Joanna Czarnowska, Institute of Mathematics, University of Gdańsk, ul. Wita Stwosza 57, 80-952 Gdańsk, Poland

On the Intermediate Value Property of Multivalued Maps

We denote by $P(R)$ and $K(R)$ the families of all non empty, and non empty and compact subsets of R respectively. For a multivalued map $F : I \to P(R)$ and any sets $A \subset I$ and $B \subset R$ we denote:

$$F(A) = \bigcup\{F(x) : x \in A\}$$

$$F^+(B) = \{x \in I : F(x) \subset B\}$$

$$F^-(B) = \{x \in I : F(x) \cap B \neq \emptyset\}.$$

A multivalued map $F : I \to P(R)$ is lower (upper) semi continuous if for any open set $V \subset R$, the set $F^-(V)$ ($F^+(V)$) is open (in I). A multivalued map F is continuous if it is both lower and upper semi continuous. A multivalued map $F : I \to P(R)$ is lower (upper) first class if for every open set $V \subset R$, the set $F^+(V)$ ($F^-(V)$) is an F_σ–set.

In paper [2] the following definition of the intermediate value property of multivalued map was given

Definition 1 A multivalued map $F : I \to P(R)$ has the intermediate value property on I if for any distinct points $x_1, x_2 \in I$ and every $y_1 \in F(x_1)$, there exists $y_2 \in F(x_2)$ such that $(y_1, y_2) \subset F((x_1, x_2))$.

If we suppose that $F : I \to P_v(R)$ then the definition given above is equivalent to that given by J. Ceder in paper [3].

The following theorem gives a characterization of compositions of continuous functions and multivalued maps.

Theorem 2 *Suppose $f : R \to R$ is continuous and onto. Then a multivalued map $G : R \to P(R)$ has the intermediate value property on R if and only if the composition $G \circ f$ has the intermediate value property on R.*

The following example shows that in Theorem 2 the function f cannot be replaced by a multivalued map F.

Example 3 Let us define $F(x) = [-x, x]$ and

$$G(x) = \begin{cases} [0,2], & x \leq 0 \\ [0,1], & x > 0 \end{cases}$$

Then F is continuous and $G \circ F(x) \equiv [0,2]$ has the intermediate value property on R but G has not.

The following theorem generalizes a similar theorem on single–valued real functions given by ZAHORSKI (see [1]).

Theorem 4 *Suppose $F : [a,b] \to K(R)$ is lower and upper first class and for any closed interval P, the set $F^{-}(P)$ has compact components. Then the following conditions are equivalent*

(1) *F has the intermediate value property on $[a,b]$*

(2) *for any open set $V \subset R$ the counter image $F^{-}(V)$ is bilaterally c-dense-in-itself*

(3) *for any open set $V \subset R$ the counter image $F^{-}(V)$ is bilaterally dense-in-itself.*

The assumption about $F^{-}(P)$ is not necessary to prove the implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3), but it is necessary to prove the implication (3) $\Rightarrow$ (1).

REFERENCES

[1] A. M. BRUCKNER, *Differentiation of real functions*, Springer–Verlag, Berlin–Heidelberg–New York 1978.

[2] J. CZARNOWSKA, G. KWIECIŃSKA, *On the Darboux property of multivalued function*, Demonstr. Math., to appear.

[3] J. CEDER, *Characterizations of Darboux selections*, Rendiconti del Circolo Matematico di Palermo, Serie II, Tomo XXX(1981).