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Integration of a Functional

Let $\tau > 0$ be a real number and let $T = \mathbb{R}^{(0,\tau)}$. Hence, $T = \{x : t \mapsto x(t), t \in (0,\tau), x(t) \in \mathbb{R}\}$, where $x(0)$ and $x(\tau)$ are fixed real numbers. Let $N = \{t_1, \ldots, t_{n-1}\}$, $x_j = x(t_j)$, $x(N) = (x_1, \ldots, x_{n-1})$, $T(N) = \{x(N)\} = \mathbb{R}^{n-1}$, $I = I[N] = \{x : x(t_j) \in I_j, 1 \le j \le n-1\}$, and $I(N) = I_1 \times \ldots \times I_{n-1}$, where each I_j is an interval $[u_j, v_j)$ of $\mathbb{R}$.

$(I[N], x)$ is an *associated interval-point pair* in T if $x_j = u_j$ or $x_j = v_j$ for each j. A finite collection $\mathcal{E}$ of associated interval point pairs is a *division* of T if the $I[N]$ are disjoint and exhaust T.

For each x, let $L(x) \subset (0,\tau)$ be finite; for each $N \supseteq L(x)$, let $\delta(x(N)) > 0$ be defined for $x(N) \in T(N) = \mathbb{R}^{n-1}$, so $\delta(x(N))$ is a gauge in the sense of generalised Riemann integration in $\mathbb{R}^{n-1}$. A gauge γ in T is defined as

$$\gamma = \{(L(x), \delta(x(N)) : x \in T\}.$$

Then $(I[N], x)$ is γ-fine in T if $(I(N), x(N))$ is δ-fine in $\mathbb{R}^{n-1}$, and $\mathcal{E}$ is γ-fine if each $(I[N], x)$ of $\mathcal{E}$ is γ-fine.

Given a functional $h(I, x, N)$ of associated $(I[N], x)$, we define the integral of h in T to be α if, given $\epsilon > 0$, there exists γ such that

$$|(\mathcal{E}) \sum h(I, x, N) - \alpha| < \epsilon|$$

for every γ-fine division $\mathcal{E}$ of T.

In Feynman integration, the following functional occurs. Let $\lambda = \mu + \iota\nu$ be a complex number and let $U(\cdot)$ be a real-valued function of a real variable. Let

$$u(x, N) = \exp(-\lambda \sum_{j=1}^{n} U(x_j)(t_j - t_{j-1}))$$

and, if U is continuous,

$$u(x) = \exp(-\lambda \int_0^\tau U(x(t))dt), \ x \text{ continuous,}$$

$$= 0 \text{ otherwise.}$$

Let

$$w_\lambda(I,N) = \int_{I(N)} \exp(\lambda \sum_{j=1}^{n} \frac{(x_j - x_{j-1})^2}{t_j - t_{j-1}}) \prod_{j=1}^{n} (-\frac{\pi}{\lambda}(t_j - t_{j-1}))^{-1/2} dx(N)$$

We are interested in the existence of $\int_T u(x,N) w_\lambda(I,N)$ and $\int_T u(x) w_\lambda(I,N)$, which we write as $\int_T u(x,N) dw_\lambda$ and $\int_T u(x) dw_\lambda$, respectively.

Let

$$\tau_j = \frac{j\tau}{2^m},\ 1 \le j \le 2^m - 1,\ y_j = x(\tau_j),\ M = \{\tau_1, \dots, \tau_{m-1}\},$$

$$y = (y_1, \dots, y_{m-1}) \in I\!R^{m-1}.$$

Then M is fixed (unlike N), $u(x,M)$ is called a *cylinder functional*, and we have

$$\int_T u(x,M) dw_\lambda =$$

$$= \int_{I\!R^{m-1}} u(y_1, \dots, y_{m-1}) \exp(\lambda \sum_{j=1}^{n} \frac{(y_j - y_{j-1})^2}{\tau_j - \tau_{j-1}}) \prod_{j=1}^{n} (-\frac{\pi}{\lambda}(t_j - t_{j-1}))^{-1/2} dy$$

provided

- $\mu \le 0,\ \nu \ge 0,\ \mu, \nu$ not both zero,
- U is continuous (except, perhaps, for a null set of reals), and
- the finite-dimensional integral exists.

If, in addition,

- the sequence of finite dimensional integrals converges to α as $m \to \infty$,

then

$$\int_T u(x,N) dw_\lambda,\ \int_T u(x) dw_\lambda \text{ both exist and equal } \alpha.$$

The proof uses Henstock's criteria for limits under the integral sign ([2], 120-125).

References

1. Bullen et al. (eds.), *New Integrals*, Springer, 1990.
2. Henstock, *The General Theory of Integration*, Oxford, 1991.
3. Muldowney, *A General Theory of Integration in Function Spaces*, Longman, 1987.