Real Analysis Exchange Vol. 17 (1991–92)

Harvey Rosen, Department of Mathematics, University of Alabama, Tuscaloosa, AL 35487.

QUASI-COMPONENTS OF PREIMAGES OF A CONNECTIVITY FUNCTION $I^2 \to I$

Very little is known about what properties are satisfied by the composite of any two connectivity functions. In 1959, Stallings [6] asked under what circumstances will the composition of almost continuous functions be almost continuous? He indicated how to construct an example of connectivity functions $I^2 \to I$ and $I \to I$ whose composite fails to be a connectivity function $I^2 \to I$, where $I = [0,1]$. In 1973, Kellum [3] gave an example of almost continuous functions $f : I^n \to I^m$ and $g : I^m \to I^n$ so that $g \circ f : I^n \to I^n$ has no fixed point and is not almost continuous. When $n = m = 1, f$ and g are then connectivity functions [6].

It is well known that if a function $h : X \to Y$ is the composite of connectivity functions $f : X \to Y$ and $g : Y \to Y$, then h must be a Darboux function. In this note we give a simple example of an almost continuous Darboux function $h : I^2 \to I$ for which the converse is false. To verify this example, we rely on either of two useful results about quasi-components. Kellum's question about whether the converse is true when $X = Y = I$ is still unanswered. We also give a sufficient condition on quasi-components in order for a function $I^2 \to I$ to be Darboux.

A function $f : X \to Y$ is defined to be a <u>Darboux</u> (<u>connectivity</u>) function if $f(C)$ (the graph of the restriction $f|C$) is connected for every connected subset C of X. We say $f : X \to Y$ is <u>almost continuous</u> if each open neighborhood of the graph of f in $X \times Y$ contains the graph of a continuous function from X into Y. A function $f : X \to Y$ is <u>peripherally continuous</u> if for each $x \in X$ and each open neighborhood U of x and V of $f(x)$, there is an open neighborhood W of x in U such that $f(bd(W)) \subset V$. According to [6], if $X = I^n$ and $n \geq 2$, then W and $bd(W)$ can be chosen to be connected. For functions $f : I^n \to I^m$, $n \geq 2$, we have: peripheral continuity $\Leftrightarrow$ connectivity $\Rightarrow$ almost continuity [2], [7], [6].

A set $A \subset X$ has <u>external dimension</u> 0 if for every $p \in X - A$, each open neighborhood of p contains an open set about p whose boundary misses A. If $Q \subset B \subset X$, we say Q is a <u>quasi-component</u> of B provided Q is a maximal set which lies in one of two separated sets D or E whenever $B = D \cup E$.

The following result comes out of Whyburn's work [8].

Theorem 1. *If $f : I^n \to Y$ is a peripherally continuous function and A is a closed subset of Y, then each quasi-component of $f^{-1}(A)$ is a subcontinuum of I^n.*

Proof. By Theorem 3.1 of [8], $f^{-1}(A)$ has external dimension 0. According to Corollary 2.12 of [8], the quasi-components and the components of $f^{-1}(A)$ are the same. By Theorem 1 of [4], each component of $f^{-1}(A)$ is closed in I^n and therefore compact.

The next theorem is a consequence of the following result in [5]: If $f : I^2 \to I$ is a connectivity function and z is an interior point of $f(I^2)$, then any point of $f^{-1}([0, z))$ and any point of $f^{-1}((z, 1])$ lie in different quasi-components of $I^2 - f^{-1}(z)$. An example is given in [5] to show that the conclusion of this result is false for Darboux functions.

Theorem 2. *If $f : I^2 \to I$ is a connectivity function and $g : I \to I$, then for every subset C of I and for every quasi-component Q of $(g \circ f)^{-1}(C), f(Q)$ lies in a component of $g^{-1}(C)$.*

Proof. If $f(Q)$ does not lie in a component of $g^{-1}(C)$, there exist points $a, b \in Q$ and $z \notin g^{-1}(C)$ such that $f(a) < z < f(b)$. Since f is Darboux and I^2 is connected, $f(I^2)$ is a subinterval of I. Therefore z is an interior point of $f(I^2)$ because $z \in (f(a), f(b)) \subset f(I^2)$. According to [5], a and b lie in different quasi-components Q_1 and Q_2 of $I^2 - f^{-1}(z)$. $Q \subset (g \circ f)^{-1}(C) \subset I^2 - f^{-1}(z)$. There are separated sets D and E whose union is $I^2 - f^{-1}(z)$ such that $Q_1 \subset D$ and $Q_2 \subset E$. Therefore Q cannot be a quasi-component of $(g \circ f)^{-1}(C)$ because $D \cap (g \circ f)^{-1}(C)$ and $E \cap (g \circ f)^{-1}(C)$ are separated sets neither of which contains Q but whose union is $(g \circ f)^{-1}(C)$.

Example. Define the almost continuous function

$$u(x) = \begin{cases} \dfrac{1 + \sin \frac{1}{x}}{2} & \text{if } \ x \in (0, 1] \\ \dfrac{1}{2} & \text{if } \ x = 0 \end{cases},$$

and define the almost continuous Darboux function

$$h(x, y) = \begin{cases} u(x) & \text{if } \ (x, y) \in (0, 1] \times I \\ u(y) & \text{if } \ (x, y) \in \{0\} \times I \end{cases}.$$

Then $h : I^2 \to I$ is not the composite of any two connectivity functions $f : I^2 \to I$ and $g : I \to I$.

