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Robert W. Vallin, Department of Mathematics, Slippery Rock University, Slippery Rock, PA 16057, email: rwv@sru.bitnet

AN INTRODUCTION TO SHELL POROSITY*

Abstract

Porosity and several different variations of porosity have been used to describe small sets for some time. A common link between these notions is that off of the real line these small sets are not necessarily disconnected. In this paper, shell porosity is introduced and some properties of shell porous sets are investigated. This includes the fact that in a complete metric space, any closed set which may be expressed as a countable union of shell porous sets must be totally disconnected.

1. Introduction and Historical Remarks

Porosity, under different names, has been used by analysts since the early part of the twentieth century. In 1920, A. Denjoy [5] used a notion similar to what is now called the porosity index in his study of properties of trigonometric series. A. Khintchine [14], in 1924, used porosity for describing arguments involving density. E. P. Dolženko [7] gave us the current nomenclature in 1967. He needed porosity to describe a subset of the measure zero, nowhere dense sets.

We begin with some definitions.

Definition 1.1 *Let E be a set in $\mathbb{R}$ and let $a < b$. Define $\lambda(E; a, b) = \lambda(E; b, a)$ as the length of the largest subinterval in $(a, b) \cap E^c$, where E^c denotes the complement of E. If x is any point in $\mathbb{R}$, we define the right hand porosity of E at x as*

$$p^+(E; x) = \limsup_{h \to 0+} \frac{\lambda(E; x, x+h)}{h}.$$

*Many of the results presented here are from the author's Ph.D. dissertation written under the direction of Michael J. Evans

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Similarly, the left hand porosity of E at x is given by

$$p^-(E;x) = \limsup_{h\to 0+} \frac{\lambda(E;x-h,x)}{h}.$$

Finally, the porosity of E at x is

$$p(E;x) = \limsup_{h\to 0} \frac{\lambda(E;x,x+h)}{|h|} = \max\{p^+(E;x), p^-(E;x)\}.$$

Note that for any x, $p(E;x) \in [0,1]$.

Introducing some terminology associated with these definitions, we say E is porous at x if $p(E;x) > 0$. If $p(E;x) = 1$, then E is called strongly porous at x. The set E is called porous if it is porous at each of its points. Lastly, the set E is called σ-porous if it can be expressed as a countable union of porous sets. While it is clear that a σ-porous set must be both of measure zero and of first category, the converse is not true. L. Zajíček [18] was the first to construct a nowhere dense, measure zero perfect set which fails to be σ-porous.

The σ-porous sets have been used to describe several types of exceptional sets. Dolženko [7], in his study of cluster sets, showed that given a function, f, from the half-plane into the complex plane, the set of $x \in \mathbb{R}$such that there exists two Stolz angles, θ_1, and θ_2, with $C_f(\theta_1,x) \neq C_f(\theta_2,x)$ is a σ-porous set. C. L. Belna, M. J. Evans and P. D. Humke [4] have shown that for a continuous function, $f:\mathbb{R}\to\mathbb{R}$, the set of points where f is not differentiable, but the symmetric derivative exists is a σ-porous set. Later, Evans and Humke [8] proved that if f is monotone, the collection of points where the upper, or lower, left and right derivatives are not equal is a σ-porous set.

Some other applications of porosity include J. Foran's construction of a non-averaging set which is both strongly porous and of Hausdorff dimension one [11], and Zajíček's [19, 20] and V. Kelar's [13] investigations of topologies generated by porosity and strong porosity.

It is obviously possible for a set, E, to be porous at x while either $p^+(E;x) = 0$ or $p^-(E;x) = 0$. We say a set is bilaterally porous at a point if both $p^+(E;x) > 0$ *and* $p^-(E;x) > 0$. Still, this does not guarantee us any relationship between the location of the gaps in E to the right of x and the gaps to the left of x. To counter this we bring in the notion of symmetric porosity.

Definition 1.2 *Let $E \subset \mathbb{R}$and let x be any point. For $R > 0$ define $\gamma(E;x,R)$ as the supremum of*

$$\{h > 0 : \exists\, t > 0,\ t+h < R, (x-t-h, x-t)\cap E = \emptyset \text{ and } (x+t, x+t+h)\cap E = \emptyset\}.$$

We define the symmetric porosity of E at x as

$$p^s(E;x) = \limsup_{R\to 0+} \frac{\gamma(E;x,R)}{R}.$$

The definitions of strong symmetric porosity and σ-symmetric porosity follow analogously. It is this symmetric porosity which is directly related to Denjoy's bilateral porosity index [6]. Using symmetric porosity, Zajíček [22] improved on Belna, Evans and Humke's result by showing the set is actually σ-symmetrically porous.

Moving on to a general metric space, porosity is defined as follows:

Definition 1.3 *Let E be a set in the metric space (X,d). By $B_x(r)$ we mean the open ball centered at x of radius r, i. e. $\{y \in X : d(x,y) < r\}$. For $x \in X$ and $R > 0$ define $\gamma(E;x,R)$ as the supremum of*

$$\{h > 0 : \exists\, z \in X \text{ with } B_z(h) \subset B_x(R) \cap E^c\}.$$

Furthermore, define the porosity of E at x as

$$p(E;x) = 2\limsup_{R\to 0+} \frac{\gamma(E;x,R)}{R}.$$

As with porosity in $\mathbb{R}$, this general definition of porosity is blind to the symmetry of the "holes" in E. Shell porosity is our way of incorporating this symmetry.

First, we must define what we mean by shells. Let $x \in X$ and $0 < r_1 < r_2$. The open shell about x of radii r_1 and r_2 is given by

$$S_x(r_1,r_2) = B_x(r_2) \setminus \overline{B_x(r_1)}.$$

Now we may introduce shell porosity.

Definition 1.4 *For $R > 0$, we define $\Lambda(E;x,R)$ as the supremum of*

$$\{h > 0 : \exists\, t > 0 \text{ with } t + h < R \text{ and } S_x(t,t+h) \cap E = \emptyset\}.$$

The shell porosity of E at x is given by

$$p^s(E;x) = \limsup_{R\to 0+} \frac{\Lambda(E;x,R)}{R}.$$

Note that this is not abusive notation for if $X = \mathbb{R}$, then shell porosity is symmetric porosity.

The purpose of this paper is to introduce shell porosity and discuss several of its properties. To begin with, we will show that, in the space of compact sets from a complete metric space endowed with the Hausdorff metric, strongly shell porous sets are typical. That is, they form a dense G_δ set. Next, we will compare shell porosity with the definition of porosity in a general metric space. Several properties of porous sets do not apply to shell porous sets but shell porosity also has some features lacking in porous sets. We will then conclude with comparisons and contrasts with two other definitions of porous sets: hyperporous and totally porous.

2. The Abundance of Shell Porous Sets

Many papers have been published, showing that in a certain sense the "typical" set encountered in many situations is either porous or σ-porous; e.g., see [3], [4], [8], [16], [17], and [21]. This section contains another such result. We show that the typical compact subset of the real line $\mathbb{R}$is shell porous. Actually, we shall show more, but before we state the precise result, some definitions and terminology are in order.

Let (X, d) be a complete metric space and recall that a set A is called ***shell porous*** if it has positive shell porosity at each of its points. We shall say A is ***η-shell porous*** if it has shell porosity at least η at each of its points. Our main goal here is to show that the "typical" compact set in X is strongly shell porous, i.e., is 1-shell porous. We shall now clarify what we mean by "typical."

Let $\mathcal{C}$ be the collection of nonempty compact subsets of X. For A,$B \in \mathcal{C}$, let

$$\overline{\rho}_B^A = \inf\{\epsilon > 0 : B \subset \bigcup_{x\in A} B_x(\epsilon)\}.$$

The *Hausdorff distance between A and B* is

$$\rho(A, B) = \max\{\overline{\rho}_A^B, \overline{\rho}_B^A\}.$$

It can readily be shown that $\mathcal{K} = (\mathcal{C}, \rho)$ is a complete metric space [2, 15]. Hence, any dense G_δ subset of $\mathcal{K}$ is residual in $\mathcal{K}$. We shall show that the collection of strongly shell porous compact sets in X is a dense G_δ subset of $\mathcal{K}$. It is in this sense that we assert that strongly shell porous sets are typical.

