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LOCAL CONVEX HULLS OF A CURVE, AND THE VALUE OF ITS FRACTAL DIMENSION

1. Introduction

Throughout this paper, Γ will denote a bounded, simple planar curve, and is defined as the image of a continuous, injective function γ, the parametrization, defined on an interval $[a, b]$, with values in the plane. Rectifiable curves have finite length. Fractal curves have infinite length, and their fractal dimension lies between 1 and 2.

The only notion of fractal dimension that we consider here is the *Minkowski–Bouligand dimension*

$$\Delta(\Gamma) = \limsup_{\epsilon \to 0} \left(2 - \frac{\log \mathcal{A}(\cup_{x \in \Gamma} B_\epsilon(x))}{\log \epsilon} \right) , \tag{1}$$

where $B_\epsilon(x)$ is the closed disk of centre x and radius ϵ, $\cup_{x\in\Gamma} B_\epsilon(x)$ is the ϵ–*Minkowski sausage* of Γ, and $\mathcal{A}$ denotes the area (2–dimensional Lebesgue measure)[1], [4].

For strictly self–similar or self–affine curves, it is usually possible to give the exact value of $\Delta(\Gamma)$. For non–theoretical curves, a classical method consists in evaluating the quantity $\mathcal{A}(\cup_{x\in\Gamma} B_\epsilon(x))$ for a large range of values of ϵ, and trying to estimate $2 - \Delta(\Gamma)$ as a limit, usually by constructing the log–log plot of $\mathcal{A}(\cup_{x\in\Gamma} B_\epsilon(x))$ versus ϵ. This method is justified only for curves such that $\mathcal{A}(\cup_{x\in\Gamma} B_\epsilon(x)) \simeq \epsilon^{2-\Delta(\Gamma)}$. An alternative, the *box counting method*, which is, in principle, equivalent, consists in calculating the number $N_\epsilon(\Gamma)$ of disjoint squares of side ϵ covering Γ. Then

$$\Delta(\Gamma) = \limsup_{\epsilon \to 0} \frac{\log N_\epsilon(\Gamma)}{|\log \epsilon|} . \tag{2}$$

The *compass method* calculates the maximal length l_ϵ of a regular polygonal curve of step ϵ, whose vertices belong to Γ; then

$$\Delta(\Gamma) = \limsup_{\epsilon \to 0} \left(1 + \frac{\log l_\epsilon}{|\log \epsilon|} \right) . \tag{3}$$

All these methods have the same drawback: they give very imprecise results for the dimension (convergences are too slow), due to the fact that they are too general in nature, and not well adapted to the particular structure of Γ. Numerical evidences show that it is possible to find algorithms with faster convergence, in the particular case where Γ is the graph of a continuous function (*variation method* [**8**], [**3**]). Until now, no such algorithm had been found for general, simple curves. The purpose of this paper is to present an algorithm that generalizes the variation method, and is adaptable to the local geometry of a very large family of contours in the plane, by the use of convex hulls. We are only concerned here with the theoretical foundations of this algorithm, reserving the numerical results for publication elsewhere.

Nevertheless, in order to use our method, we must assume that Γ is *expansive*, a notion introduced in section 3. The well-known fractal curves are expansive, and practically all digitalized curves (geographical coastlines, aggregate contours, signal datas, rupture profiles, etc...) may be considered as expansive, within the scales of observation. We study in section 2 the *global shape* of a curve, by means of two quantities: the *size*, comparable to the diameter, and the *deviation*, which measures the perturbation of a curve. A curve with deviation 0 is a segment. These notions are applied in section 4 to the local study of Γ. We show in section 5 that, in cases where Γ is expansive, the rate of convergence to 0 of the local size and deviation may determine the value of $\Delta(\Gamma)$: this gives the *variable steps method, with constant deviation*. Particular cases are studied in section 6 (graphs of continuous functions) and in section 7 (self-similar curves), where a precise definition is given of the notion of "statistical self-similarity". Finally, section 8 presents a general algorithm for the computation of the fractal dimension, the *local convex hulls method.*

1.1 Notations For any set E in the plane,

$\mathcal{K}(E)$ denotes the convex hull of E

$\operatorname{diam} E$ is the diameter of E

∂E is its boundary.

The length of a curve Γ is $L(\Gamma)$, the area of a domain D is $\mathcal{A}(D)$.

The distance between two points x, y is denoted by $\operatorname{dist}(x, y)$.

If $\mathcal{X}$ is a set, and f, g are two functions such that $\mathcal{X} \to \mathbf{R}^{+*}$, we write

$$f \preceq g$$

when there exists a constant c such that $f(\omega) \leq c\, g(\omega)$ for all $\omega \in \mathcal{X}$, and

$$f \simeq g$$

if together $f \preceq g$ and $g \preceq f$.

2. Global analysis of a curve

Let $\mathcal{F}$ be the family of all simple, bounded curves in the plane, non reduced to one point. Using the Hausdorff distance, $\mathcal{F}$ is considered as a metric space. The continuity of a function $\mathcal{F} \to \mathbf{R}^+$ is understood according to this topology.

