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AN ANALOGUE OF CHARZYŃSKI'S THEOREM

§1. Introduction. In this note we wish to show how some recent methods introduced by Chris Freiling can be used to prove what we could call an *even* analogue of a well-known theorem of Charzyński [3]. The original problem addressed by Charzyński was to determine the continuity properties of a function f that satisfies the condition

$$\lim_{h\to 0} \frac{f(x+h)-f(x-h)}{2h} = 0 \tag{1}$$

at every point x. Mazurkiewicz [8] had already demonstrated that for a measurable function f this condition could hold only if the discontinuity points of f were nowhere dense, while Sierpiński had shown that in this situation the discontinuity points of f were at most denumerable. Charzyński relaxed these conditions by removing the measurability hypothesis and asking instead that

$$\limsup_{h\to 0} \left|\frac{f(x+h)-f(x-h)}{2h}\right| < +\infty \tag{2}$$

at each point x. Originally he proved that this hypothesis was enough to show that the set of points of discontinuity was countable and nowhere dense. It was conjectured by Szpilrajn that the set was in fact scattered (clairsemé) and thus we are led to the final version of the theorem as it appeared in Charzyński's classical paper in Fundamenta Mathematica in 1931: any function which satisfies the condition (2) at every point x is continuous everywhere excepting only at the points of some scattered set.

The proper context for discussing these kind of ideas is in terms of the properties of the even and odd parts of a function. The expression

$$f(x+t) = \frac{1}{2}[f(x+t)+f(x-t)] + \frac{1}{2}[f(x+t)-f(x-t)]$$

defines the *even* and the *odd* parts of the function f at the point x. It is natural in many contexts to examine the continuity and differentiability properties of f by studying those properties in these two parts. Thus we might be led to an investigation of "even continuity" and "odd continuity", and of "even differentiability" and "odd differentiability". Unfortunately the terminology has not evolved in this way. Continuity of the odd part of a function f is known as *symmetric continuity* while f is said to be *symmetric* if the even part is continuous. The derivative of the odd part of f at $t = 0$ is exactly the symmetric derivative of the function f at the point x. On the other hand differentiability of the even part of f at $t = 0$ is equivalent to the requirement that

$$\lim_{t \to 0} \frac{f(x+t) + f(x-t) - 2f(x)}{t} = 0 \tag{3}$$

and this condition is usually called the *smoothness* of the function f at the point x. This is the even analogue of (1). The analogue of (2) is the condition

$$\limsup_{t \to 0} \left| \frac{f(x+t) + f(x-t) - 2f(x)}{t} \right| < +\infty \tag{4}$$

which is sometimes called *quasi-smoothness.* For a survey of these and other even and odd properties of functions see [7].

While there are differences between the even and odd properties there are also close parallels. The differences are most notable in the study of the differentiation properties of functions satisfying some symmetric condition. The condition (2) requires f to be almost everywhere differentiable; in contrast there are continuous, nowhere differentiable functions f that satisfy (4) even uniformly. The stronger condition (3) on a measurable function requires the derivative to exist on a c–dense set but it can be of measure zero.

For continuity properties there are striking similarities however. Perhaps the best known example is a theorem of Stein and Zygmund [12, Lemma 9, p. 266]. (Note that there is an oversight in the original proof, as observed in [2], but it is easily amended.) They show that for a measurable function f that is symmetrically continuous at every point of a measurable set E, f is continuous almost everywhere in E; in the same lemma and with very much the same kind of arguments they show that "symmetrically continuous" may be replaced by "symmetric".

Armed with these and other analogies between the even and odd properties of a function one should be inevitably led to try for the even analogue of the Charzyński theorem. The addition of some regularity hypothesis, such as measurability, to the even analogues is always necessary because of the existence of nonmeasurable additive functions; this is not necessary in the original Charzyński theorem since the condition (2) already implies that the function f is measurable and has the Baire property.