Let $P(\eta) = \{F \in \mathcal{C}$: F is η-shell porous$\}$, and let $\mathcal{F}$ denote the collection of all nonempty finite subsets of X. From the definitions, it is easy to see that if $0 < \alpha < \beta \leq 1$, then

$$P(\alpha) \supset P(\beta) \supset P(1) \text{ and } P(\beta) = \bigcap_{\gamma<\beta} P(\gamma).$$

We also have the following elementary lemma.

Lemma 2.1 *For each $\eta \in [0,1]$, $\mathcal{F} \subset P(\eta)$, and $\mathcal{F}$ is dense in $\mathcal{K}$.*

PROOF. Clearly $\mathcal{F} \subset P(\eta)$ for every η. To see that $\mathcal{F}$ is dense, let $\epsilon > 0$ and $C \in \mathcal{C}$. Choose a finite subset of $\{B_x(\epsilon) : x \in C\}$, say

$$\{B_{x_1}(\epsilon), B_{x_2}(\epsilon), \dots, B_{x_n}(\epsilon)\}$$

to cover C. Clearly, if $F = \{x_1, x_2, \dots, x_n\}$, then $F \in \mathcal{F}$ and $\rho(C, F) < \epsilon$.

Theorem 2.1 *The collection of all strongly shell porous compact subsets of X is a dense G_δ subset of $\mathcal{K}$.*

PROOF. For each natural number n and each $0 < \eta < 1$, let

$$\begin{aligned} P_n(\eta) \;=\; & \{F \in \mathcal{C} : \forall x \in F \;\exists R_x \in (0, 1/n), h_x > 0, \text{ and } t_x > 0 \\ & \text{such that } t_x + h_x \le R_x, h_x/R_x > \eta, \text{ and } \overline{S_x(t_x, t_x + h_x)} \subset F^c\}. \end{aligned}$$

Choose any $F \in P_n(\eta)$ and let $x \in F$ with h_x, t_x, and R_x as above. Let

$$r_x = \frac{h_x - \eta R_x}{2} > 0.$$

For each $y \in B_x(r_x)$ let

$$h_y = h_x - 2d(x, y) > 0$$

and

$$t_y = t_x + d(x, y) > 0.$$

Then we see that

$$t_y + h_y < t_x + h_x < R_x, \tag{1}$$

$$\frac{h_y}{R_x} > \frac{h_x - 2r_x}{R_x} = \eta, \tag{2}$$

and

$$\overline{S_y(t_y, t_y + h_y)} \subset \overline{S_x(t_x, t_x + h_x)}, \tag{3}$$

where this set inclusion can be seen as follows: if $z \in \overline{S_y(t_y, t_y + h_y)}$, then

$$d(z, x) \le d(z, y) + d(x, y) \le t_y + h_y + d(x, y) = t_x + h_x,$$

and

$$d(z, x) \ge d(z, y) - d(x, y) \ge t_y - d(x, y) = t_x.$$

The collection $\{B_x(r_x) : x \in F\}$ is an open cover for F and, hence, there is a finite subset $\{x_1, x_2, \ldots, x_p\}$ of F so that the open set

$$G \equiv \bigcup_{i=1}^{p} B_{x_i}(r_{x_i}) \setminus \bigcup_{i=1}^{p} \overline{S_{x_i}(t_{x_i}, t_{x_i} + h_{x_i})}$$

contains F. Now, if $y \in G$, then there is an i, $1 \le i \le p$, such that $y \in B_{x_i}(r_{x_i})$, and consequently, from (1), (2) and (3) we know that there exist positive numbers t_y and h_y such that

$$t_y + h_y \le R_{x_i} < \frac{1}{n},$$

$$\frac{h_y}{R_{x_i}} > \eta,$$

and

$$\overline{S_y(t_y, t_y + h_y)} \subset G^c.$$

Thus any compact subset of G is in $P_n(\eta)$.

Let δ denote one-half the usual "distance" from the compact set F to the closed set G^c. So $\delta > 0$, and if S is any nonempty compact set such that $\rho(F, S) < \delta$, then $\overline{\rho}_S^F < \delta$, and hence

$$S \subset \bigcup_{x \in F} B_x(\delta) \subset G.$$

Consequently, $S \in P_n(\eta)$. Therefore, $P_n(\eta)$ is open in $\mathcal{K}$, and this is true for every natural number n and every $0 < \eta < 1$.

Now, let $\alpha \in (0,1)$. Then since

$$P(\alpha) = \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} P_n(\alpha - 1/m),$$

$P(\alpha)$ is a $\mathbf{G}_\delta$ set in $\mathcal{K}$. Thus

$$P(1) = \bigcap_{n=1}^{\infty} P(1 - 1/n)$$

is a $\mathbf{G}_\delta$ set and from the lemma is also dense in $\mathcal{K}$. Hence, our proof is complete.

After obtaining this result we discovered that P. M. Gruber [12] had independently proved virtually the same observation in 1989. He did not introduce a nomenclature for shell porosity, but a careful reading of his work shows that he has indeed obtained Theorem 2.1.

Applied to $\mathbb{R}$with the usual metric, this result yields the fact that the collection of strongly symmetrically porous compact subsets form a dense G_δ in the space of compacta of $\mathbb{R}$with the Hausdorff metric, and, consequently, the larger collection of compact sets in $\mathbb{R}$which are bilaterally strongly porous is also residual in the same space. This latter result was originally announced by L. M. Larson at the Summer Symposium on Real Analysis in Esztergom, Hungary, August, 1987 [2] and the proof presented here is modelled on his original argument.

3. Comparisons of Porosity and Shell Porosity

In this section we shall compare and contrast the notions of porosity and shell porosity. We shall do this for sets in $\mathbb{R}^n$ and in the more general setting of a metric space.

First, in $\mathbb{R}$it is fairly easy to construct a set which is bilaterally strongly porous at one point, but which is not symmetrically porous at that point. A deeper concern would be whether a set which is bilaterally strongly porous at each of its points could fail to be a symmetrically porous set, or even a σ-symmetrically porous set. Recently, Evans, Humke and K. Saxe [9] have shown that the latter situation is possible: that is, they constructed a symmetric Cantor set in $\mathbb{R}$which is bilaterally strongly porous and showed that this contains a residual set which is not σ-symmetrically porous. Thus, in contrast to the abundance of symmetrically porous sets set forth in the previous section, this reult seems to indicate that the symmetrically porous sets in $\mathbb{R}$are in some sense less populous than the porous ones.

This view of symmetric porosity as being more restrictive than porosity is further enhanced by another example constructed in [9]. Specifically, Zajíček [21] had observed that given any $0 < c < 1$ and any σ-porous set A in $\mathbb{R}$, A can be expressed as the union of a sequence of sets, A_n, such that the porosity of each A_n at each of its points is at least c. Evans, Humke and Saxe showed the existence of a symmetrically porous set which cannot be expressed as a countable union of sets, each having symmetric porosity greater than four-fifths at each of its points.

Here, and in the next section, we wish to continue this comparison of shell porosity and porosity and more general notions of porosity. In addition to showing that analogues of the Evans-Humke-Saxe examples exist in $\mathbb{R}^n$ we shall compare and contrast porosity and shell porosity by examining a space with two equivalent metrics, looking at the products of porous and shell porous sets, utilizing the notion of very-porous and very-shell porous sets,

[2]See *Real Analysis Exchange*, Volume 13, 1987-88, pp. 116-118, for an outline of that presentation.

investigating what we shall call "sections" of shell porous sets, and observing the relationships between a set being connected and its being either porous or shell porous. Some notation and definitions will be needed for this endeavor.

We define product spaces and the product metric as follows. Given two spaces (X, d_1) and (Y, d_2) we denote $(X \times Y, d)$ as the set of all pairs (x, y) with $x \in X$ and $y \in Y$ along with the product, or box, metric d, defined by

$$d((x_1, y_1), (x_2, y_2)) = \max\{d_1(x_1, x_2), d_2(y_1, y_2)\}.$$

The definitions of porous and shell porous can be changed to those for very-porous and very-shell porous by replacing the lim sup in the definitions with lim inf. Specifically, the v-porosity and v-shell porosity of a set E in the metric space (X, d) are given respectively by

$$p_v(E; x) = \liminf_{r \to 0+} \frac{\gamma(x, r, E)}{r} \quad \text{and} \quad p_v^s(E; x) = \liminf_{r \to 0+} \frac{\Lambda(x, r, E)}{r}.$$

The first way we will analyze shell porous sets is by looking at the behavior of a shell porous set in a space X where X has been given two equivalent metrics. Following the terminology of Barnsley [2] we shall say that two metrics d and d^* on X are equivalent if there exists $m > 0$ and $M > 0$ such that $md^*(x, y) \leq d(x, y) \leq Md^*(x, y)$ for every x and y in X. Unlike a porous set, in order to be shell porous at a point with both metrics a certain condition must be met.

