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Lech Drewnowski, Institute of Mathematics, A. Mickiewicz University, Matejki 48/49, 60-769 Poznan, Poland.[1]

Iwo Labuda, Department of Mathematics, University of Mississippi, University, MS 38677, U.S.A.[2]

ON MINIMAL CONVEX USCO AND MAXIMAL MONOTONE MAPS

0. Foreword. (December 4, 1989).

A set-valued map F, from a topological space X into a locally convex space Y, is said to be convex usco if it is upper semicontinuous and takes convex compact nonempty values; it is minimal convex usco if there is no other convex usco map whose graph is strictly contained in the graph of F. In the following paper, minimal convex usco maps are studied and it is shown that usual multivalued Namioka type thoerems extend to such maps.

The paper was written at the end of 1986 but remained unpublished. In the meantime L. Jokl's paper (Minimal convex-valued weak*-usco correspondences and the Radon-Nikodym property, Commentationes Math. Univ. Carolinae, 28, 2(1987) 353-376) appeared, in which most of our results were obtained independently. However, we believe that many readers will find our approach to be more efficient and more readable than Jokl's. Let us mention, that it was our approach that was partially adopted by R. R. Phelps in his 1988 University of Washington Lectures (Convex Functions, Monotone Operators and Differentiability, Lecture Notes in Mathematics 1364, Springer 1989).

In view of the circumstances under which the paper is being published, we thought it appropriate to keep its form intact and, accordingly, what follows is the original paper without any change.

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[1] The paper was written while L.D. was a visiting professor at the Department of Mathematics, University of South Carolina, Columbia, S.C. 29208.

[2] The research was partially done while I.L. had a S.E.R.C. grant at the Department of Mathematics, University College London, Gower Street, London WC1E 6BT, U.K.

1. Introduction.

Here we explain the terminology used in this paper, discuss some multivalued Namioka type results that motivate our research and, finally, outline the contents of the paper.

Our main convention is that the term "*map*" will be used throughout as a shorthand for "set-valued map".

Let X be an arbitrary topological space, $Y = (Y,\rho)$ a Hausdorff space, and let $F:X \to Y$ be a map. The following terminology is standard.

F is *upper semicontinuous at* a point x of X (*usc at* x) if for every open set V containing F(x) there exists a neighborhood U of x such that

$$F(U) = \bigcup\{F(u): u \in U\} \subset V.$$

F is *upper semicontinuous* (*usc*) if it is usc at each point of X. F is *usco* if it is usc and takes only nonempty compact values. The *graph* of F is defined by

$$Gr(F) = \{(x,y) \in X \times Y: y \in F(x)\}.$$

Given another map $G:X \to Y$, we say that G *is contained in* F, and write $G \subset F$, if $G(x) \subset F(x)$ for each x in X; equivalently, if $Gr(G) \subset Gr(F)$. The map F is said to be *minimal usco* if it is usco and does not contain properly any other usco map from X into Y. (Likewise, given a family $\mathcal{F}$ of maps from X into Y, we may speak of maps that are *minimal in* $\mathcal{F}$.)

For future reference, we record here a useful result from [2] (cf. also [4]), and two of its easy consequences.

1.1 PROPOSITION. *If a map* $F:X \to Y$ *is usc and compact-valued, then its graph is a closed subset of* $X \times Y$, *and every map* $G:X \to Y$ *contained in* F *and having a closed graph is usc and compact-valued.*

1.2 COROLLARY. *If a map* $F:X \to Y$ *has a closed graph and every point* x *in* X *has a neighborhood* U *such that* F(U) *is a relatively compact subset of* Y, *then* F *is usc and compact-valued.*

1.3 COROLLARY ([2],[6]). *Every usco map* $F:X \to Y$ *contains a minimal usco map* $G:X \to$

Given a map $F:X \to Y$, we define

$$\Xi_0(F) \equiv \Xi_0(F;\rho) = \{x \in X: F(x) \text{ is a singleton and } F \text{ is } \rho\text{-usc at } x\}.$$

It is easy to see that if the topology ρ of Y is metrizable, then $\Xi_0(F)$ is a G_δ subset of X. [In fact, if d is a metric consistent with ρ, then for each n the set

G_n = {x: d-diam(F(U)) < 1/n for some neighborhood U of x} is open in X, and $\Xi_0(F) = \bigcap G_n$.]

By a (multivalued) *Namioka type theorem* we shall mean any result stating that, under some appropriate assumptions about X, Y, and a given family $\mathcal{F}$ of maps from X into Y, every map $F \in \mathcal{F}$ satisfies the following condition:

(N) $\Xi_0(F)$ is a dense subset of X.

