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A CONVERSE TO A THEOREM OF SIERPINSKI ON ALMOST SYMMETRIC SETS

A collection of sets of reals, $\mathscr{L}$, is called a "notion of largeness" if it is closed under supersets. When we are considering a notion of largeness, a set will be called large if it is in $\mathscr{L}$ and small if its complement is in $\mathscr{L}$. We say that a set is large in an interval I, if the intersection of its complement with I is small. A real function f will be called **$\mathscr{L}$–symmetric** (resp. **locally $\mathscr{L}$–symmetric**) iff for each real x, the set of h for which $f(x+h)=f(x-h)$ is large (resp. large in some neighborhood of zero). We say a set of reals is $\mathscr{L}$–symmetric (or locally $\mathscr{L}$–symmetric) when its characteristic function is. For a real function f, define the **$\mathscr{L}$–symmetric derivative** (when it exists) to be $\lim^{\mathscr{L}}_{h\to 0}$ $(f(x+h)-f(x-h))/2h$ where $\lim^{\mathscr{L}}_{h\to 0}$ denotes that for each x a small set of h's may be ignored in the limit. Note that if $\mathscr{L}$ is a non–trivial filter (ie. $\mathbb{R}\in\mathscr{L}$, $\phi\notin\mathscr{L}$, and the intersection of two sets in $\mathscr{L}$ is also in $\mathscr{L}$) then $\lim^{\mathscr{L}}_{h\to 0}$ cannot take on two different values. Similarly we have the upper and lower $\mathscr{L}$–symmetric derivates.

Let C be a collection of closed intervals and I an open interval. For each x in I, let $E_x = \{h \mid [x-h,x+h]\subset I \text{ but } [x-h,x+h]\notin C\}$. C will be called a **locally $\mathscr{L}$–symmetric cover** of I, if for each x in I there is an $\epsilon>0$ such that $E_x \cap (0,\epsilon)$ is small. C is called a **globally $\mathscr{L}$–symmetric cover** of I if for each x in I, E_x is small. A collection of closed intervals, C, is **transitive** if $[x,y]\in C$ and $[y,z]\in C$ imply $[x,z]\in C$.

<u>**Examples**</u>:

(a) $\mathscr{L}=\{\mathbb{R}\}$. Then "$\mathscr{L}$–symmetric" is abbreviated "full symmetric" or just "symmetric".

(b) $\mathscr{L}$ is the collection of sets whose complement has cardinality less than κ. Then "$\mathscr{L}$–symmetric" is abbreviated "co$<\kappa$–symmetric". When $\kappa=\aleph_1$ we abbreviate this as "co–countably symmetric". The case where $\kappa=2^{\aleph_0}$ is what Sierpinski calls "almost symmetric". [6]

(c) $\mathscr{L}$ is the collection of sets of full Lebesgue measure. Then "$\mathscr{L}$–symmetric" is referred to as "essentially symmetric" (cf. [1]).

(d) $\mathscr{L}$ is the collection of sets which have zero as a point of density. Then "$\mathscr{L}$–symmetric" is referred to as "approximately symmetric".

(e) $\mathscr{L}$ is the collection of sets which have zero as a point of outer density. Then "$\mathscr{L}$–symmetric" is what Sierpinski calls "approximately symmetric in the large sense" Note that in this case $\mathscr{L}$ is not necessarily a filter and $\lim^{\mathscr{L}}$ is not necessarily uniquely defined.

<u>Theorem</u> **(Sierpinski [6])**: (ZFC) There exists a non–measurable set which is co$<2^{\aleph_0}$–symmetric.

As Sierpinski pointed out, this immediately implies the following:

<u>Corollary A</u>: (ZFC) There is a non–measurable set which is approximately symmetric in the large sense.

<u>Corollary B</u>: (ZFC) There is a non–measurable function whose approximately symmetric derivative in the large sense is everywhere zero.