Theorem 3.1 *Let (X, d) be a metric space and suppose that the metric d^* is equivalent to d. Let $E \subset X$ and $x \in X$ be such that the shell porosity of E at x in the d metric satisfies $p^s(E; x) = 1 - \psi$, where $\psi < m/M$. Then the shell porosity of E at x in the d^* metric satisfies $p_*^s(E; x) \geq 1 - (M/m)\psi$.*

PROOF. Since $p^s(E; x) = 1 - \psi$, there must exist two sequences $\{r_n\}$ and $\{R_n\}$ with

$$0 < r_n < R_n, \; R_n \to 0, \; S_x(r_n, R_n) \subset E^c$$

and

$$\frac{R_n - r_n}{R_n} = 1 - \frac{r_n}{R_n} \to 1 - \psi.$$

Since $\psi < m/M$, for n large enough we have $r_n/m < R_n/M$.

Since $d(x, y)$ and $d^*(x, y)$ are equivalent, then for any radius r

$$B_x^*(r/M) \subset B_x(r) \subset B_x^*(r/m)$$

where $B_x^*(r)$ and $B_x(r)$ are the open balls of radius r in (X, d^*) and (X, d) respectively.

Using a large n along with r_n and R_n from above, we know $B_x^*(R_n/M) \subset B_x(R_n)$ and $B_x(r_n) \subset B_x^*(r_n/m)$. Thus we have $S_x^*(r_n/m, R_n/M) \subset E^c$, since

$$B_x(r_n) \subset B_x^*(r_n/m) \subset B_x^*(R_n/M) \subset B_x(R_n).$$

So for (X, d^*), we have

$$p_*^s(E;x) \geq \lim_{n\to\infty} \frac{\frac{R_n}{M} - \frac{r_n}{m}}{\frac{R_n}{M}} = 1 - \frac{M}{m} \lim_{n\to\infty} r_n/R_n = 1 - \frac{M}{m}\psi.$$

A consequence of this is the following corollary, which we shall see later does not hold when strongly shell porous is replaced with strongly porous.

Corollary 3.1 *If a set is strongly shell porous at a point in one metric, then it will be strongly shell porous at that point in any equivalent metric.*

PROOF. This is an application of Theorem 3.1 with $\psi = 0$.

If the assumption that $p^s(E;x) > 1 - m/M$ in Theorem 3.1 is omitted, then it is possible for $p^s(E;x)$ to be positive and $p_*^s(E;x)$ to be zero as the following example illustrates.

Example 3.1 *There is a set in the plane which has shell porosity at the origin of $1 - 1/\sqrt{2}$ using the box metric, and shell porosity at the origin of 0 using the euclidean metric.*

PROOF. In $\mathbb{R}^2$ with the box metric,d, let

$$E = \{x : \sqrt{2}/2^{k+1} \leq d(0,x) \leq 1/2^k,\ k \in N\}.$$

Using the box metric and the fact that $S_{(0,0)}(\frac{1}{2^k}, \frac{\sqrt{2}}{2^k}) \subset E^c$

$$p_B^s(E;0) = \lim_{k\to\infty} \frac{\frac{\sqrt{2}}{2^k} - \frac{1}{2^k}}{\frac{\sqrt{2}}{2^k}} = 1 - \frac{1}{\sqrt{2}}.$$

However, for the euclidean metric note that for any k value both the point $(0, \frac{\sqrt{2}}{2^k})$ and the point $(\frac{1}{2^k}, \frac{1}{2^k})$ lie on the ball centered at the origin with radius $\frac{\sqrt{2}}{2^k}$. This means that we cannot place a shell centered at the origin which will not touch any point in E. Thus $p^s(E;0) = 0$.

This idea can be extended as follows:

Example 3.2 *There exists a set in the plane which is shell porous using the box metric, but not shell porous in the euclidean metric.*

PROOF. This example will be based on the construction in Example 3.1. Now, however, we will have a set which consists of isolated points except for one limit point which will be the origin.

For each natural number k, let $l_k = \frac{\sqrt{2}}{k2^{k+1}}$ and let N_k be the greatest integer less than or equal to $k(\sqrt{2}-1)$, i.e. the greatest integer less than or equal to

$$\frac{\frac{1}{2^k} - \frac{\sqrt{2}}{2^{k+1}}}{l_k}.$$

Define E to be the union of the four sets

$$\begin{aligned} E_0 &= \{(0,0)\} \\ E_1 &= \{(\tfrac{\sqrt{2}}{2^k},0) : k = 0,1,\ldots\} \\ E_2 &= \cup_{k=1}^{\infty}\{(\tfrac{1}{2^k} - il_k, 0) : i = 0,1,\ldots,N_k\} \\ E_3 &= \cup_{k=1}^{\infty}\{(\tfrac{1}{2^k}, \tfrac{1}{2^k} - \tfrac{i}{4^k}) : i = 0,1,\ldots,2^k\}. \end{aligned}$$

We claim that E is a shell porous set in the box metric, yet fails to be shell porous at the origin in the Euclidean metric. First, since every point in E other than the origin is isolated, it follows that E is shell porous at each such point in both the box and euclidean metrics. Next, to see that E is shell porous at the origin in the box metric again note that each box metric shell $S_{(0,0)}(\frac{1}{2^k}, \frac{\sqrt{2}}{2^k}) \subset E^c$ and

$$\lim_{k\to\infty} \frac{\frac{\sqrt{2}}{2^k} - \frac{1}{2^k}}{\frac{\sqrt{2}}{2^k}} = 1 - \frac{1}{\sqrt{2}} > 0.$$

Finally, we shall show that E is not shell porous at the origin in the euclidean metric. To this end let $S_{(0,0)}(t,h)$ denote any euclidean shell contained in E^c. Then there is a k such that

$$\frac{\sqrt{2}}{2^{k+1}} < h \le \frac{\sqrt{2}}{2^k}.$$

If $\frac{\sqrt{2}}{2^{k+1}} < h \le \frac{1}{2^k}$, then since $S_{(0,0)}(t,h) \cap E_2 = \emptyset$, $h - t < l_k$. This yields

$$\frac{h-t}{h} < \frac{l_k}{\frac{\sqrt{2}}{2^{k+1}}} = \frac{1}{k}.$$

Next, suppose $\frac{1}{2^k} < h \le \frac{\sqrt{2}}{2^k}$. Since the circle of radius $\frac{\sqrt{2}}{2^k}$ centered at the origin contains the point $(\frac{1}{2^k}, \frac{1}{2^k}) \in E$, it follows that since $S_{(0,0)}(t,h)$ must miss E_3, we have $h - t < \frac{1}{4^k}$ and so

$$\frac{h-t}{h} < \frac{\frac{1}{4^k}}{\frac{1}{2^k}} = \frac{1}{2^k} < \frac{1}{k}.$$

Consequently, E has shell porosity 0 at the origin in the euclidean metric.

This behavior is in sharp contrast to ordinary porosity as illustrated by the following theorem, which is probably quite well known. Its statement and proof are included for completeness and to further illustrate the difference between shell porosity and porosity.

Theorem 3.2 *Let (X,d) be a metric space and assume that the metric d^* is equivalent to d. If $E \subset X$ and $x \in X$, then*

$$\frac{m}{M}p(E,x) \leq p_*(E,x)$$

and hence E is porous at x in one metric if and only if it is porous at x in the other metric.