Evidently, if (N) holds for a map F, then it also holds for *any* map $G \subset F$ such that $G(x) \neq \emptyset$ whenever $F(x) \neq \emptyset$; thus, for a map F to satisfy (N), neither the global upper semicontinuity with respect to some other topology on Y, nor any sort of "regularity" of its values F(x), are really necessary. Nevertheless, it appears to be highly plausible that the most natural setting for a Namioka theorem is when a bitopological space (Y,τ,ρ) is given (where the topology τ is usually weaker than ρ), and $\mathcal{F}$ consists of maps that are τ-usco and are *a priori* known to be "small", in one sense or another. In fact, all the general Namioka type theorems known at the moment fit into this natural scheme and, actually, deal exclusively with *minimal* τ-usco maps. We refer the reader to [10] for an up-to-date account on this topic, and recall here only the Namioka type result of Christensen and Kenderov ([3], 1.6) which is relevant to our research.

1.4 THEOREM. *Let* X *be a Baire space, and let* $Y = (Y,\rho)$ *be a Banach space with the Radon-Nikodym property. Let* τ *denote the weak topology of* Y, *or the weak* topology of* Y *in case* Y *is a dual Banach space. Then every minimal* τ*-usco map* $F:X \to Y$ *satisfies condition* (N).

The proof of this theorem in [3] followed an idea employed earlier by Kenderov [9] in establishing a similar result for monotone maps. We recall (see [11] for more information) that if X is a Banach space and $D \subset X$, then a map $F:D \to X^*$ is said to be *monotone* if

$$\langle x - y, x^* - y^* \rangle \geqslant 0 \quad \text{for all } (x,x^*) \text{ and } (y,y^*) \text{ in } Gr(F).$$

If, in addition, any monotone map $G:D \to X^*$ such that $F \subset G$ coincides with F, then F is called *maximal monotone* (over D). We will say that F is w*-usc [w*-usco] if it is usc [usco] when D is considered with its norm topology and X* with its weak* topology. By the Kuratowski-Zorn Principle, every monotone map $F:D \to X^*$ is contained in a maximal monotone map $M:D \to X^*$.

Kenderov's result can be now formulated as follows.

1.5 THEOREM. *Let* X *be a Banach space whose dual* X* *has the Radon-Nikodym property, and let* D *be an open subset of* X. *If* $F:D \to X^*$ *is a monotone map assuming only non-empty values, then the set of points* $x \in D$ *such that* $F(x)$ *is a singleton and* F *is norm-to-norm usc at* x *is a dense* G_δ *subset of* D.

Of course, this is a Namioka type theorem again. Moreover, since it is enough to have it proved for *maximal* monotone maps, and these are known to be w*-usco [8], thus modified it seems to be just another instance of the general scheme described above. Except that we don't know yet in what sense such maps are "small". Anyway, even in the one-dimensional case, a maximal monotone map need not be minimal w*-usco! Indeed, if $f:R \to R$ is a discontinuous increasing function, then the map $M:R \to R^* = R$ defined by $M(x) = [f(x-),f(x+)]$ is maximal monotone and usco, but it is not minimal usco.

In spite of these obstacles, one tends to have a feeling that Theorem 1.4 is "more general" than Theorem 1.5. As a matter of fact, the present paper has originated from an attempt to make this vague idea precise. We accomplish this via the concept of minimal convex-usco maps.

Let, as before, X be a topological space, and let Y be a Hausdorff locally convex space. A map $F:X \to Y$ is *convex usco* if it is usco and $F(x)$ is convex for each $x \in X$; F is *minimal convex-usco* if it is convex usco and does not contain properly any other convex usco map from X into Y. Evidently, the following analog of Cor.1.3 holds.

1.6 PROPOSITION. *Every convex usco map* $F:X \to Y$ *contains a minimal convex usco map* $G:X \to Y$.

We investigate minimal convex usco maps in Sections 2 and 4. In Section 2 we show that, roughly, they arise as "convexifications" of minimal usco maps. From this we deduce easily that, under very mild assumptions, *a multivalued Namioka type theorem holds for minimal convex usco maps iff it holds for minimal usco maps*. In particular, we have an exact analog of Theorem 1.4 for minimal convex usco maps.

In Section 3, devoted to monotone and maximal monotone maps with an *open* domain D, we first reprove quickly a few essentially known results about such maps, in the form that suits our purposes. We proceed then to show that a maximal monotone map containing a given monotone map is *unique*. Also, which is more important for us, we show that *maximal monotone maps are minimal convex* w*-*usco*, and so Kenderov's result can be readily derived from the (convex version of the) result of Christensen and Kenderov.