Proof. Assume there exist two such connectivity functions f and g for which $h = g \circ f$. Let $C = \{0,1\}, G = u^{-1}(C)$, and $Q = \{0\} \times G$. Then $G = \{\frac{2}{(2n+1)\pi} : n = 0,1,2,\ldots\}$ and $(g \circ f)^{-1}(C) = Q \cup (G \times I)$. Q is a quasi-component of $(g \circ f)^{-1}(C)$. According to Theorem 2, $f(Q)$ lies in a component K of $g^{-1}(C)$. $C = h(Q) = g(f(Q)) \subset g(K) \subset C$. Therefore $g(K) = C$, in contradiction to the fact that g is a Darboux function.

An obvious consequence of this example is that h is not a connectivity function. This also follows directly from the fact that the set $M = \{(0,0)\} \cup (I \times \{1\}) \cup (G \times I)$ is connected, but the graph of $h|M$ is not connected.

Another proof for the above example can be given which does not use Theorem 2. Instead, Theorem 1 can be applied to see that the set $A = g^{-1}(C) \cap f(I^2)$ is not closed in I because the quasi-component Q of $f^{-1}(A)$ is not a subcontinuum of I^2. Let $p \in \bar{A} - A$. Since A has external dimension 0 [8], there is in the real line a sequence of nested intervals (a_n, b_n) about p such that for each $n, |a_n - b_n| < \frac{1}{n}, a_n, b_n \notin A$, and (a_n, b_n) contains a point $p_n \in A - (a_{n+1}, b_{n+1})$. For each n, there is a point x_n for which $f(x_n) = p_n$. Let $q \in Q$ and choose n_0 so that $f(q) \notin [a_{n_0}, b_{n_0}]$. Since f is peripherally continuous, there exists an open neighborhood W of q with connected boundary such that $f(bd(W))$ misses $[a_{n_0}, b_{n_0}]$. There is a nonnegative integer m such that for all integers $k \geq m, bd(W)$ meets $L_k = \left\{\frac{2}{(2k+1)\pi}\right\} \times I$, and therefore the connected subset $f(L_k)$ of A misses $[a_{n_0}, b_{n_0}]$. If $N = bd(W) \cup (\bigcup_{k=m}^{\infty} L_k)$, then $\overline{f(N)}$ misses (a_{n_0}, b_{n_0}) and so $p \notin \overline{f(N)}$. By Theorem 2 of [4], $f(\bar{N}) \subset \overline{f(N)}$ because f is peripherally continuous and N is connected. If $x_i, x_j \in L_k$ and $k \geq 0$, then both p_i and p_j belong to the connected subset $f(L_k)$ of $A \cap (a_i, b_i) \cap (a_j, b_j)$, which implies $p_i = p_j$ and $x_i = x_j$. It follows that for n large enough, $x_n \in \overline{\bigcup_{k=m}^{\infty} L_k} \subset \bar{N}$ and so $p_n = f(x_n) \in f(\bar{N}) \subset \overline{f(N)}$. Since p_n converges to p, then $p \in \overline{f(N)}$, a contradiction.

We end with a sufficient condition for a function to be Darboux.

Theorem 3. *Suppose $f : I^2 \to I$. If for every point z of $f(I^2)$, any point of $f^{-1}([0,z))$ and any point of $f^{-1}((z,1])$ lie in different quasi-components of $I^2 - f^{-1}(z)$, then f is a Darboux function.*

Proof. Suppose K is a connected subset of I^2 for which $f(K)$ is not connected. There exist points $a, b \in K$ and $z \notin f(K)$ such that $f(a) < z < f(b)$. Since K is a connected subset of $I^2 - f^{-1}(z), K$ is a subset of some quasi-component Q of $I^2 - f^{-1}(z)$. Then Q contains the point a of $f^{-1}([0,z))$ and the point b of $f^{-1}((z,1])$.

References

[1] Bert D. Garrett and Kenneth R. Kellum, *Almost continuous real functions*, Proc. Amer. Math. Soc. 33 (1972), pp. 181-184.

[2] Melvin R. Hagan, *Equivalence of connectivity maps and peripherally continuous transformations*, Proc. Amer. Math. Soc. 17 (1966), pp. 175-177.

[3] Kenneth R. Kellum, *Almost continuous functions on I^n*, Fund. Math. 79 (1973), pp. 213-215.

[4] Paul E. Long, *Properties of certain non-continuous transformations*, Duke Math. J. 28 (1961), pp. 639-645.

[5] Harvey Rosen, Richard G. Gibson, and Fred Roush, *Extendable functions and almost continuous functions with a perfect road*, Real Analysis Exchange, to appear.

[6] John Stallings, *Fixed point theorems for connectivity maps*, Fund. Math. 47 (1959), pp. 249-263.

[7] Gordon T. Whyburn, *Connectivity of peripherally continuous functions*, Proc. Nat. Acad. Sci. U.S.A. 55 (1966), pp. 1040-1041.

[8] ____________, *Loosely closed sets and partially continuous functions*, Mich. Math. J. 14 (1967), pp. 193-205.

Received February 4, 1991