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The Exact Borel Class Where a Density Completeness Axiom Holds

Richard O'Malley [1] defined the following density property and then showed that the F_σ subsets of $\mathbf{R}$ have this property. We say that a collection, $\mathbf{A}$, of subsets of $\mathbf{R}$ has *the O'Malley density property* if whenever a non-empty bounded set $A \in \mathbf{A}$ has right (left) density 1 at each of it's points, then there is a point in A^c at which A has left (right) density 1. In [1] O'Malley proved the following theorem (restated here using our terminology):

Theorem 1 (O'Malley) . *The F_σ subsets of $\mathbf{R}$ have the O'Malley density property.*

O'Malley established several consequences of this result and asked whether the restriction to F_σ was necessary. This last question was repeated in the form of a query at the *14^{th} Summer Symposium in Real Analysis* held in San Bernardino, June, 1990 where a handsome reward for a resolution was offered ($50 by O'Malley and $10 by one of the organizers; see [2]). The purpose of this paper is to claim the prize!

We begin with what we believe is a new proof of Theorem 1 above. Then we establish a similar and stronger density property for the $\mathbf{G}_\delta$ sets. Namely, if A is a non-empty bounded $\mathbf{G}_\delta$ set which has positive left lower density at each of it's points, then there is a point $x \in A^c$ and a $y > x$ such that A has full measure in (x, y). These ideas are expanded in Section 2 to establish the O'Malley density property for $\mathbf{G}_{\delta\sigma}$ sets.

The last section of the paper is devoted to constructing a non-trivial open set $A \subset (0,1)$ such that for every $x \in [0,1]$ if A has right density 1 at x, then A has left density 1 at x. This shows both that the O'Malley density property does not hold for the $\mathbf{F}_{\sigma\delta}$ sets[1] and that the stronger version for $\mathbf{G}_\delta$ can not be extended to $\mathbf{F}_\sigma$ sets [2] We conclude with a question and offer the generous sum of \$60 for it's resolution. The densities referred to in this question are defined in Section 1 below. This question is:

> *Are there two open disjoint non-empty sets, A and B, whose union has full measure and such that for each $x \in \mathbf{R}$, $d_-(A,x) = d_+(A,x)$ and $d^-(A,x) = d^+(A,x)$?*

1 The $\mathbf{F}_\sigma$ and $\mathbf{G}_\delta$ sets have the O'Malley density property

We begin by proving Theorem 1, but to do so we first need to establish some notation. If E is a measurable subset of $\mathbf{R}$, we define the relative measure of E in the interval I as $\Delta(E,I) = \frac{\mu(E\cap I)}{\mu(I)}$ where μ denotes Lebesgue measure. The *right lower density* of E at x is then $d_+(E,x) = \liminf_{h\to 0}\Delta(E,(x,x+h))$; The *upper density (density) on the right* at x is defined similarly but with limsup (lim) in place of liminf. Densities on the left are defined and denoted in the obvious way.

Theorem 1 (O'Malley) . *The F_σ subsets of $\mathbf{R}$ have the O'Malley density property.*

Proof: Suppose $E \in \mathbf{F}_\sigma$ is bounded and non-empty. Using the Lusin-Menchov Theorem we can write $E = F_1 \cup F_2 \cup \ldots$ where $F_1 \subset F_2 \subset \ldots$ each F_n is closed, $F_1 = \{a_1\}$ is a singleton, and if $x \in F_n$ then $d_+(F_{n+1},x) = 1$. Define

$$R_n(a) = \{x \in F_{n+1} : \ x > a \ and \ if \ y \in (a,x), \ then \ \Delta(E,(y,x)) \geq 1 - \frac{1}{n}\}$$

[1] Adjoin to A all points at which A has right density 1. This does not add any measure to A by Lebesgue's Density Theorem. It is easy to see that the resulting set is $\mathbf{F}_{\sigma\delta}$, has left density one at each of it's points and yet does not have right density 1 at any point of the complement.

[2] Let A_o be the union of all intervals [a,b) in which A has full measure. A_o is $\mathbf{F}_\sigma$ (in fact it is open in the Sorgenfrey topology), has left density 1 at each of it's points, but at no point in A_o^c does A_o have full measure on the right.