Finally, in Section 4, we give some characterizations of minimal convex usco maps. In particular we show that a map F is minimal convex usco iff all its compositions y*F with continuous linear functionals are minimal convex usco.

2. Minimal convex usco maps as convexifications of minimal usco maps.

In this section, X is a topological space, Y is a vector space, and τ, ρ are two Hausdorff locally convex topologies on Y. In our considerations, we will be dealing with either the locally convex spaces (Y,τ) and (Y,ρ), or the "bitopological' locally convex space (Y,τ,ρ). (Usually τ is weaker than ρ, but we do not need this here.) We say that ρ is τ-*polar*, or that the triplet (Y,τ,ρ) is *polar*, if (Y,ρ) has a base of neighborhoods of the origin consisting of τ-closed sets (which can be chosen to be absolutely convex as well). We say that the space (Y,τ) or (Y,τ,ρ) has *property* (C) if the τ-closed convex hull $\overline{\mathrm{co}}^{\tau} K$ of every τ-compact set $K \subset Y$ is τ-compact. (We omit the superscript τ, and write simply $\overline{\mathrm{co}}\ K$, when there is no risk of ambiguity.)

Given a map $F:X \to Y = (Y,\tau)$, we define its *convexification* $\overline{\mathrm{co}}\ F \equiv \overline{\mathrm{co}}^{\tau} F:X \to Y$ pointwise by

$$(\overline{\mathrm{co}}\ F)(x) = \overline{\mathrm{co}}\ F(x).$$

2.1 PROPOSITION. *Suppose that* (Y,τ,ρ) *is polar.*
If a map $H:X \to Y$ *is* ρ*-usc at a point* $x \in X$, $F = \overline{\mathrm{co}}^{\tau} H$ *and* $F(x)$ *is* ρ*-compact, then the map* F *is* ρ*-usc at* x.

PROOF. Since ρ is τ-polar, the locally convex vector topology $\nu = \inf\{\tau,\rho\}$ is Hausdorff, $\nu \leqslant \rho$, and ρ is easily seen to be ν-polar. Since $\nu \leqslant \tau$, $F = \overline{\mathrm{co}}^{\tau} H \subset \overline{\mathrm{co}}^{\nu} H = G$; moreover, $F(x) = G(x)$ by the ρ-compactness of $F(x)$. Hence if G is ρ-usc at x, so is F. In what follows we may therefore assume that $\tau \leqslant \rho$.

Let $\mathcal{V}$ be a base of τ-closed absolutely convex neighborhoods of 0 in (Y,ρ). Since $F(x)$ is ρ-compact, the sets $F(x) + V$ ($= \emptyset$ if $F(x) = \emptyset$), where $V \in \mathcal{V}$, form a base of ρ-neighborhoods of $F(x)$. Now, as H is ρ-usc at x, for each $V \in \mathcal{V}$ there is a neighborhood U of x such that for all $u \in U$,

$$H(u) \subset H(x) + V \subset F(x) + V$$

and hence, since $F(x) + V$ is τ-closed (because $\tau \leqslant \rho$) and convex,

$$F(u) = \overline{\mathrm{co}}^{\tau} H(u) \subset F(x) + V.$$

Thus F is ρ-usc at x.

Applying the above proposition with $\tau = \rho$, we obtain the following two results.

2.2 COROLLARY. *If a map* $F:X \to Y = (Y,\tau)$ *is minimal convex usco, then* $F = \overline{\mathrm{co}}\ H$ *for every usco map* $H:X \to Y$ *contained in* F.

2.3 COROLLARY. *If* $Y = (Y,\tau)$ *has property* (C) *and a map* $H:X \to Y$ *is usc and compacy valued, so is its convexification* $F = \overline{co}\, H$.

In the next result, "minimal [convex] usc" means "minimal in the class of usc maps from X into Y which assume nonempty closed [and convex] values".

2.4 PROPOSITION. *Let* $Y = (Y,\tau)$. *If a map* $H:X \to Y$ *is minimal usc, and the map* $F = \overline{co}\, H$ *is usc, then* F *is minimal convex usc.*

PROOF. Suppose it is not so. Then there exists a convex nonempty-set-valued usc map $G \subset F$ such that $G(x) \neq F(x)$ for some x in X. By the Second Separation Theorem for convex sets ([7],7.3.4), there exists a continuous real-valued linear functional y^* on Y such that for some r,

$$\sup y^*(G(x)) < r < \sup y^*(F(x)). \tag{1}$$

Denote $L = \{y: y^*(y) < r\}$ and $M = \{y: y^*(y) > r\}$. Then

$$H(x) \cap M \neq \emptyset; \tag{2}$$

otherwise $H(x) \subset \overline{L}$, hence $F(x) = \overline{co}\, H(x) \subset \overline{L}$, contradicting (1).