If one assumes in addition to the axioms of ZFC the continuum hypothesis, CH, then;

<u>Corollary C</u>: (ZFC+CH) There is a non–measurable set which is co–countably symmetric.

<u>Corollary D</u>: (ZFC+CH) There is a non–measurable function whose approximately

symmetric derivative is everywhere zero.

In his paper [6] Sierpinski also gives an alternate proof of the first two corollaries. This proof, which is much simpler than the proof of Sierpinski's Theorem, is attributed to M.S. Ruziewicz and uses Hamel basis.

We will provide here a short proof of Sierpinski's whole theorem using a Hamel basis. We will also establish the following converse:

__Theorem__: Let $\mathscr{S}$ be a non–trivial filter on $\mathbb{R}$, which is translation and reflection invariant, which is closed under countable intersections, and such that every singleton is small. Suppose further that for some cardinal κ:

(i) The intersection of fewer than κ large sets is never empty.

(ii) Every non–small set contains a subset of size $<\kappa$ which is also non–small.

Let C be a transitive locally $\mathscr{S}$–symmetric cover of an open interval, I. Then there is a small set E such that C contains of all the closed intervals whose endpoints are in I–E.

The proof of this theorem, which will be provided, does not contain any new ideas but merely combines ideas from [2], [4], and [7]. It immediately implies the following corollaries:

__Corollary 1__: Let $\mathscr{S}$ be as in the previous theorem. Then any function whose $\mathscr{S}$–symmetric derivative is non–negative on an open interval I, is non–decreasing on a large set in I.

__Proof__: Let f be a function with a positive $\mathscr{S}$–symmetric derivative on an open interval I. Then the collection of intervals [a,b] for which f(b)>f(a) forms a transitive locally $\mathscr{S}$–symmetric cover of I. Therefore Corollary 1 follows for positive derivatives. If f has a nonnegative derivative, then consider the functions $f_n(x) = f(x)+x/n$ for $n=1,2,\ldots$. Then each $f_n(x)$ has a positive derivative and hence a set L_n which is large in I, such that f_n is monotone on L_n. But then since $f_n \to f$, f must be monotone on $\cap L_n$.

__Note__: In the foregoing proof it is not necessary for the limit to exist. Thus the theorem also holds for a non–negative $\mathscr{L}$–symmetric lower derivate. Note also that if $\mathscr{L}$ is translation and reflection invariant then the characteristic function of any large set has an $\mathscr{L}$–symmetric derivative which is everywhere zero. Thus whenever $\mathscr{L}$ is translation invariant we refer to the conclusion of Corollary 1 as **"Monotonicity for the $\mathscr{L}$–symmetric derivative"**. On the other hand let **"Monotonicity for the approximately symmetric derivative"** denote the proposition that every function with a non–negative approximately symmetric derivative on an open interval I, is non–decreasing on a set of full Lebesgue measure.

If $\mathscr{L}$ is the collection of sets of co–cardinality less than κ and κ is an uncountable cardinal less than $2^{\aleph_0}$ then (i) and (ii) hold for the cardinal κ^+. If we combine this with (the proof of) Sierpinski's Theorem then we immediately get:

__Corollary 2__: Let κ be any uncountable cardinal. Then Monotonicity for the co$<\kappa$–symmetric derivative holds iff $\kappa \neq 2^{\aleph_0}$.

__Corollary 3__: (ZFC) Monotonicity for the co–countably symmetric derivative is equivalent to the negation of the continuum hypothesis.