It is easy to see that $R_n(a)$ is closed in (a,∞) and that if $\Delta(F_{n+1},(a,x)) > 1-\frac{1}{n}$ then $R_n(a)\cap(a,x)\neq\emptyset$. (For a proof see Lemma 1 below.)

Let $a_1\in F_1$, $a_{n+1}=supR_n(a_n)$, and $a=limit(a_n)$. Note that if $a_n\in F_n$ then a_{n+1} exists and as $R_n(a_n)$ is closed a_{n+1} is in F_{n+1}. If $a_n\leq x<a_{n+1}<a$, then $\Delta(E,(x,a_{n+1}))>1-\frac{1}{n}$ because $a_{n+1}\in R_n(a_n)$. It follows that $\Delta(E,(x,a))>1-\frac{1}{n}$ and hence, $d_-(E,a)=1$.

It remains to show $a\notin E$. Suppose to the contrary, that there is an n such that $a\in F_{n+1}$. If $x\in(a_n,a)$ then, as before, $\Delta(E,(x,a))>1-\frac{1}{n}$ implying that $a\in R_n(a_n)$ and contradicting the choice of a_{n+1}.

Theorem 2 *Suppose $F\in\mathbf{F}_\sigma$ is nonempty, bounded below and $\mu(F\cap(x-h,x))>0$ for every $x\in F$ and every $h>0$. Then there exists a $y\in F^c$ such that $d^+(F,y)=1$.*

Proof: Let $1>\epsilon_1>\epsilon_2>\ldots\to 0$, and write

$$F^c=G_1\cap G_2\cap\ldots \text{ where } \mathrm{G}_1\supseteq\mathrm{G}_2\supseteq\ldots,$$

and each G_n is open. Let (a_1,b_1) be any component of G_1 containing a point $g_1\notin F$. Then, $b_1\in F$ so that $\mu(F\cap(g_1,b_1))>0$. Let d_1 be a density point of F in (g_1,b_1) and let h_1 be such that $\Delta(F,[d_1,d_1+h_1])>1-\epsilon_1$. Let $c_1=inf\{c:[c,d_1]\subset F\}$. Assume $c_1\in F$ since otherwise we are done. Then $c_1>g_1>a_1$ and c_1 is a limit point, from below, of both F and F^c. Choose $n_1>1$ large enough so that a component, (a_2,b_2), of G_{n_1} is contained in $(c_1-h_1\epsilon_1,c_1)\cap(a_1,c_1)$. Now continue inductively with (a_1,b_1) replaced by (a_2,b_2), etc. If $y=\bigcap_{n=1}^{\infty}(a_n,b_n)\in F^c$ then an easy computation shows that $\Delta(F,(y,d_n+h_n))\mapsto 1$. This implies that $d^+(F,y)=1$ and the proof is complete.

The $\mathbf{G}_\delta$ form of this theorem mentioned in the introduction is obtained by interpreting Theorem 2 using the complements of the sets listed in the statement of that theorem. So interpreted, this theorem becomes:

Theorem 3 *Suppose $E\in\mathbf{G}_\delta$ is non-empty, bounded, and $d_+(E,y)>0$ for every $y\in E$. Then there exists a $z\in E^c$ and an $h>0$ such that $\Delta(E\cap[z-h,z])=1$.*

As a corollary we obtain the following theorem.

Theorem 4 *The $\mathbf{G}_\delta$ subsets of $\mathbf{R}$ have the O'Malley density property.*

2 O'Malley Density for $G_{\delta\sigma}$ Sets

Fix a measurable set E. A key tool for our investigation is the following collection of sets:

$$R_n(a) = \{x > a : if\ z \in (a,x)\ then\ \Delta(E,(z,x)) \geq 1 - \frac{1}{n}\}$$

It is easy to see that $R_n(a)$ is closed in any closed interval to the right of a and that if $n > m$ then $R_n(a) \subset R_m(a)$. Note too that if $x \in R_n(a)$ and $(x,y) \subset E$ then $y \in R_n(a)$. We need some slightly deeper properties of these sets for our investigation, however.