On the other hand, as $G(x) \subset L$ and L is open, there is a neighborhood U of x such that

$$G(U) \subset L. \tag{3}$$

Now, since H is minimal usc, (2) implies that $H(u) \subset M$ for some $u \in U$ (see [4], Prop. 4.6 and Remark 4.7). It follows that $F(u) = \overline{co}\, H(u) \subset \overline{M}$, which contradicts (3)

2.5 PROPOSITION. *If* $Y = (Y,\tau)$ *has property* (C) *and a map* $H:X \to Y$ *is minimal usco, then the map* $F = \overline{co}\, H$ *is minimal convex usco.*

PROOF. Apply Cor. 2.3 and Prop. 2.4.

For a topological space X and a triplet $Y = (Y,\tau,\rho)$, let $N(X;Y)$ [resp., $N_c(X;Y)$] denote the following condition (cf. Introduction):

For each minimal τ-usco [resp., minimal convex τ-usco] map $F:X \to Y$, the set
$\Xi_0(F) \equiv \Xi_0(F;\rho) = \{x \in X: F(x) \text{ is a singleton and } F \text{ is } \rho\text{-usc at } x\}$
is dense in X.

2.6 THEOREM. *Let a topological space* X *and a triplet* $Y = (Y,\tau,\rho)$ *be given.*

(a) *If* Y *is polar, then* $N(X;Y) \Rightarrow N_c(X;Y)$.

(b) *If* Y *has property* (C), *then* $N_c(X;Y) \Rightarrow N(X;Y)$.

PROOF. Let $F:X \to Y$.

(a) Suppose F is minimal convex τ-usco, and choose (by Cor. 1.3) a minimal τ-usco map $H \subset F$. Then $F = \overline{co}^{\tau} H$, by Cor. 2.2. Clearly, $\Xi_0(F) \subset \Xi_0(H)$. To prove the reverse containment, let $x \in \Xi_0(H)$. Then $F(x) = H(x)$ is a singleton, and Prop. 2.1 shows that F is ρ-usc at x. Thus $\Xi_0(F) = \Xi_0(H)$, and the latter set is dense by $N(X;Y)$.

(b) Suppose F is minimal τ-usco. Then $G = \overline{co}^{\tau} F$ is minimal convex τ-usco by Cor. 2.5. Since $F \subset G$, we have $\Xi_0(G) \subset \Xi_0(F)$; and as $\Xi_0(G)$ is dense by $N_c(X;Y)$, so is $\Xi_0(F)$.

2.7 COROLLARY. *Let* $Y = (Y,\tau,\rho)$ *be a Banach space [a dual Banach space] with the norm topology* ρ *and the weak [weak*] topology* τ. *Then, for every topological space* X, *conditions* $N(X;Y)$ *and* $N_c(X;Y)$ *are equivalent.*

2.8 COROLLARY. *An exact analog of Theorem 1.4 holds for minimal convex* τ*-usco maps.*

3. Maximal monotone maps.

In the present section, $X = (X,||\cdot||)$ denotes a real Banach space, X^* its dual space, and w^* the weak* topology of X^*.
We assume throughout that D is a nonempty *open* subset of X, and consider only maps $F:D \to X^*$ such that $F(x) \neq \emptyset$ for all x in D. Recall from the Introduction that such a map is said to be w^*-usc if it is usc when D is equipped with its norm topology, and X^* with its weak* topology.

The following three results are essentially known but, for sake of clarity, we prefer to state them here, in the form we need later, and with concise proofs.
We start with an important result due to Rockafellar [12]. Our proof of it is the same as in Pascali and Sburlan [11], pp. 103-104, except that the Lemma, which is a crucial ingredient of this proof, is deduced here directly from the Banach-Steinhaus theorem.

3.1 PROPOSITION. *Every monotone map* $F:D \to X^*$ *is locally bounded; that is, each point* x *in* D *has a neighborhood* U *such that* $F(U)$ *is a bounded (hence relatively* w^**-compact) subset of* X^*.