2.1 Definition *We call* size *a continuous function* $p : \mathcal{F} \to \mathbf{R}^+$, *such that*

- $$\left|\frac{p(\mathbf{\Gamma})}{\operatorname{diam} \mathbf{\Gamma}} - 1\right| \preceq \frac{\mathcal{A}(\mathcal{K}(\mathbf{\Gamma}))}{(\operatorname{diam} \mathbf{\Gamma})^2} \tag{4}$$
- $$\mathbf{\Gamma}_1 \subset \mathbf{\Gamma}_2 \Longrightarrow p(\mathbf{\Gamma}_1) \leq p(\mathbf{\Gamma}_2) \tag{5}$$
- *If F is a similitude, of contraction ratio c, then*
$$p(F(\mathbf{\Gamma})) = c\,p(\mathbf{\Gamma}) \,. \tag{6}$$

In particular, the size of a segment is always equal to its length.
A trivial example of a size function is

$$p(\mathbf{\Gamma}) = \operatorname{diam} \mathbf{\Gamma} \,.$$

For this definition of p, a segment has the same size as a half-circle of the same diameter. The following examples are more sensitive to a perturbation of the curve:

2.2 Example Let

$$p(\mathbf{\Gamma}) = \frac{1}{2} L(\partial \mathcal{K}(\mathbf{\Gamma})) \tag{7}$$

be the half-perimeter of the convex hull. This function clearly verifies (5) and (6). As to (4), it is possible to show that

$$\operatorname{diam} \mathbf{\Gamma} \leq p(\mathbf{\Gamma}) \leq \operatorname{diam} \mathbf{\Gamma} + 2\,\frac{\mathcal{A}(\mathcal{K}(\mathbf{\Gamma}))}{\operatorname{diam} \mathbf{\Gamma}} \,. \tag{8}$$

Indeed, let x_1, x_2 be two points of $\mathbf{\Gamma}$, such that $\operatorname{dist}(x_1, x_2) = \operatorname{diam} \mathbf{\Gamma}$. Since they are both on the curve $\partial\mathcal{K}(\mathbf{\Gamma})$), we have

$$2 \operatorname{diam} \mathbf{\Gamma} \leq L(\partial \mathcal{K}(\mathbf{\Gamma})) \,. \tag{9}$$

Let us contruct the smallest rectangle K_1, containing $\mathbf{\Gamma}$, with two sides parallel to the segment $x_1 x_2$. Let l be the width of K_1. Since $\mathcal{K}(\mathbf{\Gamma}) \subset K_1$,

$$L(\partial \mathcal{K}(\mathbf{\Gamma})) \leq L(\partial K_1) = 2(l + \operatorname{diam} \mathbf{\Gamma}) \,. \tag{10}$$

Since $\mathbf{\Gamma}$ meets the four sides of the rectangle,

$$\mathcal{A}(\mathcal{K}(\mathbf{\Gamma})) \geq \frac{1}{2} \mathcal{A}(K_1) = \frac{1}{2} l \operatorname{diam} \mathbf{\Gamma} \,. \tag{11}$$

Inequalities (9), (10), (11) prove (8). Therefore, p is a size function.

2.3 Remark Let us recall the probabilistic interpretation of $L(\partial\mathcal{K}(\mathbf{\Gamma}))$ [6]: If $\mathbf{\Gamma}$ is included in a domain D of area 1, let us consider the events

$$\begin{aligned}&\mathcal{E}_1 : \text{a random straight line cuts } \mathbf{\Gamma}\\ &\mathcal{E}_2 : \text{a random straight line cuts } D\,.\end{aligned}$$

Then $L(\partial\mathcal{K}(\mathbf{\Gamma}))$ is the probability of $\mathcal{E}_1$, conditional on $\mathcal{E}_2$.

Now, we need to establish a notion related to the "width" for rectangles.

2.4 Definition *For all* $\mathbf{\Gamma} \in \mathcal{F}$ *we call* deviation *of* $\mathbf{\Gamma}$ *a continuous function* $q : \mathcal{F} \to \mathbf{R}^+$, *such that*

- $$q(\mathbf{\Gamma}) \simeq \frac{\mathcal{A}(\mathcal{K}(\mathbf{\Gamma}))}{\operatorname{diam}\mathbf{\Gamma}} \tag{12}$$
- $$\mathbf{\Gamma}_1 \subset \mathbf{\Gamma}_2 \Longrightarrow q(\mathbf{\Gamma}_1) \leq q(\mathbf{\Gamma}_2) \tag{13}$$
- *If* F *is a similitude, of contraction ratio* c, *then*

$$q(F(\mathbf{\Gamma})) = c\,q(\mathbf{\Gamma})\,. \tag{14}$$

In particular, the deviation of a segment is always equal to 0.
A trivial example of a deviation function is

$$q(\mathbf{\Gamma}) = \frac{\mathcal{A}(\mathcal{K}(\mathbf{\Gamma}))}{\operatorname{diam}\mathbf{\Gamma}}\,.$$

2.5 Remark For any size function $p(\mathbf{\Gamma})$, and deviation function $q(\mathbf{\Gamma})$, we have

$$\mathcal{A}(\mathcal{K}(\mathbf{\Gamma})) \simeq p(\mathbf{\Gamma})\,q(\mathbf{\Gamma})\,. \tag{15}$$

2.6 Example The *breadth* of a convex body E in the plane [7] is the minimum $b(E)$ of the distance between two parallel straight lines which contain E between them. One can show that

$$q(\mathbf{\Gamma}) = b(\mathcal{K}(\mathbf{\Gamma}))$$

is a deviation function, by proving the inequalities

$$\frac{1}{2}q(\mathbf{\Gamma})\operatorname{diam}\mathbf{\Gamma} \leq \mathcal{A}(\mathcal{K}(\mathbf{\Gamma})) \leq 2\,q(\mathbf{\Gamma})\operatorname{diam}\mathbf{\Gamma}\,.$$

Here, the constant $1/2$ on the left is the best possible constant (obtained with an equilateral triangle, for example), but the constant 2 on the right is not.