Lemma 1 *Suppose I lies to the right of a and is contiguous to $R_n(a)$. Then $\Delta(E^c, I) \geq \frac{1}{n}$.*

Proof: By contiguous we mean that I is a complementary interval with left endpoint of I either equal to a or in $R_n(a)$. Let $I = (b,c)$ and $x \in I$. As $x \notin R_n(a)$ there is a $z \in (a,x)$ such that $\Delta(E,(z,x)) < 1-\frac{1}{n}$. We may assume $z \geq b$ because if $z < b$ then $\Delta(E,(z,b)) \geq 1 - \frac{1}{n}$ so that $\Delta(E,(b,x)) < 1 - \frac{1}{n}$. It is easy to see that $\inf\{z \geq b : \Delta(E,(z,x)) \leq 1 - \frac{1}{n}\} = b$ and as x is arbitrary, the lemma is proved.

Theorem 5 *Let $n_1 > n_2 n_3$ and $\Delta(E,(a,x)) \geq 1 - \frac{1}{n_1}$. Then,*

$$\Delta(R_{n_2}(a),(a,x)) \geq 1 - \frac{1}{n_3}.$$

Proof: Suppose that $\Delta(R_{n_2}(a),(a,x)) < 1 - \frac{1}{n_3}$. It follows from Lemma 1 that in each interval, $I \subset (a,x)$, contiguous to $R_{n_2}(a)$, $\Delta(E^c, I) \geq \frac{1}{n_2}$. Hence,

$$\Delta(E^c,(a,x)) \geq \Delta(R^c_{n_2}(a) \cap E^c,(a,x)) \geq \frac{1}{n_2 n_3},$$

and as $n_1 > n_2 n_3$ this contradicts the fact that $\Delta(E,(a,x)) \geq 1 - \frac{1}{n_1}$.

Corollary 1 *If $d^+(E,a) = 1$ then $d^+(R_n(a),a) = 1$ for each n=1,2,....*

Lemma 2 *Suppose $n > m$. Then for every $y \in R_n(a)$ and for every $z \in [a,y]$, $\Delta(R_m(a),(z,y)) \geq 1 - \frac{m}{n}$.*

Proof: Suppose that there is a $y \in R_n(a)$ and a $z \in [a,y]$ such that $\Delta(R_m(a),(z,y)) < 1 - \frac{m}{n}$. Then there is a set of mutually exclusive left-half open intervals, $\{I_i\}$, in (z,y) such that each $I_i \subset \mathbf{R}\backslash R_m(a)$ and $\Delta(\cup I_i,(z,y)) > \frac{m}{n}$. As in the proof of Lemma 1, this implies that there is a set of mutually exclusive intervals J_j such that $\Delta(E^c,J_j) \geq \frac{1}{m}$ and $\cup I_i \subset \cup J_j$. Hence, $\Delta(E^c,(z,y)) > \frac{1}{m}\frac{m}{n} = \frac{1}{n}$. This contradicts the fact that $y \in R_n(a)$ and completes the proof of the lemma.

Theorem 6 *The $G_{\delta\sigma}$ subsets of* $\mathbf{R}$ *have the O'Malley density property*

Proof: Suppose that E is a nonempty $G_{\delta\sigma}$ set with $d_+(E,a) = 1$ for every $a \in E$. Suppose too that there is an interval where E^c has positive measure to the right of an interval where E has positive measure. If $E = r - int(E) \equiv \{x \in E : \text{for some } \epsilon > 0,\ [x,x+\epsilon) \subset E\}$ then let I be any component of E which is bounded above. The right endpoint, e, of I is in E^c (since $E = r - int(E)$) and is such that $d_-(E,e) = 1$. Hence, we may assume $E\backslash r - int(E) \neq \emptyset$. We also assume that if E has full measure in an interval (a,b), then E contains $(a,b]$ for otherwise we are done. Our aim is to find an increasing sequence $x_0 < x_1 < ...$ of points from E such that for each $z_n \in (x_n,x_{n+1})$, $\Delta(E,(z_n,x_{n+1})) \mapsto 1$ as $n \mapsto \infty$. To insure that the limit, x^*, of this sequence is not in E some care must be taken in defining the x_n. First write:

