RESEARCH ARTICLES

Real Analysis Exchange
Vol. 18(2), 1992/93, pp. 321–329

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QUASI-UNIFORM CONVERGENCE AND $\mathcal{L}$-SPACES

1. Introduction

Trying to explain when a limit of a sequence of real valued continuous functions is continuous, C. Arzelà [1] introduced a new type of convergence of a sequence of functions (which was later named quasi-uniform by E. Borel). Later on, at least three different types of convergence were introduced for the same purpose by several authors [5, 11, 14] . In this note, we shall compare all of them and investigate their elementary properties.

We shall use the standard terminology and notations as introduced e.g. in [4, 6, 10]

Let $(S,\leq)$ be an (upwards) directed set. Let $(X,\mathcal{O})$ be a topological space, $(Y,\mathcal{U})$ a uniform space and $f, f_s : X \longrightarrow Y$ for $s \in S$. The net $\{f_s, s \in S\}$ is *Arzelà quasi-uniformly* (ArQU, for short) convergent to f on X if it converges pointwise to f on X and

$$(\forall U \in \mathcal{U})(\forall s_0 \in S)(\exists S_0 \subseteq S, S_0 \text{ finite})(\forall s \in S_0)\\ (s \geq s_0 \ \& \ (\forall x \in X)(\exists t \in S_0)[f_t(x), f(x)] \in U). \tag{1}$$

The original result of [1] for a closed interval and a sequence of functions can be easily generalized as

Theorem 1 (C. Arzelà) *Let $f_s, s \in S$ be continuous.*

(a) If $\{f_s, s \in S\}$ ArQU converges to f on X then f is continuous.

Received by the editors June 3, 1991

(b) If X is compact, f is continuous and $\{f_s, s \in S\}$ pointwise converges to f on X then $\{f_s, s \in S\}$ also ArQU converges to f.

PROOF. The proof is immediate and essentially contained in C. Arzelà [1].

Later B. Gagaeff [5] investigated when a limit of a sequence of Borel measurable functions of the additive class α is a function of the same class. He modified Arzelà's notion. The net $\{f_s, s \in S\}$ is *Gagaeff quasi-uniformly* (GaQU, for short) convergent to f on X if it converges pointwise to f on X and if

$$(\forall U \in \mathcal{U})(\exists X_s, s \in S \text{ open}, X = \bigcup_{s \in S} X_s) \\ (\forall s)(\forall x \in X_s)[f_s(x), f(x)] \in U. \tag{2}$$

Let us recall that $\mathcal{O} \subseteq \mathcal{P}(X)$ is called a σ-topology on X if $\emptyset, X \in \mathcal{O}$, $\mathcal{O}$ is closed under finite intersections and countable unions (see [10] for details). The α^{th} additive Borel class is a σ-topology. The main result of [5] can be stated as

Theorem 2 (B. Gagaeff) *Let $(X, \mathcal{O})$ be a σ-topological space, (Y, ρ) being a separable metric space. Let $f_n : X \longrightarrow Y$ be continuous, $n = 0, 1, 2, \ldots$ and $f_n \to f$ on X pointwise. Then f is continuous if and only if $\{f_n, n \in \mathbb{N}\}$ converges GaQU to f on X.*

PROOF. The proof is essentially contained in [5].

Moreover, B. Gagaeff [5, p.187] presents a remark by M. Szpilrajn that the GaQU convergence in Theorem 2 can be replaced by another type of convergence which was later explicitly introduced by H. Poppe [13]. The net $\{f_s, s \in S\}$ is *Szpilrajn-Poppe quasi-uniformly* (SPQU, for short) convergent to f on X if it is pointwise convergent to f on X and

$$(\forall U \in \mathcal{U})(\forall x \in X)(\forall s_0 \in S)(\exists s \geq s_0) \\ (\exists V, \text{neighbourhood of } x)(\forall y \in V)[f_s(y), f(y)] \in U. \tag{3}$$

Evidently, SPQU implies GaQU.

