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SYMMETRIC VARIATION

E. Freund [4] has recently in this Exchange investigated the structure of functions having zero symmetric variation on an interval. It is our purpose here to extend that study by addressing some natural related problems. By elementary methods and by some well-known techniques we are able to present properties of functions having zero, finite or σ–finite symmetric variation on an arbitrary set.

Let us recall first the definition of the symmetric variation (cf. [4] or [7]). Let $E \subset \mathbb{R}$ and let δ be a positive function on E. Then we write

$$\mathrm{S}_\delta(f, E) = \sup \sum_{i=1}^{n} |f(x_i + h_i) - f(x_i - h_i)|$$

where the supremum is with regard to all sequences $\{[x_i - h_i, x_i + h_i]\}$ of nonoverlapping intervals with centers $x_i \in E$ and with $h_i < \delta(x_i)$. We write then

$$\mathrm{VS}_f(E) = \inf \mathrm{S}_\delta(f, E)$$

where the infimum is taken over all positive functions δ on E. This expression is called the *symmetric variation* of f on E and the set function VS_f is called the *symmetric variational measure* associated with f. It is not difficult to see that VS_f is an outer measure on the real line.

Freund [4, Theorem 2.3] shows that if $\mathrm{VS}_f((a,b)) = 0$ then there is a constant function g so that

$$\{x \in (a,b);\ f(x) \neq g(x)\} \tag{1}$$

is countable and so that each set

$$\{x \in (a,b);\ |f(x) - g(x)| > \epsilon\} \qquad (\epsilon > 0) \tag{2}$$

has countable closure. Note that the set in (1) cannot be further described: any countable set may appear as such for some constant function g and some f with $\mathrm{VS}_f((a,b)) = 0$.

We now investigate the conditions $\mathrm{VS}_f((a,b)) < +\infty$, $\mathrm{VS}_f(E) = 0$ and $\mathrm{VS}_f(E) < +\infty$. The condition $\mathrm{VS}_f((a,b)) < +\infty$ can be handled almost exactly as in [4]. If we apply the elementary covering theorem from [6] we can easily prove the following theorem.

THEOREM 1 *Suppose that* $\mathrm{VS}_f((a,b)) < +\infty$. *Then there is a function* g *of bounded variation on* (a,b) *so that*

$$\{x \in (a,b);\ f(x) \neq g(x)\}$$

is countable.

Moreover it can be shown too that the sets (2) have countable closure as in the Freund result. Here with a little extra effort more can be said about the set (1) though. By applying the covering theorem from Freiling [3] (rather than the simpler one in [6]) we obtain immediately that this set is splattered. (A splattered set, in Freiling's colourful language, is a set all of whose nonempty subsets contain a point isolated on one side at least.) In fact the set can be shown to be scattered (all of its nonempty subsets contain an isolated point) by using the same covering theorem but arguing rather more carefully.

This theorem, especially in its scattered version, is related to a similar but much older result of Charzyński [2]: *Suppose that*

$$\limsup_{h \to 0+} h^{-1}|f(x+h) - f(x-h)| \leq M$$

at every point $x \in (a,b)$. *Then there is a Lipschitz function* g *with Lipschitz constant no more than* M *so that*

$$\{x \in (a,b);\ f(x) \neq g(x)\}$$

is scattered.

The analogy cannot be pushed too far. One might hope that the condition $\limsup_{h \to 0+} h^{-1}|f(x+h) - f(x-h)| < +\infty$ at every point $x \in (a,b)$ would force f to agree with a reasonable function g except on a small set. The example $f(x) = \cos x^{-1}$ shows that f cannot agree even with a continuous function outside of anything fairly small.

Notice that for the function $f(x) = \cos x^{-1}$ the measure VS_f is not finite but it is σ-finite. One asks then what properties this will impose on f. An answer is available by applying some techniques due to Charzyński [2].

THEOREM 2 *Suppose that* VS_f *is* σ*-finite on an interval* (a, b)*. Then there is a dense set of subintervals of* (a, b) *on each of which* f *has bounded variation.*

PROOF. We begin by observing that f is symmetrically continuous at each point of (a, b) excepting only the points of a countable set C. This follows by much the same arguments that show a function of bounded variation in the ordinary sense has countably many ordinary discontinuities.