$$E = \cup_{n=1}^{\infty} E_n \text{ where } E_n = \cap_{k=1}^{\infty} G_{n,k}$$

and each $G_{n,k}$ is open. We also assume that for each n and k, $G_{n,k+1} \subset G_{n,k}$, and $E_n \subseteq E_{n+1}$. Let $x_0 \in E\backslash r - int(E)$. Then there is a first n_0 such that $x_0 \in G_{n_0,k_0}$ for some k_0. Note that it does not necessarily follow that $x_0 \in E_{n_0}$. We associate the pair (n_0,k_0) with x_0. There is an $\epsilon_0 < 1$ such that $[x_0,x_0+\epsilon_0) \subset G_{n_0,k_0}$. If $(x_0,x_0+\epsilon_0) \subset R_{n_0+1}(x_0)$, then it follows from the Lebesgue Density Theorem that E has full measure in $[x_0,x_0+\epsilon_0)$; so by assumption, $[x_0,x_0+\epsilon_0] \subset E$ contradicting the fact that $x_0 \notin r - int(E)$. $e' \in E^c \cap (x_0,x_0+\epsilon_0)$ satisfies the conclusion of the theorem. Hence, we may assume $(x_0,x_0+\epsilon_0) \not\subset R_{n_0+1}(x_0)$. Let $(y_0,y_0+\delta_0)$ be contiguous to $R_{n_0+1}(x_0)$ in $[x_0, x_0+\epsilon_0)$. It follows from Lemma 1 that $\Delta(E^c,(y_0,y_0+\delta_0)) \geq \frac{1}{n_0+1}$. Suppose $y_0 \in E$. Then $d_+(E,y_0) = 1$ so it follows from Corollary 1 that $d_+(R_m(y_0),y_0) = 1$ for every m. But, if $y_0' \in R_{n_0+1}(y_0) \cap (y_0,y_0+\delta_0)$, then $y_0' \in R_{n_0+1}(x_0)$. This, however, contradicts the fact that $(y_0, y_0+\delta_0) \cap R_{n_0+1}(x_0) = \emptyset$. Hence, $y_0 \notin E$.

Let $z_0 = \max\{x_0, y_0 - \delta_0\}$. As $y_0 \in R_{n_0+1}(x_0)$, $\Delta(E, (z_0, y_0)) \geq 1 - \frac{1}{n_0+1}$. It follows from Lemma 2 that $\Delta(R_{n_0-1}(x_0), (z_0, y_0)) \geq \frac{2}{n_0+1}$. Hence, $R_{n_0-1}(x_0) \cap (z_0, y_0)) \cap E \neq \emptyset$. If $R_{n_0-1}(x_0) \cap (z_0, y_0)) \cap E \backslash r-int(E) = \emptyset$, choose $x \in R_{n_0-1}(x_0) \cap (z_0, y_0) \cap E$. Then $x \in r-int(E)$, so that x is in an interval of E whose right endpoint (by assumption) is also in E. But, this endpoint cannot be in $r-int(E)$ and hence, must be greater or equal to y_0 contradicting the fact that $y_0 \notin E$. Hence, $R_{n_0-1}(x_0) \cap (z_0, y_0)) \cap E \backslash r-int(E) \neq \emptyset$. We let x_1 be any element of this set and continue inductively. of that interval is y_0. If y_0 is the right endpoint of an interval from E, then as $y_0 \notin E$, $d_-(E, y_0) = 1$ and y_0 is the point we're looking for. If $R_{n_0-1}(x_0) \cap (z_0, y_0)) \cap E \backslash r-int(E) \neq \emptyset$, we let x_1 be any element of this set and continue inductively.