THEOREM 1 *Let f be a measurable function that satisfies the condition*

$$\limsup_{h\to 0}\left|\frac{f(x+h)+f(x-h)-2f(x)}{h}\right| < +\infty \tag{5}$$

at every point x. Then f is continuous at every point with the exception only of a scattered set.

This theorem can very nearly be found in the literature. Auerbach [1] shows that for an integrable function and any $\alpha > 0$ the condition

$$\lim_{h\to 0}\left|\frac{f(x+h)+f(x-h)-2f(x)}{h^{\alpha}}\right| = 0 \tag{6}$$

holding at every point will require the set of discontinuity points to be measure zero and nowhere dense. Neugebauer [10] applies this to measurable, smooth functions and obtains the same conclusion. In a later paper [11] he shows that the set is countable. Evans and Larson [4] carry this to the final version by showing that the set of discontinuities can be characterized as scattered. Their proof, which is a modification of the original Charzyński proof, shows that the same conclusion holds for measurable, quasi-smooth functions if the set of discontinuities is granted to be countable. Thus the theorem is available just by proving that quasi-smooth functions have a countable set of discontinuities, a result whose proof could be fashioned after the original Charzyński proof for the odd symmetry case.

We shall give two proofs of Theorem 1. The first in Section 2 uses the methods introduced by Freiling; the second proof appeals to Theorem 3 below so that Theorem 1 then becomes an easy corollary of the original Charzyński theorem. A direct proof however should not be unwelcome since the Charzyński theorem requires itself a rather difficult proof; the proof given here can be viewed as an interesting application of Freiling's monotonicity theorem from [5]. Note that this proof and Freiling's proof of the Charzyński theorem in [5] are equally non-elementary since both require an appeal to the Khintchine theorem.

To show that Theorem 1 cannot be further improved we shall present as well the following assertion from [4].

THEOREM 2 (Evans–Larson) *Let S be a scattered subset of the interval $(0,1)$. Then there is a smooth, measurable function f defined on that interval that is discontinuous precisely at the points in the set S.*

Finally one might ask whether there is some more transparent reason why the continuity properties of functions satisfying an even symmetric growth condition so closely parallel the continuity properties associated with odd conditions. The following elementary theorem may help display the reason. (A similar version for $o(\phi(h))$ holds too.)

THEOREM 3 *Let ϕ be a positive function on an interval $(0,\eta)$ and let f be a function possessing a dense set of points of continuity. If at every point x of a set E*

$$|f(x+h)+f(x-h)-2f(x)| = O(\phi(h)) \tag{7}$$

as $h \to 0+$ then there is a measurable function g such that

$$|g(x+h)-g(x-h)| = O(\phi(h)) \tag{8}$$

as $h \to 0+$ at each $x \in E$ and f is continuous precisely at the points at which g is continuous.

There are a number of applications of Theorem 3 that come to mind. Theorem 2 above now can be viewed as a corollary of Theorem 3 together with the Charzyński theorem: any measurable function that satisfies the condition (5) must be symmetric and so is continuous on a dense set.

The classically known continuity properties of symmetric functions in turn can be deduced from similar results for symmetrically continuous functions. All that is required is to show that the hypotheses require continuity on a dense set. If f is symmetric and also bounded then an elementary argument of Mazurkiewicz [9] will show this. If f is symmetric and measurable then one can show that f is bounded on a dense set of intervals and so conclude that f has a dense set of points of continuity. If f is symmetric and has the Baire property then by essentially the same methods one can show that f is bounded on a dense set of intervals and so yet again f has a dense set of points of continuity. Thus the theorems [10, Theorem 1, p. 24] and [4, Theorem 1.2, p. 252] can be made to follow from a statement about symmetrically continuous functions.