PROOF. Say $p(E;x) = \rho > 0$. So there exists three sequences $\{R_n\}$, $\{r_n\}$, and $\{z_n\}$ with $R_n > r_n$, $R_n \to 0$, such that

$$B_{z_n}(r_n) \subset B_x(R_n) \cap E^c \textit{ and } \frac{2r_n}{R_n} \to \rho.$$

With $B^*_x(r)$ denoting the open ball in the d^* metric $B^*_{z_n}(\frac{r_n}{M}) \subset B_{z_n}(r_n)$ and $B_x(R_n) \subset B^*_x(\frac{R_n}{m})$; so

$$B^*_{z_n}(\frac{r_n}{M}) \subset (B^*_x(\frac{R_n}{m}) \cap E^c)$$

and

$$p_*(E;x) \geq \lim_{n\to\infty} 2(\frac{r_n/M}{R_n/m}) = 2\frac{m}{M}\lim_{n\to\infty}\frac{r_n}{R_n} = \frac{m}{M}\rho > 0,$$

yielding $\frac{m}{M}p(E,x) \leq p_*(E,x)$.

Whereas strong shell porosity is preserved by equivalent metrics, strong porosity need not be. The following simple example demonstrates this.

Example 3.3 *There is a set in the plane which is strongly porous at the origin in the euclidean metric, but is not strongly porous at the origin in the box metric.*

PROOF. First we define two sequences. Let $\{h_n\}$ and $\{r_n\}$ be sequences with $h_n = 1/n$ and $r_n = (1-\frac{1}{n})h_n$. In $\mathbb{R}^2$ let z_n be the point $(0,h_n)$. Next, define the open balls $B_n = B_{z_n}(r_n)$. Finally, take a subsequence $\{n_k\}$ of the natural numbers so that for $j \neq k$, $B_{n_j} \cap B_{n_k} = \emptyset$. Let $E = \mathbb{R}^2 \setminus \{\cup_j B_{n_j}\}$. Then for the euclidean metric,

$$p(E;(0,0)) = \lim_{j\to\infty}\frac{2r_{n_j}}{h_{n_j}+r_{n_j}} = \lim_{n\to\infty}\frac{2(1-\frac{1}{n})}{2-\frac{1}{n}} = 1.$$

However, using the box metric,

$$p_B(E;(0,0)) = \lim_{n\to\infty} \frac{\sqrt{2}(1-\frac{1}{n})}{1+\frac{1}{\sqrt{2}}(1-\frac{1}{n})} = \frac{2}{1+\sqrt{2}} < 1.$$

We now turn our attention to products concerning porous and shell porous sets. This first result is well known.

Theorem 3.3 *Let (X,d_1) and (Y,d_2) be metric spaces. If $E \subset X$ has porosity η at $x \in X$, then $E \times Y$ has porosity at least η at $(x,y) \in (X \times Y, d)$.*

PROOF. Since $p(E;x) = \eta$ there exists $\{R_n\}$, $\{r_n\}$ and $\{z_n\}$ such that

$$R_n > r_n > 0,\ R_n \to 0,\ B^X_{z_n}(r_n) \cap E = \emptyset,\ B^X_{z_n}(r_n) \subset B^X_x(R_n),$$

and

$$\frac{2r_n}{R_n} \to \eta > 0.$$

Take $(x,y) \in E \times Y$. For R_n, r_n, and z_n above

$$(z_n,y) \in B^{X\times Y}_{(z_n,y)}(r_n) \subset B^X_{z_n}(r_n) \times Y \subset E^c \times Y = (E \times Y)^c$$

and

$$B^{X\times Y}_{(z_n,y)}(r_n) \subset B^{X\times Y}_{(x,y)}(R_n).$$

So $p_{X\times Y}(E \times Y;(x,y)) \geq \lim_{n\to\infty} \frac{2r_n}{R_n} = \eta$.

An easy corollary of this theorem is the following:

Corollary 3.2 *If $E \subset (X,d_1)$ has porosity η at $x \in E$ and $F \subset (Y,d_2)$ has porosity γ at $y \in F$ then*

$$p(E \times F;(x,y)) \geq \max\{\eta,\gamma\}.$$

The next example shows the sharp contrast in that shell porosity is *not* preserved when taking products.

Example 3.4 *We can construct a set $E \subset \mathbb{R}$ which is strongly shell porous at two points, x and y, yet $E \times E$ is not shell porous at (x,y) using the box metric on $\mathbb{R}^2$.*

PROOF. First we define the following set in $\mathbb{R}$. Let $E_1 = \{x : 1/(2n+1)! \leq d(x,1) \leq 1/(2n)!,\ n = 1,2,3,\ldots\}$ and $E_2 = \{x : 1/(2n+2)! \leq d(x,3) \leq$

$1/(2n+1)!$, $n = 1,2,3,\ldots\}$. Define E by $E = \mathbb{R}\setminus\{E_1 \cup E_2\}$. It is easy to see that both

$$p^s(E;1) = \lim_{n\to\infty} \frac{1/(2n)! - 1/(2n+1)!}{1/(2n)!} = 1$$

and

$$p^s(E;3) = \lim_{n\to\infty} \frac{1/(2n+1)! - 1/(2n+2)!}{1/(2n+1)!} = 1.$$

However, $E \times E \subset \mathbb{R}^2$ is not shell porous at (1,3) using the box metric. If $S_{(1,3)}(r_1, r_2)$ is a shell which is in $(E \times E)^c$ then both

$$(1 - r_1 - r_2, 1 - r_1) \cap E = \emptyset,\ \ (1 + r_1, 1 + r_1 + r_2) \cap E = \emptyset \ \ (\dagger)$$

and

$$(3 - r_1 - r_2, 3 - r_1) \cap E = \emptyset,\ \ (3 + r_1, 3 + r_1 + r_2) \cap E = \emptyset. \ \ (\ddagger)$$

But this is a contradiction. From (†) we can see there exists an n_0, a natural number, such that $\{(1-r_1-r_2, 1-r_1) \cup (1+r_1, 1+r_1+r_2)\} \subset \{x : 1/(2n_0+1)! \leq d(x,1) \leq 1/(2n_0)!\}$. This says $\{(3-r_1-r_2, 3-r_1) \cup (3+r_1, 3+r_1+r_2)\} \cap E \neq \emptyset$. Therefore, no shells centered at (1,3) are contained in $(E \times E)^c$ and

$$p^s(E \times E; (1,3)) = 0.$$

If we wish to obtain some sort of positive result concerning the shell porosity of a product, we may do so by requiring at least one of the sets to be very shell porous.

Theorem 3.4 *Let (X, d_1) and (Y, d_2) be metric spaces. Let $F \subset Y$ have v-shell porosity $p > 0$ at $y \in Y$, and let $E \subset X$ have shell porosity $q > 1 - p$ at $x \in X$. Then in $(X \times Y, d)$ the set $E \times F$ has shell porosity at least $q + p - 1$ at (x, y), where d denotes the product metric.*

Proof. Let $\epsilon > 0$. From the v-shell porosity of F we know there exists $\delta > 0$ such that for all h, $0 < h < \delta$, there exists t_1 and $t_2, 0 < t_1 < t_2 < h$ so the Y-shell $S_y^Y(t_1, t_2)$ is in F^c and $\frac{t_2 - t_1}{h} > p - \epsilon$.

Also there exists δ' such that $0 < \delta' < \delta$ and for all $h, 0 < h < \delta'$, there exists h_1, h_2, $0 < h_1 < h_2 < h$ so that the X-shell $S_x^X(h_1, h_2)$ is in E^c and $\frac{h_2 - h_1}{h_2} > q - \epsilon$.

Now fix h, $0 < h < \delta'$ and let $0 < h_1 < h_2 < h$ be as above. Choose $0 < t_1 < t_2 < h_2$ such that $S_y^Y(t_1, t_2) \subset F^c$ and $\frac{t_2 - t_1}{h_2} > p - \epsilon$. Let $(s_1, s_2) = (t_1, t_2) \cap (h_1, h_2)$. The length of (s_1, s_2) is at least

$$(h_2 - h_1) - [h_2 - (t_2 - t_1)] > [(q - \epsilon) - (1 - (p - \epsilon))]h_2 = [p + q - 1 - 2\epsilon]h_2.$$

This leads to $\frac{s_2-s_1}{s_2} > \frac{s_2-s_1}{h_2} > p+q-1-2\epsilon$. Note also that $S_y^Y(s_1,s_2) \subset S_y^Y(t_1,t_2)$ and $S_x^X(s_1,s_2) \subset S_x^X(h_1,h_2)$. Hence

$$\begin{aligned} S_{(x,y)}^{X\times Y}(s_1,s_2) &\subset (S_x^X(s_1,s_2)\times Y)\cap(X\times S_y^Y(s_1,s_2)) \\ &\subset (E^c\times Y)\cap(X\times F^c)\subset(E\times F)^c. \end{aligned}$$

Thus $p^s(E\times F;(x,y)) \geq p+q-1-2\epsilon$ and, since ϵ was arbitrary, the shell porosity of $E\times F$ at (x,y) is at least $p+q-1$.