Finally, M. Predoi [14, 15] has introduced a strongest notion. The net $\{f_s, s \in S\}$ is *Predoi quasi-uniformly* (PrQU, for short) convergent to f on X if

$$(\forall U \in \mathcal{U})(\forall x \in X)(\exists s_0 \in S)(\forall s \geq s_0) \\ (\exists V, \text{neighbourhood of } x)(\forall y \in V)[f_s(y), f(y)] \in U. \tag{4}$$

Of course, the PrQU convergence implies both pointwise and SPQU ones.

Let us remark the essential difference between the ArQU convergence and the others: in conditions (2), (3), (4) the topology of X is strongly involved; it is not involved in (1).

2. Comparison of quasi-uniform convergences

We start with a mixing of nets. Let $(S, \leq)$ be a directed set. Let $S = S_1 \cup S_2, S_1 \cap S_2 = \emptyset$ and both S_1, S_2 being cofinal in S. Further, let $\{g_s, s \in S_1\}$ be a net of functions from X into Y. Let $f, g : X \longrightarrow Y$. We set

$$\begin{array}{lll} f_s = g_s & \text{for } s \in S_1, \\ f_s = g & \text{for } s \in S_2. \end{array} \tag{5}$$

If the net $\{g_s, s \in S_1\}$ uniformly converges to f and $f \neq g$, then the net $\{f_s, s \in S\}$ satisfies the conditions (1)–(3) and does not converge pointwise to f on X. We show, for example, that $\{f_s, s \in S\}$ satisfies (2). Let $U \in \mathcal{U}$. Since $\{g_s, s \in S_1\}$ uniformly converges to f, there exists an $s_0 \in S_1$ such that $[g_s(x), f(x)] \in U$ for every $s \in S_1, s \geq s_0$ and every $x \in X$. Set

$$\begin{array}{ll} X_s = X & \text{for } s \in S_1, s \geq s_0, \\ X_s = \emptyset & \text{otherwise.} \end{array}$$

Thus none of the conditions (1)–(3) implies pointwise convergence.

We present four examples which show that the implications PrQU $\Longrightarrow$ SPQU, SPQU $\Longrightarrow$ GaQU are the only true implications between the four investigated notions of quasi-uniform convergence.

Example 1 Let $f_n(x) = x^n$ for $x \in \langle 0, 1 \rangle$.Then $\{f_n, n \in \mathbb{N}\}$ converges GaQU, SPQU and PrQU to the zero function on $\langle 0, 1)$ but not ArQU.

Example 2 Let $f_{2n}(x) = x^n$ and $f_{2n+1}(x) = 1/(n+1)$ for $x \in \langle 0, 1 \rangle$, and $f_n(1) = 1, n = 0, 1, 2, \ldots$. Let $f(x) = 0$ for $x \in \langle 0, 1)$ and $f(1) = 1$.

Then $\{f_n, n \in \mathbb{N}\}$ converges ArQU, GaQU and SPQU to f on $\langle 0, 1 \rangle$ but not PrQU.

Example 3 Let

$$\begin{array}{ll} f_n(x) = x^n & \text{for } x \in \langle 0, 1 \rangle \cap \mathbb{Q}, \\ f_n(x) = \frac{1}{n+1} & \text{for } x \in \langle 0, 1 \rangle - \mathbb{Q}, \end{array}$$

for n even. For n odd we set

$$\begin{array}{ll} f_n(x) = x^n & \text{for } x \in \langle 0, 1 \rangle - \mathbb{Q}, \\ f_n(1) = 1, & \\ f_n(x) = 1/n & \text{for } x \in \langle 0, 1 \rangle \cap \mathbb{Q}. \end{array}$$

Set $f(x) = 0$ for $x \in \langle 0, 1)$ and $f(1) = 1$.

Then $\{f_n, n \in \mathbb{N}\}$ converges ArQU to f on $\langle 0,1\rangle$ but not GaQU, SPQU and PrQU.

