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ON ACG* FUNCTIONS

The objective of this short note is to clarify the various definitions of ACG^*. The ACG^* property is used in defining the Denjoy integral. Also, it is used in stating the controlled convergence theorem for the Denjoy integral [3,4,5;Section 7]. In view of its close affiliation with the real line, it wasthe stumbling block for a natural generalization of the Denjoy integral to higher dimensions for many years. However a different definition, called ACG^{**}, was given recently [5;p.129,6]. It allows easy generalization to higher dimensions. We shall show in what follows that they are equivalent when considered on the real line.

Let us recall the classical definition of ACG^*. Let $X \subset [a,b]$. A function F is said to be $AC^*(X)$ if for every $\epsilon > 0$ there is $\eta > 0$ such that for every finite or infinite sequence of non-overlapping intervals $\{[a_k,b_k]\}$ with a_k, $b_k \in X$

$$\sum_k |b_k - a_k| < \eta \quad \text{implies} \quad \sum_k \omega(F;[a_k,b_k]) < \epsilon$$

where ω denotes the oscillation of F over $[a_k,b_k]$. A function F is said to be ACG^* if F is continuous and [a,b] is the union of a sequence of sets $\{X_i\}$ such that the function F is $AC^*(X_i)$ for each i. It is known [5;p.29] that if F is continuous and $AC^*(X)$ then F is $AC^*(\bar{X})$ where $\bar{X}$ is the closure of X. In other words, we may always assume in the definition of ACG^* that the set X_i is closed for each i.

THEOREM 1. A function F defined on [a,b] is ACG^* if and only if [a,b] is the union of a sequence of sets $\{X_i\}$ such that for each i and for every

$\epsilon > 0$ there is $\eta_i > 0$ such that for every finite or infinite sequence of non-overlapping intervals $\{[a_k,b_k]\}$ with at least one of a_k, b_k belonging to X_i

$$\sum_k |b_k - a_k| < \eta_i \text{ implies } \sum_k |F(b_k) - F(a_k)| < \epsilon.$$

The proof is elementary (see [5;p.27]). As suggested by Chew T. S., the one-endpoint version of ACG* as described in Theorem 1 has greatly simplified the proof of many results, for example, the equivalence of the Henstock and Denjoy integrals [5;section 6].

Given $\delta(\xi) > 0$, a division or a partial division of [a,b] given by a finite collection of interval-point pairs $\{([u,v],\xi)\}$, is said to be δ-fine if $\xi \in [u,v] \subset (\xi - \delta(\xi), \xi + \delta(\xi))$ for each $([u,v],\xi)$. Here ξ is called the associated point of [u,v]. The following condition in Theorem 2 is a weaker version of that in Theorem 1.

THEOREM 2. A function F defined on [a,b] is ACG* if and only if [a,b] is the union of a sequence of sets $\{X_i\}$ such that for each i and for every $\epsilon > 0$ there are $\eta_i > 0$ and $\delta_i(\xi) > 0$ for $\xi \in X_i$ such that for every finite or infinite sequence of non-overlapping δ_i-fine intervals $\{[a_k,b_k]\}$ with at least one of a_k, b_k belonging to X_i

$$\sum_k |b_k - a_k| < \eta_i \text{ implies } \sum_k |F(b_k) - F(a_k)| < \epsilon.$$

The proof of Theorem 2 is given in Chew [1]. It is equivalent to the version given by Henstock [2;p.62] though both appear in slightly different forms. As shown in [2;p.102], it helps to provide a lucid presentation of Henstock's theory.

Next, we define yet another version. Let $X \subset [a,b]$. A function F is said to be AC**(X) if for every $\epsilon > 0$ there are $\delta(\xi) > 0$ and $\eta > 0$ such that for any two δ-fine partial divisions D_1 and D_2 of $[a,b]$ with the associated points in X, in which D_2 may be void,

$$(D_1\backslash D_2)\sum |v - u| < \eta \quad \text{implies} \quad (D_1\backslash D_2) \sum |F(v) - F(u)| < \epsilon.$$

The above sums are over $D_1\backslash D_2$. Here $D_1\backslash D_2$ denotes the collection of component intervals $[u,v]$ in $E_1 - E_2$ where E_1 is the union of intervals in D_1 and E_2 the union of intervals in D_2. A function F is said to be ACG^{**} if $[a,b]$ is the union of a sequence of $\{X_i\}$ such that the function F is $AC^{**}(X_i)$ for each i.

THEOREM 3. A function F defined on $[a,b]$ is ACG^* if and only if F is ACG^{**}.