Corollary 3.3 *Let (X,d_1) and (Y,d_2) be complete metric spaces. Let $E \subset X$ be strongly shell porous at $x \in X$ and let $F \subset Y$ have v-shell porosity $p > 0$ at $y \in Y$. Then $E\times F$ has shell porosity at least p at $(x,y) \in (X\times Y,d)$, where d denotes the box metric.*

PROOF. This is an application of Theorem 3.4 with $q = 1$.

Corollary 3.4 *If $E \subset (X,d_1)$ has shell porosity at least $q > 0$ at each of its points and $F \subset (Y,d_2)$ has v-shell porosity at least $p > 0$ at each of its points with $q > 1-p$ then $E\times F$ is a shell porous set in $(X\times Y,d)$, where d denotes the box metric.*

PROOF. This is an obvious consequence of the definition of a shell porous set along with Theorem 3.4.

Corollary 3.5 *Let E and F be subsets of $\mathbb{R}$. If the shell porosity of E is at least $q > 0$ at each of its points and F is very shell porous with v-shell porosity greater than $2-1/\sqrt{2}-q$ at each of its points then $E\times F$ is shell porous in $(\mathbb{R}^2, eucl.)$.*

PROOF. The shell porosity of $E\times F$ at any point will be greater than $2-1/\sqrt{2}$ using the box metric on $\mathbb{R}^2$. From Theorem 3.1 we know that with the euclidean metric on $\mathbb{R}^2$ the shell porosity at any point of $E\times F$ must be greater than zero, hence $E\times F$ is a shell porous set in the euclidean metric.

The next comparison we'll make deals with one of the fundamental motivations for considering shell porosity. In $\mathbb{R}$ when a set is σ-porous it is by necessity totally disconnected. However, we shall see that this is not true for a porous set in a general metric space. We will be able to show, however, that a closed σ-shell porous set in a complete metric space must also be totally disconnected. We begin by noting the following well known lemma.

Lemma 3.1 *If a set E is closed in a complete metric space (X,d), then any connected component of E is closed.*

PROOF. Assume not. Say C is a connected component of E which is not closed. Since $\overline{C}$ is not connected there exists U and V such that $U \cap \overline{C} \neq \emptyset$, $V \cap \overline{C} \neq \emptyset$, $V \cap U \cap \overline{C} = \emptyset$, and $\overline{C} = (U \cup V) \cap \overline{C}$. But then $U \cap C \neq \emptyset$, $V \cap C \neq \emptyset$, $(U \cap V) \cap C = \emptyset$, and $C = (U \cup V) \cap C$, which yields a disconnection of our connected set. Thus C is closed.

Lemma 3.2 *Say E is shell porous in a metric space (X, d). Let C be any connected set with $E \subset C$. Let G be an open set with card$(G \cap E) \geq 2$. Then E is not dense in $G \cap C$.*

PROOF. Assume not. Say E is dense in $(G \cap C)$. Let $x \in (G \cap E)$ and $y \in (G \cap E)$. Since E is shell porous at x there exists an r_1 and an r_2, $0 < r_1 < r_2 < d(x, y)$ such that $B_x(r_2) \subset G$ and $S_x(r_1, r_2) \cap E = \emptyset$. Then no point in C is in $S_x(r_1, r_2)$ else from the density of E in $(G \cap C)$ there would be an element of E in $S_x(r_1, r_2)$. So

$$C = (B_x(r_2) \cap C) \cup [(X \setminus \overline{B_x(r_1)}) \cap C],$$

contrary to the assumption that C is connected. Thus E is not dense in $(G \cap C)$.

Theorem 3.5 *If E is a closed σ-shell porous set in a complete metric space then E is totally disconnected.*

PROOF. Let D be a connected component of E. Then both D is closed and is σ-shell porous. Write $D = \cup_n D_n$, where each D_n is shell porous. From the Baire Category Theorem there exists n_0 such that D_{n_0} fails to be nowhere dense in D. So there exists an open set G such that $D \cap G \neq \emptyset$ and D_{n_0} is dense in $G \cap D$. By Lemma 3.2 we must have card$(G \cap D_{n_0}) = 1$. Thus card$(G \cap D) = 1$ else we would get a second point in the dense $(G \cap D_{n_0})$. Hence the only connected components of E are singletons; so E is totally disconnnected.

Corollary 3.6 *Any σ-shell porous set in a complete metric space is totally pathwise disconnected.*

PROOF. If there exists an x and y in E which are pathwise connected in E then there exists $\gamma : [0, 1] \to E$, with γ continuous, $\gamma(0) = x$, and $\gamma(1) = y$. Since γ is continuous, the image of [0,1] (call this $\gamma[0, 1]$) is closed and connected. Furthermore, since $\gamma[0, 1] \subset E$, it is σ-shell porous. By Theorem 3.5 the closed set $\gamma[0, 1]$ must be disconnected, a contradiction. Thus there is no path in E which connects x and y.

Example 3.5 *A set can be porous (strongly porous) and connected.*

PROOF. In $\mathbb{R}^2$ any line will be a strongly porous set and obviously connected.

Another distinction between porosity and shell porosity in $\mathbb{R}^n$ can be seen by considering sections of porous or shell porous sets as defined below.

Definition 3.1 *Let E be a set in $\mathbb{R}^n$. For $x \in \mathbb{R}^m$, $m < n$, define the section Y_x as*

$$Y_x = \{y \in \mathbb{R}^{n-m} \textit{such that } (x, y) \in E\}.$$

Theorem 3.6 *Let $E \subset \mathbb{R}^n$ be shell porous and $m < n$. Then for all $x \in \mathbb{R}^m$, Y_x is shell porous in $\mathbb{R}^{n-m}$.*

PROOF. If $Y_x = \emptyset$ then the claim is vacuously true. If $Y_x \neq \emptyset$ then there exists $y \in \mathbb{R}^{n-m}$ such that $(x, y) \in E \subset \mathbb{R}^n$. Since E is shell porous at (x, y) there exists sequences $\{R_k\}$ and $\{r_k\}$ with

$$R_k > r_k > 0,\ R_k \to 0,\ S_{(x,y)}(r_k, R_k) \cap E = \emptyset,$$

and

$$\frac{R_k - r_k}{R_k} \to \eta = p^s(E; (x, y)) > 0.$$

Because this is true we must have in $\mathbb{R}^{n-m}$ that $S_y(r_k, R_k) \cap Y_x = \emptyset$ for all $k = \{1, 2, \ldots\}$. Thus $p^s(Y_x; y) \geq \lim_{k\to\infty} \frac{R_k - r_k}{R_k} = \eta > 0$. Hence Y_x is shell porous.

The same property is not true of porous sets.

Example 3.6 *In $\mathbb{R}^2$ there exist porous sets which have sections that are not porous as a subset of $\mathbb{R}$.*

PROOF. In $\mathbb{R}^2$ let $E = \{(x, y) : x = c\}$, c some constant. For the fixed x-value, $x = c$, the section Y_c is the real line as a subset of $\mathbb{R}$which, of course in not porous.

It has yet to be mentioned that if a set is shell porous at a point we have no guarantee that the set will also be porous at that point. We now look for sufficient conditions on a metric space to guarantee that shell porosity will imply porosity.