Example 4 Let $f(x) = 0$ for $x \in \langle 0,1), f(1) = 1, f_n(x) = x^n$ for $x \in \langle 0,1\rangle, n \geq 1$ and $f_0 = f$.

Then $\{f_n, n \in \mathbb{N}\}$ converges GaQU to f on $\langle 0,1\rangle$ but not SPQU, PrQU and ArQU.

3. Sequences of continuous functions

We shall need the following

Lemma 1 *Let $f_s, s \in S$ be continuous. If $\{f_s, s \in S\}$ converges GaQU to f on X then f is continuous.*

PROOF. Let $W \in \mathcal{U}, x \in X$. Take a $U \in \mathcal{U}$ such that $U + U + U \subseteq W$. Let $X_s, s \in S$ be such as in (2). Assume $x \in X_s$. For $y \in X_s$ we have $[f_s(y), f(y)] \in U$. Since f_s is continuous there exists an open set $A \ni x$ such that $[f_s(x), f_s(y)] \in U$ for any $y \in A$. Then $[f(x), f(y)] \in W$ for any $y \in X_s \cap A$.

The following result is partially contained in [5, 13, 14].

Theorem 3 *Let $\{f_s, s \in S\}$ be a net of continuous functions pointwise converging to a function $f : X \longrightarrow Y$. Then the following are equivalent:*

(i) f is continuous;

(ii) $\{f_s, s \in S\}$ converges GaQU to f on X;

(iii) $\{f_s, s \in S\}$ converges SPQU to f on X;

(iv) $\{f_s, s \in S\}$ converges PrQU to f on X.

PROOF. Evidently (iv) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii) and by the Lemma also (ii) $\Longrightarrow$ (i). We show that (i) $\Longrightarrow$ (iv) (compare [14]).

Let $U \in \mathcal{U}, x \in X$. Take a $W \in \mathcal{U}$ such that $W + W + W \subseteq U$. Since $f_s(x) \longrightarrow f(x)$, there exists an $s_0 \in S$ such that for any $s \geq s_0, [f_s(x), f(x)] \in W$. Since f is continuous, there exists an open $A \ni x$ such that $[f(y), f(x)] \in W$ for any $y \in A$. Finally, let B be open and such that $[f_s(y), f_s(x)] \in W$ for $y \in B$ (B may depend on s). Then the set $V = A \cap B$ is the desired one satisfying (4).

By Example 1, the case of ArQU-convergence cannot be included in Theorem 3.

Let us recall that a topological space X is said to be pseudocompact (see e.g. [6]) if $C(X) = C^*(X)$, i.e. if every real valued continuous function defined on X is bounded. We do not require that the space X is completely regular (=Tychonoff, as it is supposed in [4]). It is well known that X is pseudocompact if and only if from every countable cover of X by cozero-sets one can choose a finite subcover.

The following result is a strengthening of a theorem by K.Iséki [8] (K.Iséki assumes that the space X is completely regular).

Theorem 4 *Let X be a topological space. Then the following are equivalent:*

(i) Every sequence $f_n \in C(X), n = 0, 1, \ldots$ converging pointwise to a function $f \in C(X)$ converges also $ArQU$;

(ii) X is pseudocompact.

PROOF. Assume that X is pseudocompact and $f_n \in C(X), n = 0, 1, \ldots$ converge pointwise to a function $f \in C(X)$. Let $\varepsilon > 0$, n_0 being an arbitrary natural number. We denote

$$A_n = \{x \in X; \mid f(x) - f_n(x) \mid < \varepsilon\}.$$

Then the space X is covered by the cozero-sets $A_n, n > n_0$. The set of indices of the corresponding finite subcover witnesses the ArQU convergence.

Assume now that the space X is not pseudocompact. Then there exists a function $f \in C(X)$ which is not bounded. Without loss of generality we can assume that f is non-negative. Inspired by Example 1 we set

$$g_n(x) = \left(1 - \frac{1}{f(x)+1}\right)^n, n = 0, 1, \ldots, x \in X.$$

It is easy to see that the sequence $\{g_n, n \in \mathbb{N}\}$ converges pointwise to 0 and does not converge ArQU.