§2. Proof of Theorem 1. As in [5] we are able to prove a somewhat sharper version of the theorem. We suppose that f is measurable, that the condition (5) holds at every point x except possibly for x in a countable set C and that f is symmetric at every point in C. Under these hypotheses f is symmetric everywhere and hence from classical material we know that its set of discontinuities has measure zero (from Auerbach [1] and the extension in Neugebauer [10] or alternatively from Stein and Zygmund [12, Lemma 9, p. 266] as mentioned earlier). We wish to show that this set of discontinuity points is scattered.

The proof follows from a general monotonicity theorem of Freiling [5] stated within the setting of interval functions. An interval function F is simply an extended real-valued function assigning a number $F(a,b)$ to every pair of numbers $a < b$. For an interval function F we define the following additivity conditions: we say that f is *superadditive* if whenever $a < b < c$ then

$$F(a,b) + F(b,c) \leq F(a,c).$$

It is *subadditive* if whenever $a < b < c$ then

$$F(a,b) + F(b,c) \geq F(a,c).$$

Finally it is *quasi-subadditive* if whenever $a < b < c$ then

$$F(a,b) \geq F(a,c) - |F(b,c)|$$

and

$$F(b,c) \geq F(a,c) - |F(a,b)|.$$

For reference we state a version of Freiling's monotonicity theorem.

THEOREM 4 (Freiling) *Let F be a superadditive and quasi-subadditive interval function such that*

(i) $\liminf_{t\to 0+} F(x-t, x+t)/t \geq 0$ *almost everywhere, and*
(ii) $\limsup_{t\to 0+} |F(x-t, x+t)/t| < +\infty$ *for every point* x *except possibly in a countable set* C*,*
(iii) $\liminf_{t\to 0+} F(x-t, x+t) \geq 0$ *at every point in* C*.*
Then there is a scattered set S *and such that* $F(a,b) \geq 0$ *for every* a*,* $b \notin S$*,* $a < b$*.*

Freiling has shown how this theorem can be used to give an elegant proof of Charzyński's theorem. Without much alteration the same argument provides a proof of Theorem 1 under the weaker hypotheses we have assumed above.

For any pair of real numbers a, b define $F(a,b)$ as the supremum of all numbers $t < 0$ with the property that there exists a $\delta > 0$ such that

$$|f(b+h) + f(a-h) - f(b) - f(a)| < -t$$

whenever $|h| < \delta$. The function F provides the proof of the continuity properties of the function f by virtue of the fact that should $F(a,b) = 0$ then f is continuous at a if and only if f is continuous at b. Conversely if f is continuous at both points a and then b then certainly $F(a,b) = 0$.

We check first the additivity properties of F. For any a, b, c (in any order) we show that

$$F(a,c) \geq F(a,b) + F(b,c). \tag{9}$$

If $F(a,b) = -\infty$ or $F(b,c) = -\infty$ then (9) holds trivially. Otherwise let $-\epsilon_1 < F(a,b)$ and $-\epsilon_2 < F(b,c)$. Then there are positive numbers δ_1 and δ_2 for which $|h| < \delta_1$ implies

$$|f(b+h) + f(a-h) - f(b) - f(a)| < \epsilon_1 \tag{10}$$

and $|h| < \delta_2$ implies

$$|f(c+h) + f(b-h) - f(c) - f(b)| < \epsilon_2. \tag{11}$$

Certainly (10) and (11) show that if $|h| < \min\{\delta_1, \delta_2\}$ then

$$|f(c+h) + f(a-h) - f(c) - f(a)| < \epsilon_1 + \epsilon_2.$$

From this we may conclude the relation (9). Both the superadditivity and the quasi-subadditivity of F follow now since we have not assumed any order requirements on the points a, b and c.