Continuing the induction, suppose that points $x_1 < x_2 < \ldots x_i < y_i < \ldots < y_2 < y_1$, ordered pairs of integers (n_j, k_j), and positive numbers δ_j have been defined for all $j \leq i$ and that $x_j \in R_{n_{j-1}-1}(x_{j-1})$. We also assume that $(x_j, y_j) \subseteq G_{n_j,k_j}$, $\Delta(E^c, (y_j, y_j+\delta_j)) \geq \frac{1}{n_j+1}$ and $x_{j+1} \in (y_j - \delta_j, y_j) \cap R_{n_j-1}(x_j)$.

Suppose too that $x_{i+1} \in R_{n_i-1}(x_i) \cap E \backslash r-int(E)$ has been defined so that $\max\{x_i, y_i - \delta_i\} < x_{i+1} < y_i$. We define the required quantities as follows.

1. There is a first integer n_{i+1} such that $x_{i+1} \in G_{n_{i+1},k_{i+1}}$ for some $k_{i+1} > \max\{k_j : j \leq i \text{ and } n_j = n_{i+1}\}$.

   Informally, each time we choose a pair n_i, k_i we "eliminate" all $G_{n_i,k}$ for $k \leq k_i$. When it comes time to choose n_{i+1}, k_{i+1}, we pick the first n_{i+1} such that for some k_{i+1}, $G_{n_{i+1},k_{i+1}}$ has not yet been eliminated and contains x_{i+1}.

2. Let $\epsilon_{i+1} < \frac{1}{2^{i+1}}$ be such that $[x_{i+1}, x_{i+1}+\epsilon_{i+1}) \subset G_{n_{i+1},k_{i+1}} \cap [x_{i+1}, y_i)$.

   If $(x_{i+1}, x_{i+1}+\epsilon_{i+1}) \subset R_{n_{i+1}+1}(x_{i+1})$, then it follows from the Lebegue Density Theorem that E has full measure in $(x_{i+1}, x_{i+1}+\epsilon_{i+1})$ and the result follows as above. Hence, we may assume that $(x_{i+1}, x_{i+1}+\epsilon_{i+1}) \not\subset R_{n_{i+1}+1}(x_{i+1})$.

3. Let $(y_{i+1}, y_{i+1}+\delta_{i+1})$ be contiguous to $R_{n_{i+1}+1}(x_{i+1}) \cap [x_{i+1}, x_{i+1}+\epsilon_{i+1})$.

   It follows as above that $y_{i+1} \notin E$. Let $z_{i+1} = \max\{x_{i+1}, y_{i+1} - \delta_{i+1}\}$. As $y_{i+1} \in R_{n_{i+1}+1}(x_{i+1})$, $\Delta(E, (z_{i+1}, y_{i+1})) \geq 1 - \frac{1}{n_{i+1}+1}$. It follows from Lemma 2 that $\Delta(R_{n_{i+1}-1}(x_{i+1}), (z_{i+1}, y_{i+1})) \geq \frac{2}{n_{i+1}+1}$. Hence,

$$R_{n_{i+1}-1}(x_{i+1}) \cap (z_{i+1}, y_{i+1})) \cap E \neq \emptyset.$$

   If

$$R_{n_{i+1}-1}(x_{i+1}) \cap (z_{i+1}, y_{i+1}) \cap E \backslash r - int(E) = \emptyset$$

   then $E \cap (z_{i+1}, y_{i+1})$ contains an interval. The right endpoint of that interval cannot be less than y_{i+1} since it would then belong to

$$R_{n_{i+1}-1}(x_{i+1}) \cap (z_{i+1}, y_{i+1})) \cap E \backslash r - int(E).$$

   The right endpoint of that interval also cannot be y_{i+1} since otherwise, by assumption, $y_{i+1} \in E$. Hence,

$$R_{n_{i+1}-1}(x_{i+1}) \cap (z_{i+1}, y_{i+1}) \cap E \backslash r - int(E) \neq \emptyset.$$

   We let x_{i+2} be any element of this set .