Remark. In [3] the authors investigate spaces X in which pointwise convergence on $C(X)$ coincides with quasi-normal convergence. So according to [3], a space X satisfying condition (i) of the theorem could be called an ArQU-space. Thus Theorem 4 says that a topological space is an ArQU-space if and only if it is pseudocompact.

Since a paracompact space is compact if and only if it is pseudocompact we obtain a strengthening of a result of L'.Holá and T.Šalát [7, p.127].

Corollary 1 *A paracompact space X is compact if and only if pointwise convergence of sequences on $C(X)$ coincides with $ArQU$ convergence.*

4. Some $\mathcal{L}$-spaces

Directly from the definition one can easily see that every subnet of a PrQU convergent net is also PrQU converging to the same function, i.e. PrQU convergence is downwards hereditary. This is not true for any of the ArQU, GaQU and SPQU convergences.Actually, let the net $\{g_s, s \in S_1\}$ converge pointwise but not ArQU (GaQU, SPQU respectively) to the function $f = g$. Then the net $\{f_s, s \in S\}$ constructed by (5) ArQU (GaQU, SPQU respectively) converges to f and the subnet $\{g_s, s \in S_1\}$ does not.

The set ${}^X\mathbb{R}$ of all real valued functions defined on X endowed with any of the investigated convergences can be considered as a convergence structure. Let us recall that (compare [9, p.84]) a convergence structure is said to be an $\mathcal{L}$-space if every subsequence of a convergent sequence converges to the same limit and if every constant sequence converges to its common value. According to the preceding remarks, ${}^X\mathbb{R}$ with PrQU convergence is an $\mathcal{L}$-space. However, one can easily see that in the case of the ArQU, GaQU and SPQU convergences, it need not be an $\mathcal{L}$-space.

By Theorem 3, on $C(X)$ pointwise convergence is equivalent with any of GaQU, PrQU, SPQU convergences. Since $C(X)$ with pointwise convergence is an $\mathcal{L}$-space and this convergence need not be generated by a first countable topology on $C(X)$ (see [2]), the same holds true for $C(X)$ endowed with any of the mentioned three types of QU convergences. The case of ArQU convergence is answered by the following result.

Theorem 5 *For any topological space X the following are equivalent:*

(i) $C(X)$ with ArQU convergence is an $\mathcal{L}$-space;

(ii) the pointwise and ArQU convergences coincide on $C(X)$;

(iii) every sequence of continuous functions converging on X GaQU (PrQU or SPQU) converges also ArQU;

(iv) X is pseudocompact.

PROOF. Evidently, (ii) implies (i). By Theorem 3, (ii) is equivalent to (iii) and by Theorem 4, (ii) is equivalent to (iv). Assume (i). Let $\{f_n, n \in \mathbb{N}\}$ be a sequence of continuous functions converging pointwise to $f \in C(X)$ on X. Then the sequence

$$f_0, f, f_1, f, f_2, \ldots, f, f_n, f, \ldots$$

converges ArQU to f on X. By (i), also the subsequence $\{f_n, n \in \mathbb{N}\}$ converges ArQU to f.

5. Arzelà's quasi-uniform convergence and compactness

The equivalence of conditions (i) and (ii) in the following theorem has been proved by H.Poppe [11, 12].

Theorem 6 *Let X be a completely regular topological space. Then the following are equivalent:*

(i) X is compact;

(ii) every net of continuous functions $\{f_s \in C(X), s \in S\}$ pointwise converging to a continuous function $f \in C(X)$ converges also ArQU;

(iii) every net of continuous functions $\{f_s \in C(X), s \in S\}$ pointwise converging to a continuous function $f \in C(X)$ converges also ArQU provided that $|S| \leq \aleph_0 \cdot |X|$.

PROOF. The implication (i) $\Longrightarrow$ (ii) is part b) of Theorem 1. Evidently (ii) implies (iii). We show that (iii) implies (i).