When S is the ideal of Lebesgue measure zero sets then the axioms (i) and (ii) are known to be consistent (in fact they become true when $\aleph_2$ random reals are added to a model of ZFC+CH). Therefore, we also get the following consistency results:

__Corollary 4__: If ZF is consistent then so is ZFC + Monotonicity for the essentially symmetric derivative.

It was proved in [3] that every approximately symmetric transitive cover is also essentially symmetric. Thus:

__Corollary 5__: If ZF is consistent then so is ZFC + Monotonicity for the approximately symmetric derivative.

__Proof of Main Theorem__:

Call a non–trivial filter on $\mathbb{R}$ **suitable** if it is translation and reflection invariant, is closed under countable intersections, and every singleton is small. The Theorem follows from the following three lemmas:

__Lemma 1__: Let $\mathcal{L}$ be a suitable filter on $\mathbb{R}$. Let C be a transitive locally $\mathcal{L}$–symmetric cover of an open interval I. Then C is also a globally $\mathcal{L}$–symmetric cover of I.

__Proof__: This proof is identical to a proof for symmetric covers appearing in Thomson [7], and which Thomson attributes to McGrotty. Let B be the sup of all ϵ such that $E_x \cap (0,\epsilon)$ is small. Clearly, $E_x \cap (0,B)$ is small by countable additivity. Since $\{B\}$ is also small, we have $E_x \cap (0,B]$ is small. Hence, we may assume that $[x-B,x+B] \subset I$, since otherwise $E_x = E_x \cap (0,B]$ and we are done. For each z in I let ϵ_z be as in the definition of locally $\mathcal{L}$–symmetric cover, and let $\epsilon < \min \{\epsilon_{x-B}, \epsilon_{x+B}, B\}$ be small enough that $[x-B-\epsilon,x+B+\epsilon] \subset I$. We show that $E_x \cap (B,B+\epsilon)$ is small, contradicting the choice of B. Let $h \in E_x \cap (B,B+\epsilon)$. Then $[x-h,x+h]$ is not in C. It follows by transitivity that one of the intervals $[x-h,x-2B+h]$, $[x-2B+h,x+2B-h]$, or $[x+2B-h,x+h]$ is not in C. Hence either $h-B$ is in E_{x-B} or $2B-h$ is in E_x or $h-B$ is in E_{x+B}. That is, $h \in$

$(B+E_{x-B})\cup(2B-E_x)\cup(B+E_{x+B})$ and so $E_x \cap (B,B+\epsilon)$ is small. □

<u>Lemma 2</u>: Let $\mathscr{L}$ be a suitable filter satisfying (i) and (ii) for some cardinal κ, and let C be a transitive globally $\mathscr{L}$–symmetric cover of an open interval I. Then there is a set E which is the union of $<\kappa$ many small sets such that C contains of all the closed intervals whose endpoints are in I–E.

<u>Proof:</u> This proof is identical to the proof in Preiss and Thomson [4]. We start with a non–small set X of size $<\kappa$ and assume without loss of generality that X is not small in any interval, and is closed under addition, subtraction, and division by 2. Let $w\in E$ iff $w\in I$ and $(\exists p,q \text{ in } X)\ (|p-w|\in E_p$ or $|q-(2p-w)|\in E_q)$. Notice this says that if $w\in I$ is not in E, then for any p, q in X, w can be "flipped" over p to form an interval in C and $2p-w$ can be "flipped" over q to form another interval in C, that is, as long as these intervals are in I. Since X has size $<\kappa$, E is the $<\kappa$–union of small sets.

Now, C contains every closed subinterval of I with at least one endpoint not in E and whose length is in X. To see this, for example, when the left endpoint is not in E, consider the interval $[w,w+x]$ with $w\in I-E$, $w+x\in I$, and $x\in X$, choose $p\in[w,w+x/2]$ in X and $q=p+x/2$ which is also in X by closure properties. Then since $w\notin E$, we have C contains $[w,2p-w]$ and also $[2p-w,2q-(2p-w)]$, which is the same as $[2p-w,w+x]$. Hence C contains $[w,w+x]$. The case where the right endpoint is not in E is handled similarly.