2.7 Example The *inner diameter* of a convex body E, denoted $\operatorname{diam}\operatorname{int}(E)$, is the diameter of the largest disk included in E. This is again a deviation function: indeed, it is known that

$$\frac{2}{3}b(E) \leq \operatorname{diam}\operatorname{int}(E) \leq b(E)\,,$$

where $b(E)$ is the breadth.

3. Expansive curves

When a bounded curve $\mathbf{\Gamma}$ has infinite length, it is impossible at small scales to cover it with disks having disjoint interiors, centred on $\mathbf{\Gamma}$. The more intricate $\mathbf{\Gamma}$ is, the more these disks intersect each other.

Following the same line of thought, we want to measure the common intersection of coverings by *local convex hulls*, which are much more adaptable to the local structure of $\mathbf{\Gamma}$ than disks. To further apply to the calculus of the fractal dimension, we will always consider coverings by convex bodies having comparable breadths. A curve is expansive if these local structural elements do not intersect too much. In terms of a trajectory, $\mathbf{\Gamma}$ is expansive, if the move results in a space gain with time, instead of repeatedly coming back to old positions.

3.1 Definition *Let q be a deviation function. A curve $\mathbf{\Gamma}$ is* expansive *if, for some constant c, and for all $\epsilon > 0$, we can find a covering $\{\mathbf{\Gamma}_i\}_{i\in\mathcal{I}}$ of $\mathbf{\Gamma}$ by subarcs, such that*

$$\frac{\epsilon}{c} \leq q(\mathbf{\Gamma}_i) \leq c\,\epsilon \qquad \textit{for all } i \in \mathcal{I} \tag{16}$$

and

$$\sum_{i\in\mathcal{I}} \mathcal{A}(\mathcal{K}(\mathbf{\Gamma}_i)) \leq c\,\mathcal{A}(\bigcup_{i\in\mathcal{I}} \mathcal{K}(\mathbf{\Gamma}_i))\,. \tag{17}$$

3.2 Remark Condition (17) implies that the index family $\mathcal{I}$ is finite: indeed, the left member of (17) is finite. Every term of this sum is larger than $c_1\, q(\mathbf{\Gamma}_i)^2 \geq c_1\,(\epsilon/c)^2$, for some constant c_1 depending on the choice of q. The series converges, only if there is a finite number of terms.

3.3 Notation For every family $\{U_i\}$ of sets in a topological space $\mathcal{X}$, we denote by $\omega(\{U_i\})$, the largest of all integers k such that, for at least one x in $\mathcal{X}$, x belongs to the interior of k distinct sets in the family $\{U_i\}$. In particular, $\omega(\{U_i\}) = 1$ if, for every pair of distinct sets U_i, U_j, their interiors are disjoint.

3.4 Proposition *Let $\mathbf{\Gamma} \in \mathcal{F}$, and ϵ be such that $0 < \epsilon < q(\mathbf{\Gamma})$. There exists a covering $\{\mathbf{\Gamma}_i\}_{i \in \mathcal{I}}$ by subarcs, such that $q(\mathbf{\Gamma}_i) = \epsilon$ for all i, and $\omega(\{\mathbf{\Gamma}_i\}) \leq 2$.*

Proof of Proposition (3.4) Let a direction be given along $\mathbf{\Gamma}$ (it is usually provided by a parametrization of the curve). We write

$$x < x'$$

when x, $x' \in \mathbf{\Gamma}$, x' is located "after" x, and $x \neq x'$. The arc of extremities x, x' is denoted $x^\frown x'$. We define the function $u_\epsilon : \mathbf{\Gamma} \to \mathbf{\Gamma}$ as follows:

$$u_\epsilon(x) = \sup\{\, x' \in \mathbf{\Gamma} \;:\; x' \geq x \,,\; q(x^\frown x') \leq \epsilon \,\} \,.$$

Let $A < B$ denote the extremities of $\mathbf{\Gamma}$. By continuity of the function q, we get

$$\begin{aligned} u_\epsilon(x) = B &\Longleftrightarrow q(x^\frown B) \leq \epsilon \\ u_\epsilon(x) < B &\Longrightarrow q(x^\frown u_\epsilon(x)) = \epsilon \,. \end{aligned}$$

We define by induction a sequence of points

$$\begin{aligned} x_0 &= A \\ x_{i+1} &= u_\epsilon(x_i) \,, \end{aligned}$$

until the rank N such that $q(x_N{}^\frown B) < \epsilon$ (see Figure 1).