LEMMA. *If* $(x_n) \subset X$ *and* $(x_n^*) \subset X^*$ *are sequences such that* $||x_n|| \to 0$ *and* $||x_n^*|| \to \infty$, *then for every* $r > 0$ *there exists* $z \in X$ *with* $||z|| < r$, *and a strictly increasing sequence of indices* (n_j) *such that* $x_{n_j}^*(x_{n_j} - z) \to -\infty$ *as* $j \to \infty$.

PROOF of the Lemma. Fix $r > 0$.

Case 1: $\sup x_n^*(x_n) < \infty$.

Since $||x_n^*|| \to \infty$, by the Banach-Steinhaus theorem there exist $z \in X$ with $||z|| < r$ and $n_j \uparrow \infty$ such that $x_{n_j}^*(z) \to \infty$. Then $x_{n_j}^*(x_{n_j} - z) = x_{n_j}^*(x_{n_j}) - x_{n_j}^*(z) \to$

Case 2: $\sup x_n^*(x_n) = \infty$; by passing to suitable subsequences we may assume that $x_n^*(x_n) \to \infty$.

Let $y_n^* = x_n^*/x_n^*(x_n)$; then $||y_n^*|| \geqslant ||x_n||^{-1} \to \infty$ and $y_n^*(x_n) = 1$. Applying Case 1 to the sequences (x_n) and (y_n^*), we find z with $||z|| < r$ and $n_j \uparrow \infty$ so that $y_{n_j}^*(x_{n_j} - z) \to -\infty$. Then $x_{n_j}^*(x_{n_j} - z) = x_{n_j}^*(x_{n_j})\cdot y_{n_j}^*(x_{n_j} - z) \to -\infty$.

Proof of Proposition 3.1. Suppose F is unbounded in every neighborhood of some point $x \in D$. Then there exist sequences $(x_n) \subset D$ and $(x_n^*) \subset X^*$ such that $x_n \to x$, $x_n^* \in F(x_n)$, and $||x_n^*|| \to \infty$. Applying the Lemma to the sequences $(x_n - x)$ and (x_n^*), we find $z \in X$ and $n_j \uparrow \infty$ so that $y = x + z \in D$ and $\langle x_{n_j} - y, x_{n_j}^* \rangle \to -\infty$. Take any $y^* \in F(y)$. Then

$$0 \leqslant \langle x_{n_j} - y, x_{n_j}^* - y^* \rangle = \langle x_{n_j} - y, x_{n_j}^* \rangle - \langle x_{n_j} - y, y^* \rangle \to -\infty;$$

a contradiction.

Given a map $F:D \to X^*$, we denote by $\tilde{F}$ the map from D into X^* whose graph equals the closure in $(D, ||\cdot||) \times (X^*, w^*)$ of the graph of F.

3.2 PROPOSITION. *If a map* $F:D \to X^*$ *is monotone, then the map* $\tilde{F}:D \to X^*$ *is monotone and* w^**-usco.*

PROOF (Comp. [8].). We first show that if a pair $(x,x^*) \in X \times X^*$ is such that

(+) $\qquad \langle x - y, x^* - y^* \rangle \geqslant 0$

for all $(y,y^*) \in Gr(F)$, then (+) holds for all $(y,y^*) \in Gr(\tilde{F})$.

Let $(y,y^*) \in Gr(\tilde{F})$, and let (y_α, y_α^*) be a net in $Gr(F)$ such that

$$\text{(a)}\ ||y_\alpha - y|| \to 0 \quad \text{and} \quad \text{(b)}\ y_\alpha^* - y \to 0\ (w^*).$$

Then

$$\langle x - y, x^* - y^* \rangle = \langle x - y_\alpha, x^* - y_\alpha^* \rangle + \langle y_\alpha - y, x^* - y_\alpha^* \rangle + \langle x - y, y_\alpha^* - y^* \rangle,$$

where, to the right of the equality sign, the first term is nonnegative by our hypothesis, the second tends to zero in view of (a) and because the net (y_α^*) is eventually bounded (by Prop. 3.1), and the third tends to zero in view of (b). Hence $\langle x - y, x^* - y^* \rangle \geqslant 0$, as claimed.

Applying the above fact shows that (+) holds for all $(x,x^*) \in Gr(F)$ and $(y,y^*) \in Gr(\tilde{F})$, and applying it once again gives (+) for all (x,x^*) and (y,y^*) in $Gr(\tilde{F})$. That is, $\tilde{F}$ is monotone.

By definition, $\tilde{F}$ has a closed graph; hence, by Cor. 1.2 and Prop. 3.1, $\tilde{F}$ is w*-usco.

The next result is due to Browder ([1], Theorem 1.2 ; cf. also [11], p. 106).