This completes the induction and we let $x^* = limit\ x_i$. The remainder of the proof hinges on the fact that $\{n_i\} \to \infty$. Suppose, to the contrary, that there is an N such that $n_i = N$ for a subsequence n_{i_j} of the n_i's. Then $x^* \in (x_{i_j}, y_{i_j}) \subset G_{N,k_{i_j}}$ for $j = 1, 2, \ldots$, and hence, $x^* \in E_N \subset E$. Thus, $d_+(E, x^*) = 1$. However, by Lemma 1 $\Delta(E^c, (y_{i_j}, y_{i_j} + \delta_{i_j})) \geq \frac{1}{n_{i_j}+1} = \frac{1}{N+1}$ for each $j = 1, 2, \ldots$. As $x^* \in (y_{i_j} - \delta_{i_j}, y_{i_j})$ for each j, it follows that $\Delta(E^c, (x^*, y_{i_j} + \delta_{i_j})) \geq \frac{1}{N+1}$ for each $j = 1, 2, \ldots$. We conclude that $d^+(E, x^*) \leq \frac{1}{2(N+1)}$, but this is a contradiction. If $x^* \in E$, then $x^* \in E_N$ for some N. Since $n_i \mapsto \infty$, only finitely many n_i fail to exceed N. Let $K > max\{k_i : n_i = N\}$. As $x^* \in G_{N,K}$, so is some x_j where $j > max\{i : n_i \leq N\}$. But then by 1, $n_j \leq N$ contradicting the choice of j. Hence, $x^* \in E^c$.

Finally, as $x_{i+1} \in R_{n_i-1}(x_i)$ and as $\{n_i\} \to \infty$, the definition of R_n implies that the left density of E at x^* is 1. This completes the proof of Theorem 6.

3 An Example

The purpose of this section is to prove the following theorem.

Theorem 7 *There exists a proper open subset $A \subset (0,1)$ such that for every $x \in [0,1]$ if A has left density 1 at x, then A has right density 1 at x.*

Proof: Let F denote the Cantor ternary set. For each $x \in F^c$ let $k(x) = \max\{0, 0(\mathrm{x}) - 2(\mathrm{x})\}$ where 0(x) is the number of "0's" in the ternary expansion of x prior to the first "1" and 2(x) is the number of "2's" in the expansion of x prior to the first 1. Let z(x) denote the maximum length of the string of consecutive "0's" immediately following the first "1" in one of the possibly two expansions of x . Finally, set G=$\{x \in F^c : \ z(x) \leq k(x)\}$. Clearly, G is open. Thus, the set G consists of right subintervals of components of F^c. For example, in the interval $(\frac{1}{3}, \frac{2}{3}) \subset F^c$, the $k-$value is zero and G will contain the right subinterval $(\frac{4}{9}, \frac{2}{3})$.

For any x∈(0,1), if the nth digit in the ternary expansion of x is unambiguous, we denote that digit by $(\mathrm{x})_n$. Since k is constant on any component (a,b) of F^c, we say the k-value of the interval is $k(\frac{a+b}{2})$. If $x \in G$ then $d_-(G,x) = d_+(G,x) = 1$. The only other x for which $d_-(G,x) = 1$ are in F. So assume $x \in F$. Then x has a unique ternary expansion consisting of "0's" and "2's". Let $k_n(x) =$ number of $0's$ $-$ number of $2's$ in the first n digits of the expansion of x. The proof is completed by the following two claims.

Claim 1 *If there is an $L > 0$ such that for infinitely many n, $k_n(x) < L$, then $d_-(G,x) \neq 1$.*

Proof: Let n be such that $k_n(x) < L$ and $(x)_n = 2$. There are infinitely many such n. Let $(c)_j = (x)_j$ for $j < n$ and $(c)_j = 1$ for $j \geq n$. Then $k(c) \leq L$ and as c terminates in all 1's, $\in F^c$. If (a,b) is the component of F^c containing c, then $\mu(G^c \cap (a,b)) \geq (\frac{1}{3})^{L+1}(b-a) \geq \frac{1}{2}(\frac{1}{3})^{L+1}(x-a)$. As this happens for c arbitrarily close to x, $d_-(G,x) \leq 1 - \frac{1}{2}(\frac{1}{3})^{L+1}$.