Now let us show that F satisfies the condition (ii) in the statement of Freiling's theorem. Note that $-\infty \leq F(a,b) \leq 0$. At any point x at which the condition (5) holds (i.e. at every point not in the countable set C) we may determine positive numbers δ and K so that

$$|f(x+s)+f(x-s)-2f(x)| < Ks$$

if $0 < s < \delta$. Then if $0 < t < \delta/2$ and $|h| < t$ we have from this inequality that

$$|f(x+t+h)+f(x-t-h)-2f(x)| < K(t+h)$$

and

$$|f(x+t)+f(x-t)-2f(x)| < Kt$$

and hence

$$|f(x+t+h)+f(x-t-h)-f(x+t)-f(x-t)| < 3Kt.$$

This gives

$$0 \geq F(x-t, x+t) > -3Kt$$

for all $0 < t < \delta/2$ and consequently

$$\limsup_{t\to 0+} |F(x-t,x+t)/t| < +\infty$$

as we require.

A nearly identical proof shows that property (iii) also holds because f is symmetric at every point and so at every point in C.

Finally, in order to apply the Freiling theorem it remains to verify the property (i). We recall that f is continuous almost everywhere. Fix a point c at which f is continuous; then at any other point x at which f is also continuous we know $F(c,x) = 0$. Thus we can define the function $g(x) = F(c,x)$. This function vanishes almost everywhere and so is measurable.

Because of (9) we have at any point x that

$$H(x-t,x+t) \leq H(c,x+t) - H(c,x-t) = g(x+t) - g(x-t).$$

Thus

$$\frac{g(x+t)-g(x-t)}{t} \geq \frac{H(x-t,x+t)}{t}$$

and so, from what we have already proved, the lower symmetric derivate of g is always greater than $-\infty$ outside of C. By a well-known theorem of Khintchine [6] then g is almost everywhere differentiable. At any point at which $g'(x)$ exists it is easy to see that $g'(x) = 0$.

Suppose that x is now a point of continuity of f and that $g(x) = g'(x) = 0$; almost every point x has this property. We apply (9) three times more to obtain

$$F(x,x+t) \geq F(c,x) + F(c,x+t) = F(c,x+t),$$
$$F(x,x-t) \geq F(c,x) + F(c,x-t) = F(c,x-t)$$

and

$$F(x-t,x+t) \geq F(x-t,x) + F(x,x+t).$$

Together these give

$$F(x-t,x+t) \geq g(x-t) + g(x+t).$$

Thus

$$\frac{F(x-t,x+t)}{t} \geq \frac{g(x-t)}{t} + \frac{g(x+t)}{t}.$$

Since each of these expressions on the right of the inequality tends to $g'(x) = 0$ as $t \to 0$ we must have

$$\liminf_{t\to 0+} F(x-t,x+t)/t \geq 0$$

at any such point x and hence almost everywhere. This is assertion (i) of the theorem.

We now apply the Freiling theorem and the proof of Theorem 1 is complete. There is a scattered set S so that $F(a,b) = 0$ for every a, $b \notin S$, $a < b$. Take any point $a \notin S$ at which f is continuous and we see that f is continuous at any point $b \notin S$.

§3. Proof of Theorem 2. The theorem of Charzyński is completed by an example due to Jurek and Szpilrajn. (See also the related example in Freiling [5]). They show that for any scattered set $S \subset (0,1)$ there is a function g positive on S and vanishing on $(0,1) \setminus S$ and which satisfies at every point x the inequalities $g(x+h) \leq h^2$ and $g(x-h) \leq h^2$ for sufficiently small h.

Evans and Larson have shown that similar arguments provide a smooth function with the same set of discontinuity points. Their arguments are clarified if we appeal directly to the earlier construction. Let s_1, s_2, s_3 ... be an enumeration of the set S. We define the function h on the interval $(0,1)$

$$h(x) = \sum_{s_i < x} 2^{-i-1} g(s_i).$$

This function is a saltus function that has jump discontinuities exactly at the points of S. We can adjust this function so as to be smooth by defining $f(x) = h(x)$ if $x \notin S$ and at any point x in S we write

$$f(x) = \frac{h(x+0) + h(x-0)}{2}.$$

We claim that f is smooth and that its set of discontinuity points is precisely S. The latter fact is clear. To see that it is smooth take any point $x \in (0,1)$. If $x \notin S$ then for sufficiently small h

$$|f(x+h) + f(x-h) - 2f(x)| \leq [f(x+h) - f(x)] + [f(x) - f(x-h)]$$

$$\leq \sup\{g(s_i);\ s_i \in (x-h, x)\} + \sup\{g(s_i);\ s_i \in (x, x+h)\} \leq 2h^2.$$

