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A THREE–DART RESPONSE TO AN ARGUMENT OF BAGEMIHL

In his recent article "Throwing a dart at Freiling's argument against the continuum hypothesis," F. Bagemihl claims to find a flaw in dart–throwing arguments appearing in [3][1]. The center of the disagreement seems to be around which sets of reals are "rare" (meaning in some sense having an infinitely small chance of being hit by a random dart). In [3], Freiling claims that every set of Lebesgue measure zero is "rare," and possibly many more sets as well. Bagemihl, on the other hand, admits that the set of rationals is rare but claims that this depends upon certain structural properties of the rationals and is not applicable to countable sets in general.

If one accepts the (at least mathematical) possibility of choosing a real number at random,[2] then for each class "Rare" of rare sets, we may consider the following axiom:

$$A_{\text{Rare}}: \quad (\forall f\colon \mathbb{R} \to \mathbb{R}_{\text{Rare}})(\exists\, x_1,x_2)\; x_1 \notin f(x_2)\ \&\ x_2 \notin f(x_1),$$

where $\mathbb{R}_{\text{Rare}}$ denotes the "Rare" sets of reals. Here is a justification for such an axiom: Suppose we have a fixed mapping which assigns a set in "Rare" to each real. Then when two darts are randomly thrown, the second dart will predictably miss the "Rare" set

[1]The essential mathematical content of some of the arguments of [3] appears in Sierpiński [6] and [8] and Kuratowski [4]. Although they never promoted their ideas as axioms, they probably understood the philosophical import of their ideas in much the way that Freiling [3] construes it. For more on the fascinating history of Sierpiński's Theorem (the progenitor of these results) see [9].

[2]By the reals we mean the interval [0,1]. It is important to have a bounded target for our darts since we are assuming a uniform distribution. Also, choosing reals at random might even be a physical reality. For example, consider a certain type of radioactive particle and for each real t, let p_t be the natural probability that a given particle of that type will decide to decay within an amount of time t. Now choose such a particle and let t_0 be the amount of time it takes that particle to decide to decay. Then, as far as we know, p_{t_0} is a random number from [0,1].

assigned to the first dart, and by symmetry (the two darts are random with respect to each other), the first dart will also miss the "Rare" set assigned to the second. Thus, once the mapping is given, a pair x_1, x_2 may be found by independently throwing two random darts. The axiom asserts that an outcome which is almost certain to happen should at least be possible.

Freiling, therefore, considers axioms such as $A_{\text{countable}}$ which turns out to be equivalent (in ZFC) to the negation of the continuum hypothesis. This is what Bagemihl calls a "startling" conclusion. Now, the approach one would expect to be taken to refute these axioms would be to find similar axioms based upon similar intuition which turn out to be false. However, this is not what Bagemihl does. Bagemihl finds a similar axiom which turns out to be true! How does this approach work? Here is Bagemihl's argument as we understand it. Consider the axiom $A_{\text{n.d.}}$, which would be justified if all nowhere dense sets are rare. (Now, we don't think anyone really believes that all nowhere dense sets are rare; in fact their probabilities may be arbitrarily close to 1). Nevertheless, the nowhere dense sets do include many countable ones. Thus, for example, if one believes in the "rareness" of arbitrary countable sets, then one should believe $A_{\text{n.d. \& countable}}$. Furthermore, if all countable sets were equally rare then $A_{\text{n.d. \& countable}}$ should have the same startling consequence as $A_{\text{countable}}$. But it doesn't! Since $A_{\text{n.d.}}$ is true and in fact provable (Erdös [2]) it has no startling consequences at all and therefore neither does $A_{\text{n.d. \& countable}}$. It must be, therefore, that not all countable sets are equally rare, which is the flaw in Freiling's argument.