Claim 2 *If $limit_{n\to\infty} k_n(x) = \infty$ then $d_+(G,x) = 1$*

Proof: Let $\epsilon > 0$ and let L be so large that $(1-(\frac{2}{3})^L)^3 > 1-\epsilon$. Suppose that for all $m > b$, $k_m(x) > 3L$. Let $y > x$ be so close to x that x and y first disagree at some decimal place $d > b$. We finish the proof by showing

$\Delta(G,[x,y]) > 1-\epsilon$. Case 1: $(y)_j \neq 1$ for $d \leq j \leq d+L$. Let $a_0 > a_1 > \ldots$ be the numbers obtained by replacing each "0" in a decimal place $\geq d$ (in the expansion of x) with a "1" followed by a tail end of all "0's". Then $a_n \to x$. Let $a_{-1} < \ldots a_{-p}$ be the numbers obtained by taking each "2" in a decimal place $\geq d$ and $\leq d+L$ (in the expansion of y) and following it with a tail end of all "0's". This gives

$$(x,y] = \ldots \cup [a_2,a_1] \cup [a_1,a_0] \cup [a_0,a_{-1}] \cup \ldots \cup [a_{-p+1},a_{-p}] \cup [a_{-p},y]$$

where for each $i > 0$, the left half of $(a_i,a_{i-1}]$ is a component of F^c and for each $i < 0$, the right half of $[a_i,a_{i-1})$ is a component of F^c. and (a_0,a_{-1}) is the largest component of F^c between x and y. The k-value of all of these components exceeds $2L$. Hence, the relative measure of the components of F^c with k-value $> L$ in each $[a_i,a_{i-1}]$ is greater than or equal to $\frac{1}{2}+\frac{1-(\frac{2}{3})^L}{2} > 1-(\frac{2}{3})^L$ which gives

$$\Delta(G,[a_i,a_{i-1}]) > [1-(\frac{1}{3})^L][1-(\frac{2}{3})^L].$$

Also,

$$\frac{y-a_{-p}}{y-x} < \frac{y-a_{-p}}{a_{-1}-a_0} < (\frac{1}{3})^L.$$

since y and a_{-p} agree in the first $d+L$ decimal places. Therefore,

$$\Delta(G,[x,y]) > [1-(\frac{1}{3})^L][1-(\frac{1}{3})^L][1-(\frac{2}{3})^L] > 1-\epsilon.$$

Case 2: $(y)_j = 1$ for some j such that $d \leq j \leq L+d$. In this case, let $\ldots a_2 < a_1 < a_0 < a_{-1} < \ldots\ a_{-m+1} < a_{-m}$ be defined as before except that this time a_{-m} is the left endpoint of the component (a_{-m},b_{-m}) of F^c which contains y. As in Case 1, we will be done if we can show $y-a_{-m} < (\frac{1}{3})^L(y-x)$. Now, assume that y is the left endpoint of a component of G since it is at such points in F^c where $\Delta(G,(x,y))$ is smallest. Then,

$$\begin{aligned} y-a_{-m} &= (\frac{1}{3})^{k(a_{-m})+1}(b_{-m}-a_{-m}) \\ &< (\frac{1}{3})^{2L}(b-a_{-m}) < (\frac{1}{3})^{2L}(a_{-1}-a_0) \\ &< (\frac{1}{3})^L(y-x). \end{aligned}$$

The rest follows as in Case l.

As stated in the introduction, this example provides us with the following two corollaries:

Corollary 2 *The O'Malley density property does not hold for the* $\mathbf{F}_{\sigma\delta}$ *sets.*

Proof: Let $A^* = A \cup \{x :\ d_+(A,x) = 1\}$. Then $\mu(A) = \mu(A^*)$ and as A is open and $\{x :\ d_+(A,x) = 1\} \in \mathbf{F}_{\sigma\delta}$, A^* has left density one at each of it's points and yet does not have right density 1 at any point of the complement.

Corollary 3 *There is an* F_σ *set* A *which has left density 1 at each of its points, but at no point of* A^c *does* A *have full measure on the right.*

Proof: Let A_o be the union of all intervals [a,b) in which A has full measure. A_o is $\mathbf{F}_\sigma$, has left density 1 at each of it's points, but at no point in A_o^c does A_o have full measure on the right.

References

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