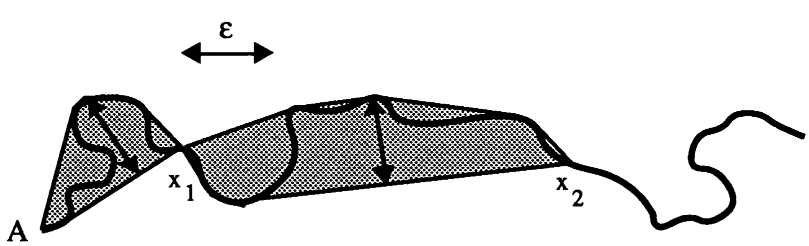

Figure 1

If $x_N = B$, then take $\mathbf{\Gamma}_i = x_{i-1}{}^\frown x_i$, $i = 1, \ldots, N$.
If $x_N < B$, then define

$$x_{N+1} = \inf\{\, x' \in \mathbf{\Gamma} \;:\; x' \leq B \,,\; q(x'^\frown B) \leq \epsilon \,\} \,,$$

which is such that

$$x_{N+1} < x_N < B \,,$$

and take $\mathbf{\Gamma}_i = x_{i-1}{}^\frown x_i$, $i = 1, \ldots, N$, $\mathbf{\Gamma}_{N+1} = x_{N+1}{}^\frown B$. The only subarcs whose interiors have common points are $\mathbf{\Gamma}_N$ and $\mathbf{\Gamma}_{N+1}$.

In either case, the sequence $\mathbf{\Gamma}_i$ is as required.

3.5 Vocabulary Using the previous notations, the sequence $\mathbf{\Gamma}_1, \ldots, \mathbf{\Gamma}_M$, $M = N$ or $N+1$, acording to the case being considered, is called *the constant deviation covering* of $\mathbf{\Gamma}$. These subarcs have, in general, unequal size — short in the irregular parts, long in the smooth parts of $\mathbf{\Gamma}$. When $\mathbf{\Gamma}$ contains a straight part, then the maximal size does not tend to 0 with ϵ.

3.6 Corollary *Let $\mathbf{\Gamma} \in \mathcal{F}$, and ϵ be such that $0 < \epsilon < q(\mathbf{\Gamma})$. Using the constant deviation covering of $\mathbf{\Gamma}$, consider the convex hulls $\mathcal{K}(\mathbf{\Gamma}_i)$ for $i - 1, \ldots, N$. If there exists a constant c, independent of ϵ, such that*

$$\sum_{i=1}^{N} \mathcal{A}(\mathcal{K}(\mathbf{\Gamma}_i)) \leq c\,\mathcal{A}(\bigcup_{i=1}^{N} \mathcal{K}(\mathbf{\Gamma}_i)) ,$$

then $\mathbf{\Gamma}$ is expansive.

Proof of Corollary 3.6 When $x_N = B$, then the Corollary follows directly from Definition (3.1). Otherwise, the arc $\mathbf{\Gamma}_{N+1}$ is needed to cover $\mathbf{\Gamma}$. Note that $c \geq 1$. We may write

$$\begin{aligned}
\sum_{i=1}^{N+1} \mathcal{A}(\mathcal{K}(\mathbf{\Gamma}_i)) &= \sum_{i=1}^{N} \mathcal{A}(\mathcal{K}(\mathbf{\Gamma}_i)) + \mathcal{A}(\mathcal{K}(\mathbf{\Gamma}_{N+1})) \\
&\leq c\,\mathcal{A}(\cup_{i=1}^{N} \mathcal{K}(\mathbf{\Gamma}_i)) + \mathcal{A}(\mathcal{K}(\mathbf{\Gamma}_{N+1})) \\
&\leq 2c \max\{\mathcal{A}(\bigcup_{i=1}^{N} \mathcal{K}(\mathbf{\Gamma}_i)) ,\ \mathcal{A}(\mathcal{K}(\mathbf{\Gamma}_{N+1}))\} \\
&\leq 2c\,\mathcal{A}(\bigcup_{i=1}^{N+1} \mathcal{K}(\mathbf{\Gamma}_i)) .
\end{aligned}$$

Thus, the family $\{\mathbf{\Gamma}_i\}$ verifies conditions (16) and (17).

4. Examples of expansive curves

4.1 Example

Let $\mathbf{\Gamma}$ be such that, for all x in $\mathbf{\Gamma}$, there exists a straight line $\mathbf{D}_x$ such that

$$\mathbf{D}_x \cap \mathbf{\Gamma} = \{x\} .$$

Then $\mathbf{\Gamma}$ is expansive.

To show this, let us first note that, for any subarc $\mathbf{\Gamma}_1 \subset \mathbf{\Gamma}$,

$$\mathcal{K}(\mathbf{\Gamma}_1) \cap (\mathbf{\Gamma} - \mathbf{\Gamma}_1) = \emptyset . \tag{18}$$

Indeed, if there exists a point x in this intersection, every line through x cuts $\mathbf{\Gamma}_1$, and therefore it cuts $\mathbf{\Gamma}$ at two points. This is impossible.

Let us now show that, if $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$ are two disjoint subarcs of $\mathbf{\Gamma}$, and U_1, U_2 are the interiors of their convex hulls $\mathcal{K}(\mathbf{\Gamma}_1)$ and $\mathcal{K}(\mathbf{\Gamma}_2)$, then

$$U_1 \cap U_2 = \emptyset . \tag{19}$$

Otherwise, there exists a point z in this intersection which is not on $\mathbf{\Gamma}$. The point z belongs to some chord A_1B_1 of $\mathbf{\Gamma}_1$, and to some chord A_2B_2 of $\mathbf{\Gamma}_2$. Either $\mathbf{\Gamma}_1$ cuts A_2B_2, or $\mathbf{\Gamma}_2$ cuts A_1B_1 (see Figure 2). In the first case, $U_2 \cap \mathbf{\Gamma}_1 \neq \emptyset$. In the second case, $U_1 \cap \mathbf{\Gamma}_2 \neq \emptyset$. Both are impossible, from (18). Therefore (19) is true.