Since we do not claim to fully understand this approach[3], one should consult [1] for more

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[3]It seems to us to say more about the ability of the axioms to discover truth in a more structured environment than it does about differences in "rareness" of countable sets. After all, there are fewer n.d. countable sets than countable sets, so $A_{\text{n.d. \& countable}}$ should not be any stronger that $A_{\text{countable}}$. And indeed, as Bagemihl has noted, it isn't. In fact, one could take the opposite point of view and consider the proof of $A_{\text{n.d.}}$ to be evidence for the truth of $A_{\text{countable}}$.

details. Nevertheless, here is a response.

Since there seems to be some concern about the "rareness" of arbitrary countable sets, let's consider only the "rarest of the rare," the finite sets[4]. Now suppose we have a mapping which assigns to each real a finite set. Then when two random darts are thrown, each will not be in the set assigned to the other. We should all be willing to accept A_{finite}. In fact A_{finite} is provably true.[5]

Now let's consider the same experiment where three darts are thrown and where the mapping assigns a finite set to each pair of reals. Then when the third dart is thrown, there is a finite set determined by the first pair of darts, and the third dart will predictably miss that set. By symmetry (the real number line cannot really tell which dart is the third one) each of the darts will predictably miss the finite set assigned to the the other two. Thus we should believe:

$$A^3_{\text{finite}}\text{: } (\forall f{:}\mathbb{R}_2 \to \mathbb{R}_{\text{finite}})(\exists x_1,x_2,x_3)\ x_1 \notin f(x_2,x_3),\ x_2 \notin f(x_1,x_3),\ \&\ x_3 \notin f(x_1,x_2).$$

However, A^3_{finite} is also equivalent to the negation of CH [3],[8].[6] In other words, if ZFC+CH is true, then we have an experiment with three outcomes, one of which is forced to happen whenever the experiment is performed, and yet each is infinitely unlikely to occur. While this does not contradict formal logic (most likely that's impossible!) it is *reductio ad absurdum* nonetheless.

So why isn't the world convinced? The most common response is that similar arguments could be given which contradict the axiom of choice. Perhaps the most convincing of these is the following:

Suppose we have a mapping which assigns a single real number to each real. Then, as before, if two darts are thrown, the second will predictably miss the real assigned to the

[4]In fact, if one believes there is even one "rare" infinite set A, then every finite set must also be "rare," since any finite set is homeomorphic to a subset of A.

[5]More evidence we are on the right track! Actually, A_{finite} is the same as saying $\mathbb{R}$ is uncountable.

[6]So there are startling consequences from n.d., but one has to throw three darts to see it.

first. Since it doesn't matter which dart was thrown first, we may as well consider them to be thrown at the same time. This is equivalent to choosing a point in the plane randomly (which is perhaps an even more natural setting for a dart experiment). Put in a planar sense, we see that any set with one point on each vertical line is "rare". Now consider a set in the unit disk of cardinality less than c. Under some rotation of the disk, this set has at most one point on each vertical line — this fact is easy to see and is also due to Sierpiński [7]. Thus any set of size less that c is just the rotation of a "rare" set and hence is also "rare". But this justifies the following axiom:

$$A_{<c}\colon\ (\forall f{:}D \to D_{<c})(\exists x_1,x_2)\ x_1 \notin f(x_2)\ \&\ x_2 \notin f(x_1),$$

which easily implies that there is no well ordering of D.

Arguments such as this cause us to lose confidence in the well–ordering principle. Others may be compelled to reject all dart–throwing arguments. There is also a middle ground: one could keep ZFC and accept as much of the dart–throwing axioms as is consistent with this, concluding that ZFC+¬CH is correct. This is analogous to the way large cardinal axioms are handled[7]. In any case, one who rejects any of these arguments should (as Bagemihl does) consider the question, "Wherein lies the fallacy?"

Bibliography

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[7]See [5]. In large cardinals, there can be no set of everything but there should be sets as large as possible. In the dart axioms, although a finite set cannot be impossible to hit, it should come as close to this as it can. In both cases the limits are tightened when one accepts the axiom of choice.

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