We must show that C contains each interval $[w,z]$ with endpoints in I–E. Let $c=(w+z)/2$ be the center of such an interval. Choose $x\in X$ such that C contains $[w+x,2c-(w+x)]$, using the fact that X is not small in any interval. But $[w,w+x]\in C$ since $w\notin E$ and $x\in X$, and similarly, $[2c-(w+x),z]=[z-x,z]\in C$. By combining these three intervals we get that $[w,z]\in C$. □

<u>Lemma 3</u>: Let $\mathscr{L}$ be a suitable filter satisfying (i) and (ii) for some κ, and C be a transitive

globally $\mathscr{L}$–symmetric cover of an open interval I. If there is a set E, which is the union of $<\kappa$–many small sets, such that $\{[a,b] \mid a,b \in I-E\} \subset C$, then there is also a small set E' such that $\{[a,b] \mid a,b \in I-E'\} \subset C$.

Proof: For each rational $\epsilon>0$ which is less that the length of I, let $L_\epsilon=\{x \mid [x,x+\epsilon]\subset I$ and C does not contain any interval $[x,h]\subset[x,x+\epsilon]$ where $h\notin E\}$ and let $R_\epsilon=\{x \mid [x-\epsilon,x]\subset I$ and C does not contain any interval $[h,x]\subset[x-\epsilon,x]$ where $h\notin E\}$. We will show that each L_ϵ is small. Suppose then that for some ϵ, L_ϵ is not small. Let Y denote a non–small subset of L_ϵ of size $<\kappa$ (using ii). Let J be an interval, [p,q], of length less than $\epsilon/4$ such that $[p,2q-p]\subset I$ and $Y\cap J$ is not small. Since $E+Y=\{e+y \mid e\in E,\ y\in Y\}$ is the $<\kappa$–union of small sets it does not contain [2q, 4q–2p]. Hence there is a c such that $2c\in[2q,\ 4q-2p]\setminus(E+Y)$, so that if $x\in Y$ then $2c-x\notin E$, and if $x\in J$ then $c\in[x,\ x+\epsilon/2]$. Therefore, if $x\in Y\cap J$ (a non–small subset of L_ϵ), then the interval [x, 2c–x] is a subset of I but not an element of C. This contradicts that C is a globally $\mathscr{L}$–symmetric cover of I and proves that each L_ϵ must be small. Similarly each R_ϵ is small.

Let $E'=(\cup L_\epsilon)\cup(\cup R_\epsilon)$. Then E' is small. Let [x,y] be a subinterval of I with endpoints not in E'. We will show that $[x,y]\in C$. Since $x\notin\cup L_\epsilon$ there is an $h<y$, $h\notin E$ such that $[x,h]\in C$. Since $y\notin\cup R_\epsilon$ there is a $k\notin E$ such that $h<k<y$ and such that $[k,y]\in C$. But also $[h,k]\in C$ since both endpoints are in I–E. Putting the three together by transitivity we get $[x,y]\in C$. Hence C contains all closed intervals with endpoints in $I-E'$. □

Example: (Sierpinski [6]) Assume ZFC. Let $H=\{h_1,h_2,...\}$ be a Hamel basis for the real numbers over the field of rationals (of length $2^{\aleph_0}$), so that $H_{even}=\{h_2,h_4,...\}$ and $H_{odd}=\{h_1,h_3,...\}$ both have full outer measure. Thus each real number has a unique representation $q_1h_{n_1}+q_2h_{n_2}+...+q_ih_{n_i}$ where each q is rational and i is finite. For each real r let s(r) denote sup $\{\alpha \mid h_\alpha$ is used in the Hamel representation of r$\}$. Let S be

$\{r|s(r)\in H_{even}\}$. If t is any real number for which $s(t)>s(r)$ (for each r the set of t for which this is false has size $<2^{\aleph_0}$), then $s(2r-t)=s(t)$ and hence $t\in S$ iff $2r-t\in S$. Therefore S is co$<2^{\aleph_0}$–symmetric.

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