Let $\{A_m\}$ be a sequence of disjoint sets covering (a, b) such that each $\mathrm{VS}_f(A_m)$ is finite. Choose a positive function δ on (a, b) so that $\mathrm{S}_\delta(f, A_m) < +\infty$ for each m. Write

$$E_{nm} = \{x \in A_m;\ \delta(x) > 1/n\}\,.$$

The countable collection $\{E_{nm}\}$ covers (a, b) and so, by the Baire theorem, there must be an interval $(c, d) \subset (a, b)$ and a set E_{NM} dense in (c, d), indeed even second category in each subinterval of (c, d). We can assume that the length of (c, d) is less than $1/N$. We shall show that f has bounded variation on (c, d); since the collection of intervals with this property is dense in (a, b) the proof is then complete.

Suppose now that $\{[x_i, y_i];\ i = 1, 2, \ldots, P\}$ are nonoverlapping intervals contained in (c, d). We shall show that

$$\sum_{i=1}^{P} |f(y_i) - f(x_i)| \leq 6\mathrm{S}_\delta(f, A_M) + 2 \tag{3}$$

and the theorem will follow.

Take $[x, y]$ as any one of these intervals $[x_i, y_i]$. We shall construct points $x = x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 = y$ from inside the interval $[x, y]$ and write I_j as the interval with endpoints x_j and x_{j+1}. Write $f(I_j) = |f(x_{j+1}) - f(x_j)|$ $(j = 1, 2, \ldots, 8)$ and then we can employ the estimate

$$|f(y) - f(x)| \leq \sum_{j=1}^{8} f(I_j).$$

Here are the details of the construction. Choose a point $z \in E_{NM}$ close to the midpoint of $[x, y]$; since E_{NM} is dense in (c, d) we can choose such a point, say in the middle 1/10th of $[x, y]$. Since E_{NM} is uncountable in this middle 1/10th of $[x, y]$ we can arrange that the points $(x + z)/2$ and $(z + y)/2$ do not belong to the countable set C, i.e. so that f is symmetrically continuous at both of these points.

Write $x = z - 2s$, $y = z + 2t$ and choose s', $t' > 0$ so that $z + t' \in E_{NM}$, $z - s' \in E_{NM}$, $z + t'$ is sufficiently close to $z + t/2$ so that

$$|f(z + 2t - 2t') - f(z + 2t')| < 1/P \tag{4}$$

and $z - s'$ is sufficiently close to $z - s/2$ so that

$$|f(z - 2s') - f(z - 2s + 2s')| < 1/P. \tag{5}$$

The inequalities (4) and (5) just employ the symmetric continuity of f at the points $z + t$ and $z - s$ by making $|t - 2t'|$ and $|s - 2s'|$ small. Now write

$$\begin{aligned}
x_1 &= x\ (= z - 2s), \\
x_2 &= z - 2s' + 2s, \\
x_3 &= z - 2s + 2s', \\
x_4 &= z - 2s', \\
x_5 &= z, \\
x_6 &= z + 2t', \\
x_7 &= z + 2t - 2t', \\
x_8 &= z + 2t' - 2t, \\
x_9 &= y\ (= z + 2t).
\end{aligned}$$

As long as $|t - 2t'|$ and $|s - 2s'|$ are not too big each of these points is inside the interval $[x, y]$. Thus we have produced 8 subintervals of $[x, y]$ and (because of (4) and (5)) we have arranged for $f(I_3) < 1/P$ and for $f(I_6) < 1/P$; notice that the centers of the remaining intervals I_1, I_2 I_4, I_5 I_7, and I_8 are in the set E_{NM}. Evidently we can do this same construction for each interval in $\{[x_i, y_i];\ i = 1, 2, \ldots, P\}$ and, using an obvious notation, we produce intervals $\{I_{ij};\ i = 1, 2, \ldots, P,\ j = 1, 2, \ldots 8\}$ with $f(I_{i3}) < 1/P$ and $f(I_{i6}) < 1/P$. All the remaining intervals have midpoints in E_{NM} and they can be split into 6 subcollections of nonoverlapping intervals each of length less than $1/N$. Thus

$$\sum_{i=1}^{P} |f(y_i) - f(x_i)| \leq \sum_{i=1}^{P}\sum_{j=1}^{8} f(I_{ij}) \leq 6\mathrm{S}_\delta(f, A_M) + P(1/P + 1/P)$$

which supplies the inequality (3) that we wished to prove.