Theorem 3.7 *Let (X, d) be a locally connected metric space which has no isolated points. If a set $A \subset X$ is shell porous at $x \in A$ then A is porous at x.*

PROOF. Say $p^s(E; x) = \eta > 0$. Then we claim $p(E; x) \geq \eta$. Since E is shell porous at x, there exists $\{R_n\}, \{r_n\}$ with

$$R_n > r_n > 0, R_n \to 0, S_x(r_n, R_n) \cap E = \emptyset$$

and

$$\frac{R_n - r_n}{R_n} \to \eta.$$

Since X is locally connected there is a neighborhood of $x, N(x)$, such that $N(x)$ is connected. Since x is not an isolated point, pick $y \in N(x)$, $y \neq x$. Let N be a positive integer such that $R_N < d(x,y)$ and if $n > N$, $B_x(R_n) \subset N(x)$. For $n \geq N$ define the following:

$$t_n = r_n + \frac{R_n - r_n}{2}(1 - 1/n),\ T_n = R_n - \frac{R_n - r_n}{2}(1 - 1/n).$$

Now if $S_x(t_n, T_n) \cap X \neq \emptyset$ for all n then for each n there exists $z_n \in S_x(t_n, T_n)$ such that $B_{z_n}(\frac{R_n - r_n}{2}(1 - 1/n)) \subset S_x(r_n, R_n) \subset E^c$. Thus

$$p(E;x) \geq \lim_{n\to\infty} 2\frac{1/2[R_n - r_n](1 - 1/n)}{R_n} = \lim_{n\to\infty} \frac{R_n - r_n}{R_n}(1 - 1/n) = \eta.$$

Say there is an $\tilde{n}$ such that $S_x(t_{\tilde{n}}, T_{\tilde{n}}) \cap X = \emptyset$. Then if $U = [(X \setminus B_x(t_{\tilde{n}})) \cap N(x)]$ and $V = [B_x(T_{\tilde{n}}) \cap N(x)]$ we have

$$U \cap N(x) \neq \emptyset, V \cap N(x) \neq \emptyset, U \cap V = \emptyset, U \cup V = N(x).$$

i. e. a disconnection of our connected set. Thus $S_x(t_n, T_n) \cap X \neq \emptyset$ for all n and E is porous at x with $p(E;x) \geq \eta$.

Both the local connectedness and the lack of isolated points are necessary for shell porosity to imply porosity as the following examples illustrate.

Example 3.7 *There is a complete metric space with no isolated points, a set A, and a point $x \in A$ such that $p^s(A;x) > 0$ but $p(A;x) = 0$.*

PROOF. Let (X,d) be $(\{(0,0)\} \cup \{x \in \mathbb{R}^2 : d(0,x) = 2^{-n}, n = 1,2,3,\ldots\}$, *euclidean*). Let $A = X$. Then $p^s(X;0) = \lim_{n\to\infty} \frac{1/2^n - 1/2^{n+1}}{1/2^n} = 1/2 > 0$. However, since $X^c = \emptyset$ we must have $p(X;0) = 0$.

Example 3.8 *There is a locally connected complete metric space (X,d), a set $A \subset X$ and a point $x \in A$ such that $p^s(A;x) > 0$ while $p(A;0) = 0$.*

PROOF. Let $(X,d) =(\mathbb{R}^2$, discrete),$A = \mathbb{R}^2$, and $x = 0$. For positive t, h such that $t + h < 1$, $S_0(t, t+h) = \{x\} \setminus \{x\} = \emptyset$ which says that $p^s(A;0) = 1$ while $p(A;0) = 0$ since $A^c = \emptyset$.

At the beginning of this section we noted two examples of Evans, Humke and Saxe [9] of sets in $\mathbb{R}$. We close this section by looking for analogous examples in $\mathbb{R}^n$. First we have

Example 3.9 *There is a shell porous set A in $\mathbb{R}^n$ which cannot be expressed as a union of a sequence of sets $\{A_n\}$ each having shell porosity at least 4/5 at each of its points.*

PROOF. Let $C \subset \mathbb{R}$ be the example of [9] which is symmetrically porous but cannot be written as the union of a sequence of sets $\{C_n\}$ each having symmetric porosity at least 4/5 at each of its points.

Let $A = C \times \bar{0}$, where $\bar{0}$ denotes the zero vector in $\mathbb{R}^{n-1}$. Then A is shell porous in $\mathbb{R}^n$ by Corollary 3.4. The conclusion follows from the properties of C and Theorem 3.6.

Next, what is the $\mathbb{R}^n$ analogue to the existence of a bilaterally strongly porous set which is not σ-symmetrically porous? Clearly a line segment in $\mathbb{R}^n$, $n > 1$, provides an example of a set which is strongly porous but not σ-shell porous, the latter via Theorem 3.5. What shall we do about the bilateral condition? If we choose to say that a set is strongly porous at x_0 in $\mathbb{R}^n$ in every direction if its intersection with every line through x_0 is strongly porous at x_0, viewed as a subset of $\mathbb{R}$, we could then exhibit a set in $\mathbb{R}^n$ which is strongly porous in every direction at each of its points, yet fails to be σ-shell porous by simply considering the graph of a parabola as a subset of $\mathbb{R}^n$. Surely, something stronger can be said and indeed we shall observe this in the next section after hyperporosity is introduced.

4. Comparisons of Shell Porosity and Other Generalized Porosities

In Section 3 we looked at shell porosity versus porosity in a general metric space. The purpose of this section is to look at shell porosity against other definitions of porosity. Specifically, we shall contrast the notion of a shell porous set with S. J. Agronsky and A. M. Bruckner's idea of a totally porous set [1] and T. Zamfirescu's notion of a hyperporous set [23]. In the context of a general metric space we can draw no implications involving shell porous sets, totally porous sets and hyperporous sets. However, when we restrict our space to be convex then we can show that there is a series of implications among the three definitions of porosity.

The first new definition of porosity we will introduce is that for a totally porous set. Agronsky and Bruckner used total porosity to show that for a convex set in a separable Banach space there is a direct relationship between a set being totally porous and its being locally compact. Specifically, in these spaces local compactness can be characterized in terms of the total porosity of the compact subsets of the space in question.

Definition 4.1 *Let (X, d) be a metric space, $B \subset X$ and $x \in B$. Let S be a sphere in X such that x is in the boundary of S. Then B is said to be porous at x with respect to S if there exists $\gamma > 0$ so that for every $\epsilon > 0$ there exists spheres $S_1 \subset S_2 \subset S$ such that $x \in \overline{S_2} \setminus S_2$, $S_1 \cap B = \emptyset$ and*

$$\epsilon > diameter S_1 \geq \gamma(diameter S_2).$$

A set B that is porous at a point $x \in B$ with respect to every sphere containing x in its boundary is called totally porous at x.

The next type of porous set we will define will be a hyperporous set. Zamfirescu, in his 1989 survey article [23], took a geometric approach to looking at porous sets in Baire spaces. He extended the idea of a totally porous set into what he named a hyperporous set.

Definition 4.2 *A set $M \subset (X, d)$ is hyperporous at $x \in M$ if there is a positive γ such that for any $z \neq x$ there is a point y with $d(x, y) + d(y, z) = d(x, z)$ such that*

$$B_y(\gamma d(x, y)) \cap M = \emptyset.$$

For a fixed z we say

$$\sup\{\gamma : \exists\, y \text{ with } d(x, y) + d(y, z) = d(x, z) \text{ and } B_y(\gamma d(x, y)) \cap M = \emptyset\}$$

is the hyperporosity at x with respect to z. The hyperporosity of M at x is then the infimum over all $z \in X$ of the hyperporosity with respect to z.

Although the idea for a hyperporous set grew from the definition of a totally porous set, the two definitions are different (not when $X = \mathbb{R}$, where they are both equivalent to the notion of bilateral porosity). The following examples show that hyperporous sets and totally porous sets are two different types of sparse sets outside of $\mathbb{R}$.

Example 4.1 *In ($\mathbb{R}^2$, euclidean) let E be a line. Then E is a totally porous set which is not hyperporous.*

PROOF. Let $x = (x_1, x_2)$ be a point in E and let the sphere S be such that $x \in \overline{S} \setminus S$. Given ϵ, greater than zero and less than the diameter of S, let S_2 be the sphere of diameter ϵ contained in S and with x in its boundary. If $S_2 \cap E = \emptyset$ the let $S_1 = S_2$. If $S_2 \cap E \neq \emptyset$ then if we consider the sphere, S_2, as being cut into two sections by E one of the sections must be large enough to contain a sphere of diameter $\epsilon/2$, this sphere will be our S_1. In either case, we found S_1 and S_2 satisfying

$$\epsilon > diameter\ S_1 \geq \frac{1}{2}(diameter\ S_2).$$

So E is a totally porous set.

However, for our given x, let z be another point in E. For any y on the line segment between x and z and any $\gamma > 0$, $B_y(\gamma d(x,y)) \cap E \neq \emptyset$. Thus E is not a hyperporous set. In the same vein, there exist sets which are hyperporous, but not totally porous.