3.3 PROPOSITION. *If a map* $F:D \to X^*$ *is monotone,* w*-*usc, and takes nonempty convex* w*-*closed values, then* F *is maximal monotone.*

PROOF. We have to show that if $(z,z^*) \in D \times X^*$ is such that

(1) $$\langle x - z, x^* - z^* \rangle \geqslant 0 \quad \text{for all } (x,x^*) \in Gr(F),$$

then $(z,z^*) \in Gr(F)$, i.e., $z^* \in F(z)$.

Suppose $z^* \notin F(z)$. Then, by the Hahn-Banach theorem, there exists $y \in X$ such that if $L = \{y^* \in X^*: y^*(y) < z^*(y)\}$, then $F(z) \subset L$. Since F is w*-usc at z and L is w*-open, there is a $\delta > 0$ such that if $u \in X$ and $||u|| < \delta$, then

$$z + u \in D \quad \text{and} \quad F(z + u) \subset L.$$

In particular, for all $t > 0$ such that $||ty|| < \delta$ we must have

(2) $$z + ty \in D \quad \text{and} \quad F(z + ty) \subset L.$$

On the other hand, for any such t, if $x^* \in F(z + ty)$, then (1) implies $\langle (z + ty) - z, x^* - z^* \rangle \geqslant 0$, hence $\langle y, x^* - z^* \rangle \geqslant 0$, and so $F(z + ty) \subset X^* \setminus L$, contradicting (2).

With the word "minimal" omitted, the next result can be found in [8].

3.4 THEOREM. *Every maximal monotone map* $M:D \to X^*$ *is minimal convex* w*-*usco.*

PROOF. It is easy to verify that the map co M (defined in an obvious manner) is monotone, hence M = co M by the maximality of M; that is, M is convex-valued. Furthermore, by maximality and Prop. 3.2, $M = \tilde{M}$, and M is w*-usco. Thus M is a convex w*-usco map, and its minimality follows immediately from Prop. 3.3.

3.5 COROLLARY. *For every monotone map* $F:D \to X^*$ *there is a precisely one maximal monotone map* $M = M_F:D \to X^*$ *containing* F.

PROOF. The existence of at least one such map M is provided by the Kuratowski-Zorn Principle. Suppose $M_1, M_2:D \to X^*$ are two maximal monotone maps containing F. Then, using Theorem 3.4, Prop. 1.1, and Prop. 3.3, it is easy to verify that the map $M:D \to X^*$ defined by $M(x) = M_1(x) \cap M_2(x)$ is maximal monotone. Since $M \subset M_1, M_2$, $M_1 = M = M_2$ follows by maximality.

Using the notation M_F introduced in the above result, we have also the following.

3.6. COROLLARY. *If* $F,G:D \to X^*$ *are two monotone maps such that* $F(x) \cap G(x) \neq \emptyset$ *for all* $x \in D$, *then* $M_F = M_G$.

PROOF. The map $H:D \to X^*$ defined by $H(x) = F(x) \cap G(x)$ ($\neq \emptyset$ by assumption) is monotone and contained in both F and G. It follows that $M_H \subset M_F$ and $M_H \subset M_G$. Since all these maps are maximal monotone, they must coincide.

We now show that M_F can be defined explicitly.

3.7 THEOREM. *If* $F:D \to X^*$ *is a monotone map, then*

$$M_F = \overline{co}^{w^*}(\tilde{F}):D \to X^*$$

is a unique maximal monotone map containing F.

PROOF. From Prop. 3.2 we know that the map $\tilde{F}$ is monotone and w*-usco, and Cor. 2.3 shows that the map M_F is w*-usco. Moreover, M_F is convex-valued, and it is easily seen to be monotone. Therefore, by Prop. 3.3, M_F is maximal monotone. Now, let $M:D \to X^*$ be any maximal monotone map containing F. Then $\tilde{F} \subset \tilde{M} = M$ (by Prop. 3.2) and, finally, $M_F = \overline{co}^{w^*}(\tilde{F}) \subset M$ because the values of M are convex and w*-closed (Theorem 3.5). By maximality, $M_F = M$.

Now we are ready to deduce Theorem 1.5 (Kenderov's result) from Corollary 2.8 (a "convex" analog of Theorem 1.4, the result of Christensen and Kenderov): By Cor. 2.8, the assertion of Theorem 1.5 holds for minimal convex w*-usco maps from D into X*. In particular, in view of Theorem 3.4, it holds for maximal monotone maps from D into X*, and hence for all monotone maps from D into X*.