PROOF. By taking D_2 void in the definition of ACG^{**}, we have precisely the conditions in Theorem 2. Hence the sufficiency follows. Furthermore, to prove neccessity, it is enough to prove the condition in ACG^{**} for D_2 nonvoid. First, suppose that $[a,b]$ is the union of X_i, $i = 1,2,\ldots,$ such that F is $AC^*(X_i)$ for each i. Write $X = X_i$ and following the remark before Theorem 1 we may assume X closed.

Since F is $AC^*(X)$, for every $\epsilon > 0$ there is $\eta > 0$ such that the rest of the condition holds. Choose an open set $G \supset X$ such that the measure $|G - X| < \eta$. Then for any $\xi \in X$, define $(\xi-\delta(\xi),\xi+\delta(\xi)) \subset G$. Take any two δ-fine partial divisions D_1 and D_2 with the associated points in X. Suppose $(D_1\backslash D_2)\Sigma|v - u| < \eta$. The component intervals $[u,v]$ from $D_1\backslash D_2$ can be classified into two groups, one intersecting X and the other not. If the latter, then $[u,v] \subset G - X$ and there is a point $\xi \in X$ such that $[u,v] = [\xi,v] - [\xi,u]$ or $[u,\xi] - [v,\xi]$. Again, we may assume both (ξ,v) and (ξ,u) are included in $G - X$, and similarly for the other case. In view of Theorem 1, we have

$$(D_1 \backslash D_2)\Sigma|F(v) - F(u)| < 3\epsilon$$

in which one ϵ is contributed by the intervals $[u,v]$ from $D_1 \backslash D_2$ intersecting X and 2ϵ by those not intersecting X, together giving 3ϵ. Hence F is $AC^{**}(X)$ and consequently ACG^{**}.

It is well-known that if F is ACG^* then F is differentiable almost everywhere. However the converse is not true. We shall characterize the property which gives the converse. Since it resembles Lusin's (N) condition but stronger, so we shall call it the strong Lusin condition. A function F is said to satisfy the strong Lusin condition if for every set $E \subset [a,b]$ of measure zero and for every $\epsilon > 0$ there exists $\delta(\xi) > 0$ for $\xi \in E$ such that for any δ-fine partial division D of interval-point pairs $\{([u,v],\xi)\}$ with $\xi \in E$ we have

$$(D)\sum |F(v) - F(u)| < \epsilon$$

THEOREM 4. A function F defined on [a,b] is ACG^* if and only if F is differentiable almost everywhere in [a,b] and satisfies the strong Lusin condition.

PROOF. The necessity is obvious. The sufficiency follows from the standard argument [5;p.31] that the derivative F' is Henstock integrable on [a,b] under the given conditions, and therefore F is ACG^*. We sketch the proof as follows.

Let $f(x) = F'(x)$ when $x \in [a,b] - E$ and 0 otherwise. Here $|E| = 0$. Then given $\epsilon > 0$ there is $\delta(\xi) > 0$ such that whenever $([u,v],\xi)$ is δ-fine and $\xi \in [a,b] - E$ we have

$$|F(v) - F(u) - f(\xi)(v - u)| \le \epsilon|v - u|.$$

Futhermore, for any δ-fine partial division D of interval-point pairs $([u,v],\xi)$ with $\xi \in E$ we have

$$(D)\sum|F(v) - F(u)| < \epsilon.$$

Now take any δ-fine division D of [a,b]. Split D into D_1 and D_2 in which $\xi \notin E$ and $\xi \in E$ respectively and we obtain

$$\begin{aligned} &|F(b) - F(a) - (D) \sum f(\xi)(v - u)| \\ &\le (D_1)\sum|F(v) - F(u) - f(\xi)(v - u)| + (D_2)\sum|F(v) - F(u)| \\ &< \epsilon(b - a) + \epsilon. \end{aligned}$$

Hence by definition [5;p.5] f is Henstock integrable on [a,b]. Therefore F is ACG^*, by Lemma 6.19 [5;p.34].

Now we may define the Denjoy integral as follows. A function f is said to be Denjoy integrable on [a,b] if there is a continuous function F such that $F'(x) = f(x)$ almost everywhere in [a,b] and F satisfies the strong Lusin condition. We remark that the statement of Theorem 6.22 [5;p.36] is faulty. The condition (N) there should be replaced by the strong Lusin condition. Though the strong Lusin condition simplifies slightly the definition of the Denjoy integral, to state the controlled convergence theorem we still require the uniform ACG^{**} property.

For other classical characterization of ACG^*, see [5,7]. It is interesting to note that though the classical definition of ACG^* has difficulty in extending to higher dimensions, all the versions described here do not have such handicap. As shown in the proof of the controlled convergence theorem for the higher dimensions [5;section 21,6], it is necessary to adopt the definition of ACG^{**}. The other alternative definitions are simply not rich enough in order to carry the proof through.

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