Example 4.2 *Let the metric space* (X,d) *equal* $(\mathbb{R}^2$, *discrete). If our set* E *consists of just the origin, then* E *is strongly hyperporous at* $(0,0)$, *but not totally porous at* $(0,0)$.

PROOF. To show that E is hyperporous at $(0,0)$, let z be any point in $\mathbb{R}^2 \setminus E$. For any $\gamma \in (0,1)$, and letting $y = z$, we have $d(x,y) + d(y,z) = d(x,z)$ and $B_y(\gamma d(x,y)) \cap E = \emptyset$. Thus E is strongly hyperporous at the origin. Now pick $z \neq (0,0)$ and let S be the open sphere centered at z of diameter 2. Thus $(0,0) \in \overline{S} \setminus S$. For any $\gamma \in (0,1)$ let $\epsilon < 2\gamma$. Then no ball of diameter ϵ can contain the origin in its boundary. So E is not totally porous at the origin.

In the context of a general metric space the results involving shell porosity are that there are sets which are shell porous at a point which are not hyperporous there and sets which are shell porous at a point and not totally porous there. As far as the reverse implications go, we shall see later on that even with convexity both hyperporous sets and totally porous sets need not be shell porous. We begin by relating shell porosity and hyperporosity in a non-convex space. Unlike the situation we have in Section 3 where local connectivity/shell porosity implied porosity we have the following:

Example 4.3 *There is a locally connected metric space* (X,d) *with no isolated points and a set* E *in* X *which is shell porous but not hyperporous.*

PROOF. Let $(X,d) = (\mathbb{R}^2 \setminus \{(x_1,0) : -1 < x_1 < 1\}, euclidean)$, and let E be $\{(1,0)\}$. For any $R > 0$ and any natural number $N > 1$ the shell $S_{(1,0)}(R/N, R)$ is in E^c. Letting R approach zero and N approach infinity we see that E is a strongly shell porous set. However, E is not hyperporous at $(1,0)$ since $(-1,0) \in X$ but there are no points in X on the line segment joining $(-1,0)$ and $(1,0)$ so the conditions for hyperporosity cannot hold.

Likewise we have the following:

Example 4.4 *There exists a locally connected metric space* (X,d) *with no isolated points and a set* $E \subset X$ *which is shell porous at a point but not totally porous there.*

PROOF. First, we will define our space. Let $E_1 \subset \mathbb{R}^2$ be defined as

$$E_1 = \{x \in \mathbb{R} : 2^{-2n} \leq x \leq 2^{-2n+1},\ n = 1,2,\ldots\} \times \{0\}.$$

Similarly, let E_2 be

$$E_2 = \{x \in \mathbb{R} : -(2^{-2n-1}) \le x \le -(2^{-2n}),\ n = 1, 2, \ldots\} \times \{0\}.$$

Finally, E_3 will be the set defined as

$$E_3 = \{x \in \mathbb{R}^2 : d(0, x) = 2^{-n},\ n = 1, 2, \ldots\}.$$

Putting all this together to define X we let $X = E_1 \cup E_2 \cup E_3 \cup \{(0,0)\}$. Let the set $E \subset X$ be $E_3 \cup \{(0,0)\}$. From the construction of X we can see that for all n, $S_{(0,0)}(1/2^n, 1/2^{n-1}) \cap E = \emptyset$ which leads to $p^s(E; (0,0)) \ge 1/2$. Thus we have E is shell porous at the origin.

Now we will define a sphere S with $(0,0)$ in its boundary so that for any $\epsilon > 0$ there is no sphere of diameter ϵ which will not intersect E thus showing E is not a totally porous set. Let S be the sphere centered at $(0, 1/2)$ of radius $1/2$. The only points inside of S where we could center the spheres needed are points in E. So we cannot construct the sphere S_1 as called for in the definition of a totally porous set.

From this point onward, we will require the space we work in to be a convex topological vector space over $\mathbb{R}$where the topology, τ, is generated by a metric d. We shall call this space (X, d) and refer to it as a convex metric space. So given any points x, $y \in X$ the set of points z such that $z = \alpha x + (1 - \alpha)y$, $\alpha \in (0, 1)$ must also be in X and will be referred to as the line segment between x and y.

In the context of this type of space we begin to have relationships between the different notions of porosity. We now have some strict implications among the different types of porous sets and we will begin with a relationship between shell porous sets and hyperporous sets.

Theorem 4.1 *Assume X is a convex metric space. If a set $E \subset X$ is shell porous at $x \in X$, then E is hyperporous at x.*

PROOF. Let $E \subset X$ and $x \in X$ be a point such that $p^s(E; x) = \eta > 0$. So there exists two sequences, $\{R_n\}$ and $\{r_n\}$, with

$$R_n > r_n > 0,\ R_n \to 0,\ S_x(r_n, R_n) \cap E = \emptyset$$

and

$$\frac{R_n - r_n}{R_n} \to \eta.$$

To show the hyperporosity of E at x, $p^h(E; x)$, is positive take any $z \in X$. For each n, let y_n be the point on the line between x and z with

$$d(x, y_n) = R_n - \frac{R_n - r_n}{2}.$$

Thus $y_n \in S_x(r_n, R_n)$ and $B_{y_n}(\frac{R_n - r_n}{2}) \cap E = \emptyset$. Finally, the hyperporosity of E at x with respect to z is at least

$$\lim_{n\to\infty} \frac{\frac{R_n - r_n}{2}}{d(x, y_n)} = \lim_{n\to\infty} \frac{\frac{R_n - r_n}{2}}{R_n - \frac{R_n - r_n}{2}} = \lim_{n\to\infty} \frac{\frac{R_n - r_n}{R_n}}{2 - \frac{R_n - r_n}{R_n}} = \frac{\eta}{2-\eta}.$$

From this we see that $p^h(E; x)$, the hyperporosity of E at x, is at least $\frac{\eta}{2-\eta} > 0$. So E is hyperporous at x.

This brings up a relationship between the strong shell porosity of a set and the strong hyperporosity of that set.

Corollary 4.1 *Let (X, d) be a convex metric space. If a set E is strongly shell porous at a point x, then E is strongly hyperporous at x.*

PROOF. In the proof of Theorem 4.1 we found that if a set has shell porosity η at a point then the hyperporosity of the set at that point is at least $\frac{\eta}{2-\eta}$. Apply this result with $\eta = 1$.

Another positive implication, now that we are working in a convex space, involves the relationship between hyperporous sets and totally porous sets.

Theorem 4.2 *Let (X, d) be a convex metric space. Let $E \subset X$ and x be a point in X such that E is hyperporous at x. Then E is totally porous at x.*

PROOF. Let $\eta > 0$ be the hyperporosity of E at x and let $\eta > \eta' > 0$. Let S be a sphere such that $x \in \overline{S} \backslash S$ and let $\epsilon > 0$ be given. Define the center of the sphere S as z_1 and if $\epsilon < diameter S$ define S_0 as the sphere of diameter ϵ with $x \in \overline{S_0} \setminus S_0$ and center z, where $d(x, z) + d(z, z_1) = d(x, z_1)$. If $\epsilon \geq diameter S$ then let $S = S_0$ and $z = z_1$. Given this z, from the hyperporosity of E at x, there is a y such that y is on the line segment between x and z, and $B_y(\eta' d(x, y)) \cap E = \emptyset$. Let S_2 be the sphere centered at y with diameter $2d(x, y)$ and let S_1 be the sphere centered at y with diameter $2\eta' d(x, y)$. Thus

$$x \in \overline{S_2} \setminus S_2, \quad S_1 \cap E = \emptyset$$

and

$$\epsilon > diameter S_1 \geq \eta'(diameter S_2).$$

Since ϵ and S were arbitrary, E is totally porous at x.

So in convex metric spaces we have the following implications between these three types of porous sets:

shell porous $\rightarrow$ *hyperporous* $\rightarrow$ *totally porous.* (†)

From this relationship, we see indirectly that in a convex metric space if a set is shell porous at a point then it is also totally porous there.

Now we show that the implications between the different porosities in (†) are strict, that is, there exists sets which are totally porous yet not hyperporous and sets which are hyperporous but not shell porous. Note that the former has already been shown in Example 4.1

ϵ and S leaving the last thing to note as

As mentioned before, the hyperporosity of a set at a point does not mean that the set will also be shell porous there. As the following example shows, even in the plane a set can be hyperporous and not shell porous at a point.