Note that the set complementary to the intervals in Theorem 2 where f has bounded variation may have positive measure. Indeed there is an everywhere differentiable function f whose set of points of non bounded variation has positive measure; [1, p. 73] indicates how to construct such a function.

We turn now to the conditions $\mathrm{VS}_f(E) = 0$ and $\mathrm{VS}_f(E) < +\infty$ for a measurable set E. A preliminary lemma carries most of the information needed to relate the measure VS_f and the differential structure of f. $\underline{\mathrm{SD}}\, f(x)$ and $\overline{\mathrm{SD}}\, f(x)$ denote the upper and lower symmetric derivates of f.

LEMMA 3 *If at each point x of a set E one of the two inequalities*

$$-\alpha > \underline{SD} f(x) \quad \textit{or} \quad \overline{SD} f(x) > \alpha$$

is true then $\mathrm{VS}_f(E) \geq \alpha|E|$.

PROOF. We may assume that E is bounded. Let δ be any positive function on E and let $\mathcal{C}$ denote the class of all intervals $[x, y]$ for which

$$|f(y) - f(x)| > \alpha(y - x), \quad y - x < \delta((x + y)/2), \quad (x + y)/2 \in E.$$

If either $-\alpha > \underline{\mathrm{SD}}\, f(x)$ or $\overline{\mathrm{SD}}\, f(x) > \alpha$ is true at each point $x \in E$ then $\mathcal{C}$ is a Vitali cover of E. For any $\beta < |E|$ choose a nonoverlapping collection $\{[x_i - h_i, x_i + h_i]\} \subset \mathcal{C}$ so that

$$\beta < \sum_{i=1}^{n} 2h_i = \alpha^{-1} \sum_{i=1}^{n} 2\alpha h_i$$

$$\leq \alpha^{-1} \sum_{i=1}^{n} |f(x_i + h_i) - f(x_i - h_i)| \leq \alpha^{-1}\mathrm{S}_\delta(f, E).$$

From this it follows that $\alpha\beta \leq \mathrm{VS}_f(E)$ and finally that $\alpha|E| \leq \mathrm{VS}_f(E)$ as required.

THEOREM 4 *Let f be a measurable function and let E be a measurable set. If* $\mathrm{VS}_f(E) < +\infty$ *then the ordinary derivative $f'(x)$ exists for almost every $x \in E$. If* $\mathrm{VS}_f(E) = 0$ *then $f'(x) = 0$ for almost every $x \in E$.*

PROOF. For any natural number n write

$$A_n = \left\{x \in E;\ \underline{\mathrm{SD}}\, f(x) < -n \ \text{ or } \ \overline{\mathrm{SD}}\, f(x) > n\right\}$$

and

$$B_n = \left\{x \in E;\ -n^{-1} > \underline{\mathrm{SD}}\, f(x) \ \text{ or } \ \overline{\mathrm{SD}}\, f(x) > n^{-1}\right\}.$$

Note that

$$A = \left\{x \in E;\ \underline{\mathrm{SD}}\, f(x) = -\infty \ \text{ or } \ \overline{\mathrm{SD}}\, f(x) = +\infty\right\} = \bigcap_{n=1}^{\infty} A_n$$

and

$$B = \{x \in E;\ \mathrm{SD}\, f(x) \neq 0\} = \bigcup_{n=1}^{\infty} B_n.$$

It follows directly from the lemma that $n|A_n| \leq \mathrm{VS}_f(E)$ and that $|B_n| \leq n\mathrm{VS}_f(E)$. Thus in the case that $\mathrm{VS}_f(E) < +\infty$ the set A has measure zero and it follows that the symmetric derivates of f are finite almost everywhere in E. By a well known theorem of Khintchine [5] $f'(x)$ exists for almost every $x \in E$. In the case that $\mathrm{VS}_f(E) = 0$, B has measure zero and it follows that the symmetric derivative of f vanishes almost everywhere in E and so $f'(x) = 0$ for almost every $x \in E$.

Removing the measurability hypotheses of this theorem would require some much less elementary machinery. The article [8] can be consulted for such ideas.

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