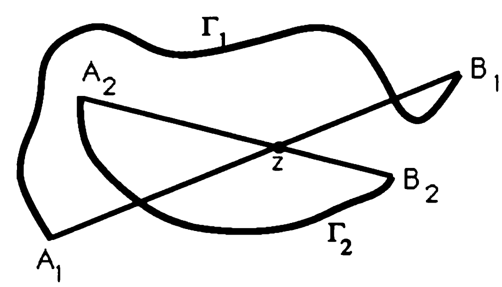

Figure 2

In conclusion, if we take, for every ϵ, the constant deviation covering $\{\mathbf{\Gamma}_i\}$ of $\mathbf{\Gamma}$, the convex hulls $\mathcal{K}(\mathbf{\Gamma}_i)$ have disjoint interiors; hence

$$\sum_{i=1}^{N} \mathcal{A}(\mathcal{K}(\mathbf{\Gamma}_i)) = \mathcal{A}(\bigcup_{i=1}^{N} \mathcal{K}(\mathbf{\Gamma}_i)) ,$$

and the conditions of Corollary (3.6) are fulfilled.

4.2 Application Let $\mathbf{\Gamma}$ be the graph of a continuous function in a cartesian system of axes. Then $\mathbf{\Gamma}$ is expansive. Indeed, every line parallel to $0y$ cuts $\mathbf{\Gamma}$ at 0 or 1 point.

4.3 Example This is a generalization of the last example:

Let $\mathbf{\Gamma}$ be such that, for some constant c, for every subarc $\mathbf{\Gamma}^*$ of $\mathbf{\Gamma}$, there exists a convex set $W(\mathbf{\Gamma}^*) \subset \mathcal{K}(\mathbf{\Gamma}^*)$, such that

$$W(\mathbf{\Gamma}^*) \cap (\mathbf{\Gamma} - \mathbf{\Gamma}^*) = \emptyset \tag{20}$$

$$\mathcal{A}(\mathcal{K}(\mathbf{\Gamma}^*)) \leq c\,\mathcal{A}(W(\mathbf{\Gamma}^*))\,. \tag{21}$$

Then $\mathbf{\Gamma}$ is expansive.

Indeed, again using the constant deviation covering of $\mathbf{\Gamma}$, we deduce from (20) that if $i \neq j$, then $W(\mathbf{\Gamma}_i)$ and $W(\mathbf{\Gamma}_j)$ have disjoint interiors, as in Example (4.1). This gives

$$\begin{aligned}\sum_{i=1}^{N} \mathcal{A}(\mathcal{K}(\mathbf{\Gamma}_i)) &\leq c \sum_{i=1}^{N} \mathcal{A}(W(\mathbf{\Gamma}_i)) \qquad \text{from (21)}\\ &= c\,\mathcal{A}(\bigcup_{i=1}^{N} W(\mathbf{\Gamma}_i))\\ &\leq c\,\mathcal{A}(\bigcup_{i=1}^{N} \mathcal{K}(\mathbf{\Gamma}_i))\,.\end{aligned}$$

Notice that in general, the convex set $W(\mathbf{\Gamma}^*)$ does not contain the entire arc $\mathbf{\Gamma}^*$ (otherwise, $W(\mathbf{\Gamma}^*) = \mathcal{K}(\mathbf{\Gamma}^*)$).

4.4 Application Example (4.3) describes a family of expansive curves, which actually contains most of the curves obtained in practice, with a constant c close to 1, inside the observation scales. It is also possible to show that self-similar curves belong to this family.

4.5 Example

Let $\mathbf{\Gamma}$ be a self-affine curve, determined by the affine transformations F_1, F_2, ..., F_N in the plane, such that $\mathbf{\Gamma} = \cup F_i(\mathbf{\Gamma})$. Moreover, we suppose that $\mathbf{\Gamma}$ is included in a convex domain D such that

(*i*) $F_i(D) \subset D$

(*ii*) *the interiors of the $F_i(D)$ are disjoint.*

Then $\mathbf{\Gamma}$ is expansive.

To show this, let us call ρ the smallest absolute value of the eigenvalues of F_1, ..., F_N. For all segments $\mathbf{S}$, and for all i, we have

$$L(F_i(\mathbf{S})) \geq \rho\, L(\mathbf{S})\,. \tag{22}$$

Every x in $\mathbf{\Gamma}$ is the limit of an imbedded family of affine copies of D:

$$D_k(x) = (F_{i_1} \circ \ldots \circ F_{i_k})(D) \, ,$$

where $\{i_1, \ldots, i_k\} \in \{1, \ldots, N\}^k$. In particular, $D_k(x)$ contains the subarc $(F_{i_1} \circ \ldots \circ F_{i_k})(\mathbf{\Gamma})$ of $\mathbf{\Gamma}$.

Let q be a deviation function. Being given ϵ, such that $0 < \epsilon < \rho$, there exists for every $x \in \mathbf{\Gamma}$ an integer K_x such that

$$q(D_{K_x+1}(x)) < \epsilon \leq q(D_{K_x}(x)) \, .$$

From (22), $q(D_{K_x+1}(x)) \geq \rho \, q(D_{K_x}(x))$, so that

$$\epsilon \leq q(D_{K_x}(x)) \leq \frac{\epsilon}{\rho} \, .$$

If x and y are two points of $\mathbf{\Gamma}$, either $D_{K_x}(x)$ and $D_{K_y}(y)$ have disjoint interiors or they are equal. This proves that $\{D_{K_x}(x)\}_{x \in \mathbf{\Gamma}}$ is a finite family of convex bodies, with disjoint interiors, covering $\mathbf{\Gamma}$. Each of them contains exactly one arc of $\mathbf{\Gamma}$, and also its convex hull. Using Definition (3.1) directly, with $c = 1$, $\mathbf{\Gamma}$ is expansive.

