 3 on the main portions one by one, for every main portion $I_k(p) \cap A_p$ of $C_p \neq \emptyset$ there exist j and l such that the main portion $I_l(pj) \cap A_{pj}$ of $C_{pj} \neq \emptyset$ satisfies

$$I_l(pj) \cap A_{pj} \subset I_k(p) \cap A_p. \tag{12}$$

Notice that $\bigcup_{|p|=k} A_p$ is an $\mathcal{F}_\sigma$ cover of A for every fixed k, thus $\lambda(\bigcup_{|p|=k} A_p) > 0$. Therefore by (9), the nonempty C_p sets are neighborhood sets. Hence B_n is neighborhood for every n. Denote $B = \bigcap B_n$, B is obviously closed. We show, that B is an neighborhood set. Let J be a rational interval, let n be the greater of the denominators of the fractions at the end points. If $B_n \cap J = \emptyset$, then $B \cap J = \emptyset$. If $B_n \cap J \neq \emptyset$, then $\lambda(B_n \cap J) > 0$ because B_n is an neighborhood set. Thus by (11) we have

$$\lambda(B_j \cap J) \geq \prod_{l=n+1}^{j} (1 - \frac{1}{4^l})\lambda(B_n \cap J) \geq \frac{2}{3}\lambda(B_n \cap J),$$

and hence by the limit with $j \to \infty$ we obtain

$$\lambda(B \cap J) \geq \frac{2}{3}\lambda(B_n \cap J) > 0,$$

showing that B is indeed an neighborhood set. By Lemma 3, $\bigcup_{|p|=n} C_p$ is a relative open set in B_n and hence by Lemma 1, $C = \bigcap_{n=1}^{\infty} \bigcup_{|p|=n} C_p$ is a $\mathcal{G}_\delta$ in B. We prove, that C is also dense in B. Let $x \in B$ fixed, $I = (x - \varepsilon, x + \varepsilon)$. Choose n such that $\frac{1}{n} < \frac{\varepsilon}{2}$. Since $x \in B_n$ and $\bigcup_{|p|=n} C_p$ is dense in B_n, there exists $p, |p| = n$ and $y \in C_p$ with $|x - y| < \frac{\varepsilon}{2}$, and hence a main portion Q_n of C_p such that $Q_n \subset I$. Thus by (12) there exists a sequence s and a sequence of main portions $Q_k = I_{j_k}(s|k) \cap A_{s|k}$ such that $Q_{k+1} \subset Q_k$ and

$\emptyset \neq \bigcap_k Q_k \subset (I \cap C)$, showing that C is indeed dense in B. Thus C is a residual $\mathcal{G}_\delta$ in B. But $\{C_p\}$ is a disjoint Souslin scheme, thus

$$C = \bigcap_{n=1}^{\infty} \bigcup_{|p|=n} C_p = \bigcup_s \bigcap_n C_{s|n} \subset \bigcup_s \bigcap_n A_{s|n} = A \subset H,$$

and hence $H \cap B$ is residual in B, making the proof complete.

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References

[1] Laczkovich, M., Problem session of the 15th Summer Symposium on Real Analysis, Real Anal. Exch., Volume 17.

[2] Kuratowski, K., *Topology,* Volume I, Academic Press, 1966.

[3] Oxtoby, J.C., *Measure and Category*, Springer-Verlag, 1971.

[4] Rogers, C.A., *Hausdorff Measures*, Cambridge University Press, 1970.