Real Analysis Exchange Vol 15 (1989-90)

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ON LEVINSON'S INEQUALITY

The purpose of this paper is to give a simple proof of a result of S.Lawrence and D.Segalman [1] for 3-convex functions. Namely, S. Lawrence and D.Segalman proved the following generalization of the well-known Levinson's inequality for 3-convex functions:

THEOREM A. Let f be a continuous function defined on $(0,2a)$ for which $\Delta_h^3 f(x) > 0$ for all x in $(0,2a)$ and $h>0$ for which $\Delta_h^3 f(x)$ is defined (i.e. for all x in $(0,2a)$ and $h > 0$ for which $x+3h < 2a$). Let $x_1,\dots,x_n$ be numbers in $(0,2a)$ such that $x_1 \leqq x_2 \leqq \dots \leqq x_n$ and $x_i + x_{n+1-i} \leqq 2a$, $i = 1,\dots,n$. Then

$$(1) \qquad \frac{1}{n}\sum_{i=1}^{n} f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \leqq \frac{1}{n}\sum_{i=1}^{n} f(2a-x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n}(2a-x_i)\right)$$

with equality if and only if either all the x_i are equal or $x_i + x_{n+1-i} = 2a$, $i = 1,\dots,n$.

Here, we shall prove the following:

THEOREM 1. Let f be a real-valued function defined on $(0,2a)$ for which $\Delta_k^2\Delta_h f(x) > 0$ for all x in $(0,2a)$ and $h > 0$, $k > 0$ such that $x + h + 2k < 2a$. Let $x_1,\dots,x_n$ be defined as in Theorem A. Then (1) is valid with the same conditions for equality.

Proof. As in [1] we have for $h = 2a-x_n-x_1$, $k = (x_n-x_1)/2$, in the case when $x_1+x_n < 2a$ and $x_1 < x_n$, i.e. $h > 0$, $k > 0$,

$$0 < \Delta_k^2\Delta_h f(x) = f(x_1+h+2k) - 2f(x_1+h+k) + f(x_1+h) - f(x_1+2k) + 2f(x_1+k) - f(x_1)$$

$$= f(2a-x_1) - 2f(2a - (x_1+x_n)/2) + f(2a-x_n) - f(x_n)$$
$$+ 2f((x_1+x_n)/2) - f(x_1).$$

If either $x_1+x_n = 2a$ or $x_1 = x_n$, we have equality in the above result. Hence

(2) $f(2a-(x_1+x_n)/2)-f((x_1+x_n)/2) \leqq \frac{1}{2}(f(2a-x_1)-f(x_1)+f(2a-x_n)-f(x_n))$,

with equality if and only if either $x_1 = x_n$ or $x_1+x_n = 2a$.

Of course, if $x_1 \leqq a$ and $x_n \leqq a$, then the above conditions are satisfied, and from (2) we have that the function $x \mapsto f(2a-x) - f(x)$ is strictly J-convex on $(0,a]$. Using this fact and inequality (2) for all relevant pairs of numbers we have:

$$\sum_{i=1}^{n} (f(2a-x_i) - f(x_i)) =$$

$$= \sum_{i=1}^{n} \frac{1}{2}((f(2a-x_i) - f(x_i)) + (f(2a-x_{n+1-i}) - f(x_{n+1-i})))$$

$$\geqq \sum_{i=1}^{n} (f(2a - (x_i+x_{n+1-i})/2) - f((x_i+x_{n+1-i})/2))$$

$$\geqq n(f(2a - \frac{1}{n} \sum_{i=1}^{n} (x_i+x_{n+1-i})/2) - f(\frac{1}{n} \sum_{i=1}^{n} (x_i+x_{n+1-i})/2))$$

$$= n(f(2a - \frac{1}{n} \sum_{i=1}^{n} x_i) - f(\frac{1}{n} \sum_{i=1}^{n} x_i)).$$

In the last inequality we used Jensen's inequality for J-convex function $x \mapsto f(2a-x) - f(x)$ and for numbers $t_i = (x_i+x_{n+1-i})/2$ since $t_i \leqq a$. Equality conditions for Jensen's inequality are $t_1 = \ldots = t_n$. So, using the equality conditions for (2) we obtain that equality in (1) is valid if and only if either all the x_i are equal or $x_i+x_{n+1-i} = 2a$, $i = 1,\ldots,n$.

Remarks: It is noted in [1] that if f is continuous and $\Delta_h^3 f(x) > 0$ for all x and $h > 0$, then $\Delta_k^2\Delta_h f(x) > 0$. The reverse implication is obvious. But, in Theorem 1 f can be discontinuous.

The above proof can be used for generalization of the above result for functions of several variables.

By using the fact that $x \mapsto f(2a-x) - f(x)$ is J-convex function on $(0,a]$ and known results for J-convex functions we can obtain many new results.

REFERENCE:

1. S.LAWRENCE and D.SEGALMAN, A Generalization of Two Inequalities Involving Means. Proc.Amer.Math.Soc. 35 (1972), 96-100.

Received March 28, 1989