5. Fractal dimension of an expansive curve

Let p and q be size and deviation functions.

5.1 Proposition *Let $\mathbf{\Gamma}$ be an expansive curve. For all ϵ, such that $0 < \epsilon < q(\mathbf{\Gamma})$, let $\{\mathbf{\Gamma}_i\}$ be a covering of $\mathbf{\Gamma}$, satisfying* (16) *and* (17)*. Let $L_\epsilon = \sum_{i \in \mathcal{I}} p(\mathbf{\Gamma}_i)$. Then*

$$\Delta(\mathbf{\Gamma}) = \limsup_{\epsilon \to 0} \left(1 + \frac{\log L_\epsilon}{|\log \epsilon|}\right) \, . \tag{23}$$

5.2 Remark Formula (23) looks very much like (3). But in the latter, all steps along the curve have equal length. The steplengths in (23) (that is, the size of the covering elements) are, in general, different. Only the deviation $q(\mathbf{\Gamma}_i)$ of every step is kept uniform (Formula (16)), equivalent to ϵ. Therefore (23) defines a new method, very different in nature from the previous ones, that we call the *variable step method, with uniform deviation.*

5.3 Remark If (ϵ_k) is a sequence tending to 0, such that

$$\lim_{k\to\infty} \frac{\log \epsilon_{k+1}}{\log \epsilon_k} = 1 ,$$

then

$$\limsup_{k\to\infty} \left(1 + \frac{\log L_{\epsilon_k}}{|\log \epsilon_k|}\right) = \limsup_{\epsilon\to 0} \left(1 + \frac{\log L_\epsilon}{|\log \epsilon|}\right) . \tag{24}$$

Proof of Proposition (5.1) Let ϵ be given. Every convex set $\mathcal{K}(\mathbf{\Gamma}_i)$ has an area equivalent to $p(\mathbf{\Gamma}_i)\,q(\mathbf{\Gamma}_i)$ (15), that is, $\epsilon\, p(\mathbf{\Gamma}_i)$. Let K_i' be the set of all points at distance $\leq \epsilon$ from $\mathcal{K}(\mathbf{\Gamma}_i)$. This set is convex too, with diameter $\operatorname{diam} \mathcal{K}(\mathbf{\Gamma}_i) + 2\epsilon$, and breadth $b(\mathcal{K}(\mathbf{\Gamma}_i)) + 2\epsilon$. Therefore,

$$\mathcal{A}(K_i') \simeq \mathcal{A}(\mathcal{K}(\mathbf{\Gamma}_i)) \simeq \epsilon\, p(\mathbf{\Gamma}_i) . \tag{25}$$

Since

$$\cup \mathcal{K}(\mathbf{\Gamma}_i) \subset \cup_{x\in\mathbf{\Gamma}} B_\epsilon(x) \subset \cup K_i' , \tag{26}$$

we have

$$\begin{aligned}
\epsilon L_\epsilon &\simeq \sum \mathcal{A}(\mathcal{K}(\mathbf{\Gamma}_i)) \quad \text{from (25)} \\
&\leq c\,\mathcal{A}(\cup \mathcal{K}(\mathbf{\Gamma}_i)) \quad \text{from (17)} \\
&\leq c\,\mathcal{A}(\cup_{x\in\mathbf{\Gamma}} B_\epsilon(x)) \quad \text{from (26)} \\
&\leq c\,\mathcal{A}(\cup K_i') \quad \text{from (26)} \\
&\leq c \sum \mathcal{A}(K_i') \\
&\leq c'\, \epsilon L_\epsilon \quad \text{from (25)} ,
\end{aligned}$$

for suitable constants c and c'. Therefore

$$\mathcal{A}(\cup_{x\in\mathbf{\Gamma}} B_\epsilon(x)) \simeq \epsilon L_\epsilon ,$$

which proves (23).

5.4 Example Let T be a triangle ABC of sides 1, 1, $\sqrt{2}$, as in Figure 3, two parameters h_1, h_2 such that $0 < h_2 < h_1 < 1$, and a chain $\{T_i\}_{1\leq i\leq N}$ of isoceles triangles of base h_1, height h_2, vertices $A_iB_iC_i$. We assume that

$$A_1 = A\ ,\ B_N = B\ , \quad \text{and } T_i \cap T_{i+1} = \{B_i\} = \{A_{i+1}\} \quad \text{for } i = 1, \ldots, N-1 .$$