3.8 REMARK. Let A be an arbitrary subset of X, $F:A \to X^*$ a monotone map with non-empty values, and D an open subset of A (e.g., D = Int A). Let $M:A \to X^*$ be a maximal monotone map containing F. By Prop. 3.1, the map $N = M|D$ is locally bounded. From this it follows, as in the proof of Prop. 3.2, that $Gr(M) \cup Gr(\tilde{N})$ is the graph of a monotone map from A into X*. By maximality, this last map coincides with M. Hence the map N has a closed graph (in $D \times X^*$); therefore, it is w*-usco. Moreover, since M(x) is convex for all x in A, Prop. 3.3 implies that N is maximal monotone (over D) In consequence, by Cor. 3.5 (or Theorem 3.7), we have $M|D = M_{F|D}$, and thus $M|D$ is determined uniquely by F (and even by $F|D$).

We do not know if a maximal monotone map containing a given monotone map is always determined uniquely.

As a corollary to the above we have the following: *The restriction of a maximal monotone map to an open set in its domain is maximal monotone over that set.* (Of course, if the domain is open, this follows directly from 3.4 and 3.3.)

4. Some characterizations of minimal convex usco maps.

Here again, X is an arbitrary topological space and Y is a Hausdorff locally convex space.

The following result is a convex analog of [4], Prop. 4.6 + Remark 4.7 (note that our space Y *is* regular). As in Prop. 2.4, "minimal convex usc" is to be understood as meaning "minimal in the class of usc maps which assume nonempty closed convex values".

4.1 PROPOSITION. *For every convex usc map $F:X \to Y$ such that $F(x)$ is closed and nonempty for all $x \in X$, the following are equivalent.*

(a) *F is minimal convex usc.*

(b) *$F|U$ is minimal convex usc for every open subset U of X.*

(c) *Whenever U is an open subset of X and C is a closed convex subset of Y such that $F(x) \cap C \neq \emptyset$ for all $x \in U$, then $F(U) \subset C$.*

(d) *Same as (c) with C a closed half-space in Y.*

(e) *For any $x \in X$, a neighborhood U of x, and an open half-space V of Y, if $F(x) \cap V \neq \emptyset$, then $F(u) \subset V$ for some $u \in U$.*

(f) *For each $x \in X$, the map F is minimal convex usc at x; that is, for every usc at x map $G:X \to Y$ such that G assumes nonempty values in a neighborhood of x and $G(x)$ is closed and convex, if $G \subset F$, then $G(x) = F(x)$.*

PROOF. (a) $\Rightarrow$ (b): Let U be an open subset of X, and let $H:U \to Y$ be a convex usc map contained in $F|U$. Then the map $G:X \to Y$ such that $G|U = H$ and $G|(X \setminus U) = F|(X \setminus U)$ is convex usc and $G \subset F$. Hence $G = F$ and, consequently, $H = F|U$. Thus $F|U$ is minimal convex usc.

(b) $\Rightarrow$ (c): The map $U \ni x \to F(x) \cap C$ is convex usc and takes nonempty values, hence (b) implies that $F(x) \cap C = F(x)$, i.e., $F(x) \subset C$ for all $x \in U$.

(c) $\Rightarrow$ (d): Obvious.

(d) $\Rightarrow$ (e): Suppose $F(u) \cap (Y \setminus V) \neq \emptyset$ for all $u \in U$; then (d) (with C = the closed half-space $Y \setminus V$) implies $F(U) \subset Y \setminus V$, contradicting the assumption in (e).

(e) $\Rightarrow$ (f): For a map G as specified in (f), suppose that $G(x) \neq F(x)$. Then, by the Second Separation Theorem for convex sets ([7], 7.3.4), there exists a continuous real-valued linear functional y^* on Y such that for some r,

$$\sup y^*(G(x)) < r < \sup y^*(F(x)).$$

Denote $W = \{y: y^*(y) < r\}$ and $V = \{y: y^*(y) > r\}$. Since G is usc at x, we can find a neighborhood U of x such that $\emptyset \neq G(u) \subset W$ for all $u \in U$. But $G \subset F$, so there is no $u \in U$ for which $F(u) \subset V$. However, $F(x) \cap V \neq \emptyset$, and we have a contradiction with (e).

(f) $\Rightarrow$ (a): Obvious.

Of course, a result similar to the above holds for minimal convex usco maps.

Our next goal is to characterize minimal convex usco maps F in terms of their compositions y^*F with continuous linear functionals y^* on the range space Y. We first prove a convex analog of Lemma 1(a) in [5].

