Janina Ewert, Department of Mathematics, Pedagogical University, Arciszewsiego 22, 76–200 Słupsk, Poland

The Topology of Semilocal Connectedness and s–Continuity of Maps on Product Spaces

A function $f : X \mapsto Y$ is called s–continuous if for each open set $V \subset Y$ whose complement is connected, the set $f^{-1}(V)$ is open [1]. We begin by recalling two known results.

Theorem 1 *[1, Th. 2.7] If f is a function from a connected space X into a topological space Y such that the graph function $\phi_f : X \mapsto X \times Y$, $\phi_f(x) = (x, f(x))$ is s–continuous, then f is s–continuous.*

Theorem 2 *[2, Th. 2.2] Let (Y_j, T_j), $j \in J$, be connected spaces and let f_j be a function from a topological space X_j into Y_j. If the product function $f : \prod_{j\in J} X_j \mapsto \prod_{j \in J} Y_j$, where $f(\{x_j\}_{j\in J}) = \{f(x_j)\}_{j\in J}$, is s–continuous, then each f_j is s–continuous.*

In [1] Kohli formulated the question of whether the converse of T heorem 1 is true; further, in [2] no comments are made concerning the converse of Theorem 2. These stand as motivation for our present considerations.

For a topological space (Y, T) we let T^* denote the topology of semilocal connectedness which is given by the subbase $\{V \in T : Y \setminus V \text{ is connected}\}$. Then a function $f : x \mapsto (Y, T)$ is s–continuous iff $f : X \mapsto (Y, T^*)$ is continuous [3, Prop.9]. This means that s–continuity of product functions is closely related to relations between suitable topologies of semilocal connectedness on product spaces. For topological spaces (Y_1, T_1), (Y_2, T_2) we have topologies $T_1^* \times T_2^*$ and $(T_1 \times T_2)^*$ on $Y_1 \times Y_2$; furthermore, as shown in the next example, these topologies are independent.

Example 1 *Let $(\mathbb{R}, T)$ be the set of real numbers with the natural topology, P the ideal of Lebesgue measure zero sets and let $Y = \bigcup_{n=0}^{\infty}[2n, 2n+1]$. By T_2 we denote the natural topology on Y induced from the real line and $T_1 = \{U \setminus H : U \in T, H \in P\}$; then $T^* = T$. Let $\{w_n : n \geq 1\}$ be the set of all rational numbers from the interval $[0,1]$ and let $B = (\bigcup_{n=1}^{\infty}\{w_n\} \times [0,1)) \cup ([0,1] \times \{1\})$. We put $U = (\mathbb{R} \times Y) \setminus B$ and $V = (\mathbb{R} \times Y) \setminus \bigcup_{n=0}^{\infty}[2n, 2n+1] \times [2n, 2n+1]$. Then we have $U \in (T_1 \times T_2)^*$, $U \notin T_1^* \times T_2^*$, $V \in T_1^* \times T_2^*$, and $V \notin (T_1 \times T_2)^*$.*

Theorem 3 *We have the following:*

1. *If* (Y_i, T_i), $i \leq n$, *are topological spaces such that each of the sets* $Y_1, \ldots, Y_n$ *has a finite number of components, then*

$$T_1^* \times T_2^* \times \ldots \times T_n^* \subset (T_1 \times T_2 \times \ldots \times T_n)^*.$$

2. *If* (Y_j, T_j), $j \in J$, *is a family of connected spaces, then*

$$\prod_{j\in J} T_j^* \subset \left(\prod_{j\in J} T_j\right)^* .$$

Part 2. of the above theorem implies Theorem 2; furthermore, we have:

Corollary 1 *Let* (Y_i, T_i), $i \leq n$, *be topological spaces such that each of* $Y_1, \ldots, Y_n$ *has a finite number of components. If* $f_i : X_i \mapsto Y_i$ *are functions such that* $f_1 \times \ldots \times f_n$ *is s–continuous, the the* f_i *are s–continuous.*

Finally, we remark that the converses to Theorem 2, Corollary 1 and Theorem 1 are not true. For instance, let $\mathbb{R}$, T, T_1 be as in Example 1 and let $f : (\mathbb{R}, T) \mapsto (\mathbb{R}, T_1)$ be the identity map. Then f is s–continuous, but neither $f \times f$ nor ϕ_f is s–continuous.

References

[1] J. K. Kohli, A class of mappings containing all continuous and all semi-connected mappings, *Proc. Amer. Math. Soc.* 72 (1978), 175–181.

[2] J. K. Kohli, S–continuous functions and weak forms of regularity and complete regularity, *Math. Nachr.* 97 (1980), 189–196.

[3] I. L. Reilly and M. K. Vamanamurthy, On the topology of semilocal connectedness, *Math. Nachr.* 129 (1986), 109–113.