This construction defines a simple, self–affine curve (we may notice that the value of the Hausdorff dimension is not known for such a curve). Each of the N affinities has eigenvalues $h_1/\sqrt{2}$ and $h_2\sqrt{2}$. For every k, the curve $\mathbf{\Gamma}$ is covered with N^k triangles, of size $\simeq h_1^k$, and deviation $\simeq h_2^k$. Letting $\epsilon = h_2^k$ and $L_\epsilon = N^k\, h_1^k$ in (23), we get

$$\Delta(\mathbf{\Gamma}) = 1 + \frac{\log N\, h_1}{|\log h_2|} .$$

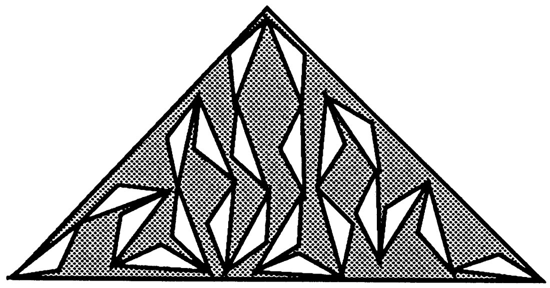

Figure 3

With

$$\alpha = \frac{\log h_1}{\log \frac{1}{N}} \quad \text{and} \quad \beta = \frac{\log h_2}{\log \frac{1}{N}} ,$$

this can be written

$$\Delta(\Gamma) = 1 + \frac{1}{\beta} - \frac{\alpha}{\beta} = \frac{\beta + 1 - \alpha}{\beta} .$$

This formula, discovered in a particular case, is actually very general in the context of curves having *uniform deviation*, as the next theorem shows.

5.5 Notations Let $\mathbf{\Gamma} = \gamma([a,b])$ be a parametrized curve. For all pairs (t_1, t_2) of real numbers, we write

$$\mathbf{\Gamma}(t_1, t_2) = \begin{cases} \gamma([t_1, t_2]) & \text{if } a \leq t_1 < t_2 \leq b; \\ \gamma([a, t_2]) & \text{if } t_1 < a; \\ \gamma([t_1, b]) & \text{if } b < t_2. \end{cases}$$

For simplicity, we use the notations $p(t_1, t_2)$, $q(t_1, t_2)$ instead of $p(\mathbf{\Gamma}([t_1, t_2]))$, $q(\mathbf{\Gamma}([t_1, t_2]))$.

5.6 Definition *Let β, $0 < \beta \leq 1$. We say that the parametrized curve $\mathbf{\Gamma}$ has* uniform deviation *of exponent β if, for all $t_1 < t_2$ in $[a,b]$:*

$$q(t_1, t_2) \simeq (t_2 - t_1)^{\beta} .$$

The property of uniform deviation depends on the parametrization of the curve.

5.7 Theorem *Let $\mathbf{\Gamma}$ be a parametrized curve, with uniform deviation of exponent β. We also assume that $\mathbf{\Gamma}$ is expansive. Let*

$$\overline{p}(\tau) = \frac{1}{b-a}\int_a^b p(t-\tau, t+\tau)\,dt$$

be the average size of the subarcs $\mathbf{\Gamma}(t-\tau, t+\tau)$, and

$$\alpha = \liminf_{\tau\to 0} \frac{\log \overline{p}(\tau)}{\log \tau} .$$

Then

$$\Delta(\mathbf{\Gamma}) = \frac{\beta + 1 - \alpha}{\beta} . \tag{27}$$

Proof of Theorem (5.7) For all ϵ, such that $0 < \epsilon \leq q(\mathbf{\Gamma})$, let $\{\mathbf{\Gamma}_i\}_{1\leq i\leq N}$ be a covering of $\mathbf{\Gamma}$, verifying (16) and (17). For all i=1, ..., N, let us call $[a_i, b_i]$ the interval of $[a,b]$ such that $\gamma([a_i,b_i]) = \mathbf{\Gamma}_i$. The family $\{[a_i,b_i]\}$ is a covering of $\mathbf{\Gamma}$. Without loss of generality, we may assume that

$$\omega(\{[a_i,b_i]\}) \leq 2 .$$

This implies that

$$b - a \leq \sum_{i=1}^{N}(b_i - a_i) \leq 2\,(b-a) . \tag{28}$$

Let $\tau = \epsilon^{1/\beta}$.

We deduce from Proposition (5.1) that

$$\begin{aligned}\Delta(\mathbf{\Gamma}) &= \limsup_{\epsilon\to 0}\left(1 + \frac{\log\sum_{i=1}^N p(\mathbf{\Gamma}_i)}{|\log\epsilon|}\right)\\ &= \limsup_{\tau\to 0}\left(1 - \frac{1}{\beta}\frac{\log N + \log\frac{1}{N}\sum_{i=1}^N p(\mathbf{\Gamma}_i)}{\log\tau}\right) .\end{aligned}$$

Using (16), and the fact that $\mathbf{\Gamma}$ has uniform deviation of exponent β,

$$q(\mathbf{\Gamma}_i) \simeq \epsilon \simeq (b_i - a_i)^\beta ,$$

so that

$$b_i - a_i \simeq \tau . \tag{29}$$

With (28), this implies

$$\tau \simeq N^{-1} , \tag{30}$$

and

$$\Delta(\mathbf{\Gamma}) = 1 + \frac{1}{\beta} - \liminf_{\tau\to 0}\frac{\log\frac{1}{N}\sum_{i=1}^N p(\mathbf{\Gamma}_i)}{\log\tau} .$$

So that all we have to prove is that, when $\tau \to 0$, or $N \to \infty$,

$$\liminf \frac{\log \frac{1}{N} \sum_{i=1}^{N} p(\Gamma_i)}{\log \tau} = \alpha \,. \tag{31}$$

a) We write

$$\int_a^b p(t-\tau, t+\tau)\, dt \le \sum_{i=1}^{N} \int_{a_i}^{b_i} p(t-\tau, t+\tau)\, dt \,.$$