4.2 PROPOSITION. *Let ϕ be a continuous affine mapping from a convex subset* C *of the locally convex space* Y *into a locally convex space* Y_1.
If a map $F:X \to Y$ *is minimal convex usco and* $F(X) \subset C$, *then also the map*

$$F_1 = \phi F:X \to Y_1$$

defined by

$$F_1(x) = \phi[F(x)]$$

is minimal convex usco.

PROOF. It is easy to verify that F_1 is convex usco. Now suppose $G_1:X \to Y_1$ is convex usco and $G_1 \subset F_1$, and consider the continuous mapping

$$\psi:Gr(F) \to Gr(F_1);\ (x,y) \to (x,\phi(x)).$$

Since $Gr(G_1)$ is a closed subset of $Gr(F_1)$ (by Prop. 1.1), $\psi^{-1}[Gr(G_1)]$ is a closed subset of Gr(F). Moreover, $\psi^{-1}[Gr(G_1)] = Gr(G)$, where $G(x) = F(x) \cap \phi^{-1}[G_1(x)]$ for all $x \in X$. By Prop. 1.1, G is usco, and since it is evidently convex-valued and $G \subset F$, we must have $G = F$. In consequence, $G_1 = \phi G = \psi F = F_1$, which proves that F_1 is minimal convex usco.

4.3 THEOREM. *Let* $F:X \to Y$ *be a convex usco map. Then the following are equivalent.*
(a) F *is minimal convex usco.*
(b) y^*F *is minimal convex usco for every* $y^* \in Y^*$.

PROOF. (a) $\Rightarrow$ (b): Apply Prop. 4.2.

(b) $\Rightarrow$ (a): If Y is a complex vector space, then from (b) and Prop. 4.2 it follow that $(\mathrm{Re}\, y^*)F:X \to R$ is minimal convex usco for all $y^* \in Y^*$. We may therefore assume that Y is a real vector space. Suppose that there exists a convex usco map $G:X \to Y$ such that $G \subset F$ and $G(x_0) \neq F(x_0)$ for some $x_0 \in X$. Then we can find $y^* \in Y^*$ so that $\sup y^*G(x_0) < \sup y^*F(x_0)$. It follows that y^*G is a convex usco map contained in y^*F and $y^*G(x_0) \neq y^*F(x_0)$ so that y^*F is not minimal, contradicting (b).

The implication (a) $\Rightarrow$ (b) in the above theorem holds true also in the non-convex case, i.e., for minimal usco maps (by [5], Lemma 1(a)), but its reverse is in genera false in that case.

4.4 EXAMPLE. Let

$$A_1 = \bigcup_{n=0}^{\infty} [2n\cdot 2\pi, (2n+1)2\pi], \qquad A_2 = \bigcup_{n=0}^{\infty} [(2n+1)2\pi, (2n+2)2\pi]$$

and

$$f_i(x) = (\sin x)\chi_{A_i}(x) \quad \text{for } x \in [0,\infty) \text{ and } i = 1,2.$$

Also, let $K = [-1,1] \times \{0\} \cup \{0\} \times [-1,1]$, $C = \mathrm{co}(K) \subset R^2$, and choose any compact set L such that $K \subset L \subset C$. Then define a map $F:[0,\infty] \to R^2$ by setting

$$F(x) = \{(f_1(x), f_2(x))\} \text{ for } x \in [0,\infty), \quad \text{and} \quad F(\infty) = L.$$

It is easily seen that F is usco; moreover, if $L = K$ it is minimal usco. For $L \neq K$ it is therefore non-minimal usco.

Now we are going to show that for every (continuous) linear functional $y^* = (c_1, c_2) \in R^2 = (R^2)^*$, the map $y^*F:[0,\infty] \to \mathbb{R}$ is minimal usco. It is usco by [5], Lemma 1(a). If $x \in [0,\infty)$, then $y^*F(x) = c_1 f_1(x) + c_2 f_2(x) = c_i \sin x$ for $x \in A_i$ $(i = 1,2)$; hence $(y^*F)((a,\infty)) = [-d,d]$ for $a \geqslant 0$, where $d = \max\{|c_1|, |c_2|\}$. For $x = \infty$,

$$\begin{aligned} y^*F(\infty) &= y^*(L) = y^*(C) = y^*[\mathrm{co}(\{(\pm 1,0),(0,\pm 1)\})] \\ &= \mathrm{co}\; y^*(\{(\pm 1,0),(0,\pm 1)\}) = \mathrm{co}\{\pm c_1, \pm c_2\} \\ &= [-d,d]. \end{aligned}$$

It follows that y^*F is minimal usco.

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