Example 4.5 *There exists a set $E \subset \mathbb{R}^2$ which is hyperporous at the origin, but $p^s(E;(0,0)) = 0$.*

PROOF. Let X be the unit disk in $(\mathbb{R}^2$, eucl.). Define E as

$$\{x = (x_1, x_2) : x_1, x_2 > 0 \text{ and } 2^{-(2n+1)} \leq d((0,0), x) \leq 2^{-2n},\ n = 0, 1, \ldots\}$$
$$\cup \{x : x_1, x_2 < 0 \text{ and } 2^{-(2n+2)} \leq d((0,0), x) \leq 2^{-(2n+1)},\ n = 0, 1, \ldots\}.$$

Now, $p^s(E;(0,0)) = 0$, but $p^h(E;(0,0)) \geq 1/3$. To see the latter, let $z \in X$ and pick $y = \alpha z$ for some $\alpha \in (0,1)$ such that $y \in E^c$ and there exists an M with $d((0,0), y) = \frac{1}{2}(2^{-M+1} - 2^{-M}) + 2^{-M} = \frac{3}{2^{M+1}}$. Note: $B_y(\frac{1}{2^{M+1}}) \cap E = \emptyset$. Also note that $\lim_{M\to\infty} \frac{\frac{1}{2^{M+1}}}{\frac{3}{2^{M+1}}} = \frac{1}{3}$. Thus the hyperporosity of E at $(0,0)$ with respect to z is at least $1/3$. Since z was arbitrary we have $p^h(E;(0,0)) \geq \frac{1}{3}$.

This last result can be strengthened to say there is a set of hyperporosity one which is not even a countable union of shell porous sets.

Example 4.6 *There exists a set in $\mathbb{R}^n$ which is strongly hyperporous but not σ-shell porous.*

PROOF. We will show the existence of this set in $\mathbb{R}^2$. The proof in $\mathbb{R}^n$ is analogous. In [9] Evans, Humke and Saxe construct in $\mathbb{R}$a symmetric Cantor set and show that it contains a residual subset which is strongly bilaterally porous (hyperporous) but not σ-symmetrically porous (σ-shell porous). Call this subset S.

Claim: In $\mathbb{R}^2$ the set $S \times \{0\}$ is strongly hyperporous but not σ-shell porous.

Pick $x \in S \times \{0\}$ and $z \in \mathbb{R}^2$. If $z = (z_1, 0)$, showing $S \times \{0\}$ is strongly hyperporous with respect to z is a restatement of the strong bilateral porosity of S in $\mathbb{R}$. Without loss of generality, let $x = (x_1, 0)$ $x_1 > 0$ and $z = (z_1, z_2)$ with $z_1 > x_1$. Using the strong bilateral porosity of S at x_1, given any $\epsilon > 0$ there exists $t > 0$ and $h > 0$ such that $t + h < \epsilon$, $(x + t, x + t + h) \cap S = \emptyset$ and $\frac{h}{t+h} > 1 - \epsilon$. Let $y = (y_1, y_2)$ be the point on the line between x and z with $y_1 = x_1 + t + h/2$ (note $y \in B_x(\epsilon)$). First, note that $B_y(d(y, (x_1 + t, 0))) \cap (S \times \{0\}) = \emptyset$. Secondly, since $h/(t + h) > 1 - \epsilon$ we have the bound

$t/h < \epsilon/(1-\epsilon)$. To find the hyperporosity at x with respect to z we look at the ratio of $d(y,(x_1+t,0))$ and $d(y,x)$. By the construction we see this ratio must be less than one. From the triangle inequality we have that $d(y,x) \leq d(y,(x_1+t,0)) + d(x,(x_1+t,0)) = d(y,(x_1+t,0)) + t$. Thus we have

$$\frac{d(y,(x_1+t,0))}{d(y,x)} \geq \frac{d(y,x)-t}{d(y,x)} = 1 - \frac{t}{d(y,x)} \geq 1 - \frac{t}{t+h/2} = 1 - \frac{\frac{t}{h}}{\frac{t}{h}+\frac{1}{2}},$$

and as ϵ approaches 0 we have the hyperporosity with respect to z approaching one.

Since this holds for all $x \in S \times \{0\}$ and $z \in \mathbb{R}^2$ the set $S \times \{0\}$ is strongly hyperporous. Now $S \times \{0\}$ cannot be σ-shell porous. If $S \times \{0\}$ was σ-shell porous, we could write $S \times \{0\} = \cup F_n$, where each F_n was shell porous. Then from Theorem 3.6 the set $E_n = \{x : (x,0) \in F_n\}$ would be shell porous. Thus S could be written as

$$S = \cup E_n, \quad E_n \textit{ shell porous}$$

and hence S would be σ-shell porous, a contradiction.

One property which places shell porous sets apart from the three other types of thin sets we have looked at (porous sets, hyperporous sets, and totally porous sets) concerns the ability of a closed set to be connected, yet porous in one of our definitions. In Theorem 3.5 we saw that a set which is closed and σ-shell porous must be totally disconnected. Also, we have already seen a set in $\mathbb{R}^2$ which is connected and both porous and totally porous. Namely, a line in $\mathbb{R}^2$ will meet these criteria. We shall next consider a set which is hyperporous and connected.

Theorem 4.3 *The classical von Koch curve is a set in the plane which is connected and hyperporous.*

PROOF. Let K be the von Koch curve placed in the first quadrant of $\mathbb{R}^2$ with one endpoint at $(0,0)$ and the other at $(1,0)$. More specifically, we may describe K in the terminology of Barnsley [2] as the attractor for the iterated function system $\{f_1, f_2, f_3, f_4\}$ where

$$f_1\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 1/3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$f_2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 1/3\begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1/3 \\ 0 \end{pmatrix}$$
$$f_3\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 1/3\begin{pmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 1/2\sqrt{3} \end{pmatrix}$$
$$f_4\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 1/3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2/3 \\ 0 \end{pmatrix}.$$

Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be the affine map

$$h\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 1/\sqrt{3}\begin{pmatrix} -\sqrt{3}/2 & 1/2 \\ -1/2 & -\sqrt{3}/2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 1/2\sqrt{3} \end{pmatrix}$$

that is, h may be viewed as a rotation (about the origin) $\frac{7\pi}{6}$ radians, followed by a change of scale of magnitude $1/\sqrt{3}$, followed by a translation sending the origin to $(1/2, 1/2\sqrt{3})$. Note, as is obvious, that $f_1(K) = E \equiv \{x = (x_1, x_2) : x \in K \text{ and } 0 \le x_1 \le 1/3\}$ is a repeat of the von Koch curve, just scaled down by 1/3. We shall argue that K is hyperporous at each point of E and then appeal to the self similarity of K to conclude that K is a hyperporous set. Toward this end we let L_1 denote the line segment joining $(1/2, 0)$ and $(17/48, 15/48\sqrt{3})$. Then $h(L_1)$ is the line segment joining $(1/4, 1/4\sqrt{3})$ and $(3/8, 1/96\sqrt{3})$. We let $L = L_1 \cup h(L_1)$. Clearly, $L \cap K = \emptyset$ and we let $d_0 = dist(L, K) > 0$. Note that for any $x \in E$ and $y \in L$, $d(x, y) \le \frac{1}{2}$. We claim that K has hyperporosity at least $2d_0$ at each point in E.

To establish this, fix $x \in E$. First consider a $z = (z_1, z_2)$ such that $z_1 > 1/2$ and $0 \le \arg z \le \pi/6$, viewing z as a complex number. The line segment joining x and z must intersect L in at least one point, call it y. We know $B_y(d_0) \cap E = \emptyset$ and $d(x, y) \le 1/2$. Next, if $z_1 > 1/2$ and $\arg z \notin [0, \pi/6]$, then we may utilize the large inlets that K determines to give us the existence of a y on the line segment between x and z such that

$$B_y(\frac{1}{4}d(x, y)) \subset E^c,$$

and $1/4 > 2d_0$. Now, if $z_1 \le 1/2$ we appeal to the self similarity of K and repeat our argument, but on a sufficiently smaller scale, using an appropriately scaled copy of L. Thus, the hyperporosity of K at x with respect to z is at least $2d_0$ and it follows that K is a hyperporous set.

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