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THE Equation $f^2+g^2=h^2$, where f, g, and h Are Derivatives

It is easy to show that the sum of squares of two derivatives is not always the square of a derivative. (Take, e.g., $f(x)=\sin\frac{1}{x}$, $g(x)=\cos\frac{1}{x}(x\neq 0)$, $f(0)=g(0)=0$.) To investigate our equation we introduce the following notation: $I=[0,1]$; D is the class of all derivatives on I; $C[C_{ap}]$ is the class of all continuous [approximately continuous] functions on I; bC_{ap} is the class of all bounded elements of C_{ap}; $M=\{f\in D;\ fg\in D$ for each $g\in bC_{ap}\}$. It can be proved that $M\cap C_{ap}$ is the class of all Lebesgue functions and that each bounded derivative is in M.

It is easy to see that $\sqrt{f^2+g^2}\in D$, if $f,g\in D$ and $g/f\in C$. This simple result leads to the question whether the relation

$$f^2+g^2=h^2,\quad f,g,h\in D \tag{1}$$

implies something about g/f, if $f\neq 0$. The following theorem points in this direction:

Theorem 1. *Let (1) hold and let*

$$\liminf \operatorname{ap} h(y)>0\ \ (y\to x, y\in I)\ \ \textit{for each}\ \ x\in I. \tag{2}$$

Then $f/h,\ g/h\in C_{ap}$.

(This follows from [1], Proposition 4.6 with $m=2$ and $|(x,y)|=\sqrt{x^2+y^2}$.) If, moreover, $f\neq 0$, then, clearly, $g/f\in C_{ap}$. Now it is natural to ask whether the relations $f,g\in D$ and $g/f\in C_{ap}$ imply that $\sqrt{f^2+g^2}\in D$. The next theorem gives a negative answer to this question.

Theorem 2. *Let $f\in D\setminus M$, $f>0$. Let $\varepsilon\in(0,1)$. Then there is a $\beta\in C_{ap}$ such that $|\beta-1|<\varepsilon$, $g=\beta f\in D$ and $\sqrt{f^2+g^2}\notin D$.*

We get, however, an h fulfilling (1) if we impose some restrictions on f and g; at the same time the requirement $g/f\in C_{ap}$ can be weakened, as Theorems 3 and 4 show.

Theorem 3. *Let* $f, g \in M$; *let* $\alpha, \beta \in C_{ap}$, $\alpha^2 + \beta^2 > 0$; *let* ψ *be a function such that* $f = \alpha\psi$, $g = \beta\psi$. *Set* $\gamma = \sqrt{a^2 + \beta^2}$, $h = \frac{\alpha}{\gamma} f + \frac{\beta}{\gamma} g$. *Then (1) holds.*

(The proof is easy.)

Theorem 4. *Let* $f \in M$, $g \in D$, $f^2 + g^2 > 0$; *let* $\alpha, \beta \in C_{ap}$ *and let* ψ *be a function such that* $f = \alpha\psi$, $g = \beta\psi$. *Suppose that there is an* $A \in (-\infty, 0)$ *such that* $g \geqq A|f|$. *Then* $\sqrt{f^2 + g^2} \in D$.

Example 5.12 in [1] shows that in Theorem 1 we cannot replace the requirement (2) by $h > 0$. However, we have Theorem 5 that points in the same direction as Theorem 1:

Theorem 5. *Let* $f \in M$, $f > 0$ *and let (1) hold. Then* $g, h \in M$.

A characterization of M and the proofs of Theorems 2, 4, and 5 will be published later.

Reference

[1] Jan Mařík and Clifford E. Weil, *Sums of powers of derivatives*, Proc. Amer. Math. Soc. 112 (1991), 807–817