Real Analysis Exchange Vol. 17 (1991-92)

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A Note on the Banach-Steinhaus Theorem[1]

§1 Introduction.

The Baire categories have been used successfully to distinguish between "large" and "small" sets in many theorems of analysis. One such theorem is the Banach-Steinhaus Theorem whose statement is the following:

Suppose a family of continuous linear operators in the Banach space X is not uniformly bounded with respect to norm. Then the set at which this family converges pointwise is of the first category; i.e. it is a countable union of nowhere dense sets.

The purpose of this note is to improve this theorem using the geometric notion of set porosity. In recent years, there has been a great deal interest in the notion of set porosity in metric spaces. (See [1], [2] and [3].) This is a more restrictive notion than "nowhere dense". It is something like "nowhere dense with estimate". (Exact defintions will be given in the next section.) The following natural question arises: In what theorems can one replace the statement that some set is of the first category by the statement that some set is σ-porous? This question is meaningful because L. Zajíček [2] proved that in any Banach space the notions of first category and σ-porosity are distinct.

In this note we shall prove that in two well-known cases the replacement can be made.

§2 Definitions.

A set M in a metric space (X, ρ) is called *porous at* $x \in X$ if there is a positive number α such that for any $\epsilon > 0$ there is a point y in the open ball $B(x, \epsilon)$ with center x and radius ϵ such that $B(y, \alpha\rho(x, y)) \cap M = \emptyset$.

If the above number α can be chosen as close to 1 as we wish, the set M is called *strongly porous at* x.

A set, M, is called *porous* if it is porous at all points of M, and *strongly porous* if it is strongly porous at every point of M.

A countable union of porous sets is called *σ-porous*, etc.

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[1] The author is grateful to the editors for their attention to his paper. The editors hope their efforts are worthy of the autor's gratitude.

§3 The Main Lemma.

LEMMA 1. *Let M be a convex nowhere dense set in a Banach space X. Then M is strongly porous.*

Proof. Fix $x_o \in X$ and $\epsilon > 0$. Take a small positive number δ. Since M is nowhere dense, there exists an open ball T such that $T \subset B(x_o, \epsilon\delta)$ and $T \cap M = \emptyset$. As M is a convex set and T is a convex body which misses M, it follows from the Hahn-Banach theorem that there exists a hyperplane Γ_c which separates M and T. This means that there exists a continuous linear functional $\phi : X \to \mathbf{R}$ such that $\phi|_M > c$, $\phi|_T \leq c$ for some $c \in \mathbf{R}$. Specifically, we take $c = \sup\{\phi(x) : x \in T\}$. In this case we have:

$$\text{dist}(\Gamma_c, \Gamma) \leq \epsilon\delta \tag{1}$$

where $\Gamma = \{x : \phi(x) = \phi(x_o)\}$ is a hyperplane containing x_o and parallel to Γ_c. It is known that if $x_o \in \Gamma$, then for each $r > 0$ there exists a point $y \in B(x_o, r)$ such that

$$\text{dist}(y, \Gamma) > r(1 - \delta) \tag{2}$$

Note that the point y' which is symmetric to y with respect to x_o also satisfies the above inequality. So we can choose $y \in B(x_o, \epsilon)$ such that,

1. $\text{dist}(y, \Gamma_c) > \epsilon(1 - 2\delta)$. (This follows from (1) and (2) by the triangle inequality.), and
2. M and y are on opposite sides of Γ_c (i.e. $\phi(y) < c$).

Therefore, we have

$$B(y, (1 - 2\delta)\rho(x_o, y)) \cap M = \emptyset$$

and the lemma is proved.

REMARK 1. K. Saxe has kindly pointed out that a somewhat more restrictive version of strong porosity is actually proved here. Specifically, for every $0 < \alpha < 1$ and every $x \in M$ there is an $\epsilon(\alpha) > 0$ such that for **every** $0 < \epsilon < \epsilon(\alpha)$ there is a y on the boundary of $B(x, \epsilon)$ such that $B(y, \alpha\epsilon) \cap M = \emptyset$. Other analogous versions of porosity can be defined in the obvious way. Lemma 1 and its corollaries in terms of this stronger version of porosity then remain in force without any changes in the proofs.

§4 Applications.

We recall the following well known theorem.

THEOREM 1. (Banach-Steinhaus) *Let X be a Banach space, let Y be a normed linear space and let Φ be a family of continuous linear operators from X to Y. Suppose that $sup\{\|\phi\| : \phi \in \Phi\} = \infty$. Then the set $E = \{x \in X :$ there is an $N = N(x)$ with $\|\phi(x)\|_Y \leq N$ whenever $\phi \in \Phi\}$ is of the first category.*

PROPOSITION 1. *The above set E is σ-strongly porous.*

Proof. This is obvious because $E = \cup_{N=1}^{\infty} E_N$ where $E_N = \{x \in X : \|\phi(x)\|_Y \leq N \ for\ every\ \phi \in \Phi\}$ is a convex nowhere dense set.

Among numerous corollaries we note the following one.

COROLLARY 1. *The set of functions which have convergent Fourier series at a specified point is σ-strongly porous in the space $C[-\pi, \pi]$.*

Proof. This follows from the above theorem and the well known fact that norms of the linear functionals $f \to S_m(f; t_o)$ (the mth partial sums of the Fourier series of f) are unbounded.

For the second application recall the following theorem of Banach.

THEOREM 2. (Banach) *Let X and Y be Banach spaces and let $A : X \to Y$ be a continuous linear operator. Then either $A(X) = Y$ or $A(X)$ is of the first category in Y.*

First note that $A(X) = \cup_{n=1}^{\infty} A(nT_o)$, where $T_o = B(0,1)$ in Y. It can be proved that either $Q = A(T_o)$ contains some open ball or Q is nowhere dense. But since Q is convex, in the second case Q is strongly porous. As a consequence, Banach's theorem has the following stronger version.

PROPOSITION 2. *Under the hypothesis of Banach's theorem, above, either $A(X) = Y$ or $A(X)$ is σ-strongly porous in Y.*

REFERENCES

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[3] T. Zamfirescu . Porosity in convexity. *Real Analysis Exch.* 15(2) (1989-90) pp.4 24-436

Received February 20, 1991