If $x \in S$ then

$$|f(x+h) + f(x-h) - 2f(x)| \leq [f(x+h) - f(x+0)] + [f(x-0) - f(x-h)]$$

and again $|f(x+h) + f(x-h) - 2f(x)| \leq 2h^2$.

In each case $|f(x+h) + f(x-h) - 2f(x)| = O(h^2)$ as $h \to 0$ and so f is smooth. Since f is monotone it is measurable and the proof is complete.

§3. Proof of Theorem 3. For the function g in the statement of Theorem 3 we have merely to take $g = \omega_f$, the oscillation of the function f, if this is finite. If there are points where it is infinite then we take $g(x) = 1$ at those points and everywhere else $g(x) = \omega_f(x)$. The function f is continuous precisely when ω_f vanishes. If f has a dense set of points of continuity then ω_f vanishes on a dense set and so $\omega_f(x) = 0$ if x is a point of continuity of

ω_f. In the other direction it is clear that if f is continuous at a point x then so too is ω_f. Finally g too has the same continuity points as f.

Here are the computations needed to check that the even symmetric condition (7) on f at a point will require that ω_f satisfies the parallel odd condition (8) there. At any point $x \in E$ and for any some $C > 0$ we may choose $0 < \delta < \eta$ so that

$$|f(x+h) + f(x-h) - 2f(x)| \leq C\phi(h) \tag{12}$$

if $0 < h < \delta$. If $0 < h_0 < \delta$ and t is any number $t < \omega_f(x + h_0)$ then we may choose sequences $\{x_n\}$ and $\{y_n\}$ converging to $x + h_0$ in such a way that $|f(x_n) - f(y_n)| > t$. Reflect these sequences about the point x by writing $x'_n = 2x - x_n$ and $y'_n = 2x - y_n$; these new sequences converge to $x - h_0$ and for sufficiently large n both of the inequalities

$$|f(x'_n) + f(x_n) - 2f(x)| \leq C\phi(h)$$

and

$$|f(y'_n) + f(y_n) - 2f(x)| \leq C\phi(h)$$

must hold because of the inequality (12). Thus we have

$$\begin{aligned} |f(x_n) - f(y_n)| \leq & \ |f(x_n) + f(x'_n) - 2f(x)| + \\ & |2f(x) - f(y_n) - f(y'_n)| + |f(y'_n) - f(x'_n)| \\ \leq & \ |f(y'_n) - f(x'_n)| + 2C\phi(h). \end{aligned}$$

We can conclude from this that $\omega_f(x - h_0) \geq t - 2C\phi(h)$. This is true for every $t < \omega_f(x + h_0)$ and hence either $\omega_f(x + h_0)$ is infinite or

$$\omega_f(x + h_0) - \omega_f(x - h_0) < 2C\phi(h).$$

Note that if $\omega_f(x + h_0) = +\infty$ then necessarily $\omega_f(x - h_0) = +\infty$.

Identical arguments show that

$$\omega_f(x - h_0) - \omega_f(x + h_0) < 2C\phi(h)$$

or else if $\omega_f(x - h_0) = +\infty$ then necessarily $\omega_f(x + h_0) = +\infty$.

Hence we have proved that ω_f has the odd symmetric property required at x at least discounting infinite values. If ω_f is not finite then replacing it by g will supply a finite function with the property required in the statement of the theorem.

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