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A historical note on the measurability properties of symmetrically continuous and symmetrically differentiable functions

H. Steinhaus asked in 1923 [12] whether there exists a function $f : \mathbf{R} \to \mathbf{R}$ such that f is discontinuous everywhere yet has an everywhere zero symmetric derivative $f^s \equiv 0$, where

$$f^s(x) = \lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h}.$$

This question was answered negatively by Z. Charzyński in 1933 [2] after some partial results by S. Mazurkiewicz [8] and W. Sierpiński [11]. In fact, Charzyński proves that if $\limsup_{h\to 0}|\frac{f(x+h)-f(x-h)}{2h}| < \infty$ for each $x \in \mathbf{R}$, then the set of points of discontinuity of f is clairsamée (scattered). In 1931, while studying a less general version of the symmetric derivative, B. Jurek [6] proved that if E is a given clairsamée set, there exists a function which is identically zero on the complement of E, positive at each point of E, and has an everywhere zero symmetric derivative. This result was discovered independently by E. Szpilrajn in 1933 [13] and after Jurek's work was pointed out to him by Jarník, Szpilrajn published a "Reconnaissance de droit d'auteur" in [14].

In 1935 F. Hausdorff [5] asked whether there is a symmetrically continuous function which is discontinuous at the points of an uncountable set or, more generally, at the points of a prescribed $\mathbf{F}_\sigma$ set. (A function is symmetrically continuous if $\lim_{h\to 0}(f(x+h) - f(x-h)) = 0$ for every x.) Hausdorff did not realize that his second question had already been answered, albeit implicitly, by the first part of Charzyński's proof. In 1937 [4], H. Fried published the theorem that every symmetrically continuous function is continuous at the points of a dense set. As an introduction to his proof he remarks that "Der Beweiss ist nach einer Methode, die Z. Charzyński angewendet hat, geführt". But, indeed, Fried's proof is merely a reiteration of the first part of the proof of Charzński's theorem; even the notation is Charzyński's. Thus, the set of points at which a symmetrically continuous function is discontinuous is of first category.

In 1971 D. Preiss [9] solved Hausdorff's first question by constructing a symmetrically continuous function with uncountably many points of discontinuity. In the same paper he proved that if $f : \mathbf{R} \to \mathbf{R}$ is symmetrically continuous, then f is continuous almost everywhere. In particular, symmetrically continuous functions are measurable and have the property of Baire. According to a theorem of M. Chlebík (announced in [3]), the set of symmetrically continuous functions is of the power 2^c and hence there are symmetrically continuous functions which are not Borel measurable. An example by Konjagin shows that the set of points where a symmetrically continuous function is discontinuous need not be σ-porous [17]. A generalization of Preiss' result was given in 1983 by C.L. Belna in [1] who proved that a function symmetrically continuous on a measurable set E is continuous a.e. in E .

In 1982 J. Uher [16] improved Charzyński's work in a substantial way. He proved that if $E \subseteq \mathbf{R}$ is measurable and at each $x \in E$ either $\overline{f}^s(x) < +\infty$ or $\underline{f}^s(x) > -\infty$, then f is differentiable at almost every point of E . (Here, $\overline{f}^s$ and $\underline{f}^s$ denote the upper and lower symmetric derivatives of f .) For measurable functions, this was proved by A. Khintchine [7] in 1927. Uher actually proves more. Namely, he shows that if each point of E is a density point of the set where either $\overline{f}^s(x) < +\infty$ or $\underline{f}^s(x) > -\infty$ and f is symmetrically semicontinuous at each point of E , then f is differentiable at almost every point of E . In 1985 H.W. Pu and H.H. Pu [10] gave a simple proof to show that every function having a finite or infinite symmetric derivative everywhere is measurable.

A function satisfying $\overline{f}^s(x) < +\infty$ is necessarily upper symmetrically semicontinuous in the sense that $\limsup_{h\to 0+}(f(x+h) - f(x-h)) \leq 0$. Similarily, $\underline{f}^s(x) > -\infty$ implies that f is lower symmetrically semicontinuous at x. Uher realized that his earlier proof actually showed that if f is symmetrically semicontinuous on a measurable set E then f is continuous almost everywhere on E. As Uher pointed out, this result, coupled with the earlier theorem of Khintchine [7] proves his 1983 theorem above. This work was published in 1986 [17].

The theory of symmetric behaviour of functions was reformulated by B.S. Thomson in [15]. In this work Thomson first proves two fundamental symmetric covering lemmas and then shows how most all of the known symmetric behaviour results (including all the aforementioned results) can be proved using these lemmas.

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