Assume that the space X is not compact. Then there is an open cover $\mathcal{A}, \bigcup \mathcal{A} = X$ such that $\bigcup \mathcal{A}_0 \neq X$ for every finite $\mathcal{A}_0 \subseteq \mathcal{A}$. By the axiom of choice there exists a function $A : X \longrightarrow \mathcal{A}$ such that $x \in A(x)$ for every $x \in X$.

Let S be the set of all finite subsets of X ordered by inclusion. Then S is upwards directed. Since the space X is completely regular, for every $s \in S$ there exists a continuous function $f_s : X \longrightarrow \langle 0, 1 \rangle$ such that

$$f_s(x) = 0 \qquad \text{for } x \in s,$$
$$f_s(x) = 1 \qquad \text{for } x \in X - \textstyle\bigcup_{y \in s} A(y).$$

Let $x \in X$. Denote $s_0 = \{x\}$. Then for $s \in S, s \supseteq s_0$ we have $f_s(x) = 0$. Thus the net $\{f_s, s \in S\}$ converges pointwise to 0 on X.

Take $\varepsilon = \frac{1}{2}$ and any finite set $\{s_0, \ldots, s_n\} \subseteq S$. Since the finite set

$$\mathcal{A}_0 = \{A(x); x \in s_0 \cup \cdots \cup s_n\}$$

does not cover the space X, there exists a point $x_0 \in X - \bigcup \mathcal{A}_0$. Then $f_{s_i}(x_0) = 1$ for $i = 0, 1, \ldots, n$. Thus the net $\{f_s, s \in S\}$ does not converge ArQU to 0 on X.

It suffices to note that $|S| \leq \aleph_0 \cdot |X|$.

We show that the estimate of the cardinality in condition (iii) of Theorem 5 cannot be improved.

Example 5 Let α be an uncountable regular cardinal. The set W of all ordinals less than or equal to α is endowed with the interval topology, i.e. the topology induced by open intervals $\{\xi \in W; \xi < \eta\}, \{\xi \in W; \zeta < \xi < \eta\}, \{\xi \in W; \xi > \eta\}, \zeta < \eta \leq \alpha$. One can easily see (compare [4, p. 174] for the case $\alpha = \aleph_1$) that W is compact.

The set $X = W - \{\alpha\}$ is not compact. We show that every net of continuous functions $\{f_s \in C(X), s \in S\}$ pointwise converging to a function $f \in C(X)$ converges also ArQU provided that $| S |< \alpha$.

We denote $W_\eta = \{\xi \in W; \xi \leq \eta\}$. Then $X = \bigcup_{\eta<\alpha} W_\eta$ and every W_η (as a closed subset of W) is compact.

We start by showing that any open cover $\{A_t; t \in T\}$ of X, $| T |< \alpha$ contains a finite subcover of X.

Let $F(\eta)$ denote a finite subset of T such that $W_\eta \subseteq \bigcup\{A_t; t \in F(\eta)\}$ (the existence follows from the compactness of W_η). Since the set of all finite subsets of T has the power less than α, α being regular, there exists a finite $T_0 \subseteq T$ and an unbounded subset Y of the set X such that $F(\xi) = T_0$ for every $\xi \in Y$. Since $\bigcup\{W_\xi; \xi \in Y\} = X$, the finite set $\{A_t; t \in T_0\}$ covers X.

Now let $\{f_s \in C(X), s \in S\}$ be a net pointwise converging to $f \in C(X)$ on X, $| S |< \alpha$. For given $\varepsilon > 0$ we denote

$$A_s = \{x \in X; | f_s(x) - f(x) |< \varepsilon\}.$$

Since the net pointwise converges, for any $s_0 \in S$ the set $\{A_s; s \geq s_0\}$ is an open cover of X. Since $| S |< \alpha$, there exists a finite subset S_0 of $\{s \in S; s \geq s_0\}$ such that $\bigcup\{A_s; s \in S_0\} = X$.

Thus the condition (1) is fulfilled.

The authors should like to thank the referees for valuable remarks and comments.

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