By the mean theorem, there exists $r_i \in [a_i, b_i]$ such that

$$\int_{a_i}^{b_i} p(t-\tau, t+\tau)\, dt = (b_i - a_i)\, p(r_i - \tau, r_i + \tau) \,,$$

so that, using (29):

$$\overline{p}(\tau) \le \frac{1}{N} \sum_{i=1}^{N} p(r_i - \tau, r_i + \tau) \,. \tag{32}$$

Using (29) again, there exists an integer K, independent of ϵ and i, such that

$$\omega(\{[r_i - \tau, r_i + \tau]\}) \le K \,.$$

Therefore, the family $\{[r_i - \tau, r_i + \tau]\}$ can be split into K families (some of them possibly empty) $\mathcal{U}_1, \ldots, \mathcal{U}_K$, of disjoint intervals. Every $u \in \mathcal{U}$ cannot meet more than K' intervals $[a_i, b_i]$, for some other constant K', so that

$$\sum_{u \in \mathcal{U}} p(u) \le K' \sum_{i=1}^{N} p(\Gamma_i) \,.$$

This gives

$$\sum_{i=1}^{N} p([r_i - \tau, r_i + \tau) \le K\, K' \sum_{i=1}^{N} p(\Gamma_i) \,.$$

With (32), we obtain

$$\overline{p}(\tau) \preceq \frac{1}{N} \sum_{i=1}^{N} p(\Gamma_i) \,. \tag{33}$$

b) For a converse inequality, note that, for some constant c, and for all $t \in [a_i, b_i]$:

$$[a_i, b_i] \subset [t - c\tau, t + c\tau] \,. \tag{34}$$

Since $\{[a_i, b_i]\}$ can be split into two families of intervals with disjoint interiors,

$$\int_a^b p(t - c\tau, t + c\tau)\, dt \ge \frac{1}{2} \sum_{1}^{N} \int_{a_i}^{b_i} p(t - c\tau, t + c\tau)\, dt \,.$$

By the mean theorem, there exists $s_i \in [a_i, b_i]$ such that

$$\begin{aligned}\int_{a_i}^{b_i} p(t - c\tau, t + c\tau)\, dt &= (b_i - a_i)\, p(s_i - c\tau, s_i + c\tau) \\ &\geq (b_i - a_i)\, p(\mathbf{\Gamma}_i) \quad \text{from (34)} \\ &\succeq \frac{1}{N}\, p(\mathbf{\Gamma}_i) \quad \text{from (30)}\,.\end{aligned}$$

Therefore

$$\frac{1}{N}\sum_{i=1}^{N} p(\mathbf{\Gamma}_i) \preceq \overline{p}(c\tau)\,. \tag{35}$$

Formulas (33) and (35) suffice to prove (31).

5.8 Remark Since $q \preceq p$, the inequality

$$\alpha \leq \beta \tag{36}$$

is always true. Moreover, when $\mathbf{\Gamma}$ is expansive, the sum $\sum \mathcal{A}(\mathcal{K}(\mathbf{\Gamma}_i))$ is bounded for all ϵ. Since

$$\begin{aligned}\sum \mathcal{A}(\mathbf{\Gamma}_i)) &\simeq \tau^{\beta}\, N\, \overline{p}(\tau) \\ &\simeq \tau^{\alpha+\beta-1}\,,\end{aligned}$$

we get

$$\alpha + \beta \geq 1\,. \tag{37}$$

When $\alpha + \beta = 1$, $\Delta(\mathbf{\Gamma}) = 2$.

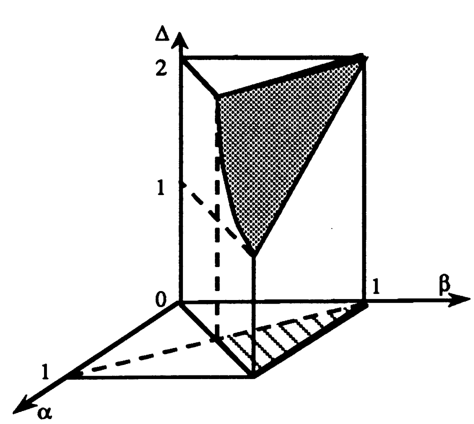

Figure 4

The pair (α, β) lies in the triangle bounded by the lines $\beta = 1$, $\alpha = \beta$, $\alpha + \beta = 1$. The surface (Figure 4) consisting of the points (α, β, Δ), where

$$\Delta = \frac{\beta + 1 - \alpha}{\beta} ,$$

is bounded by three curves:

a) The segment $\alpha + \beta = 1$, $\Delta = 2$.

b) The segment $\beta = 1$, $\Delta = 2 - \alpha$.

c) The hyperbola $\alpha = \beta$, $\Delta = \frac{1}{\alpha}$.

The last two cases contain important families of fractal curves which we study in detail in the following section.

6. Graphs of continuous, non–differentiable, functions

Let $f(t)$ be a non–constant, continuous function $[a, b] \to \mathbf{R}$, and $\mathbf{\Gamma}$ its graph. We have seen in Application (4.2) that $\mathbf{\Gamma}$ is expansive. Its length is infinite when the subset of $[a, b]$ on which f is not differentiable has measure > 0. The natural parameter of $\mathbf{\Gamma}$ is the abcissa t. Thus, $\mathbf{\Gamma}(t_1, t_2)$, where $a \leq t_1 < t_2 \leq b$, denotes the arc of $\mathbf{\Gamma}$ of extremities $(t_1, f(t_1))$, $(t_2, f(t_2))$.

Let $R(t_1, t_2)$ be the smallest rectangle whose sides are parallel to the axes, containing $\mathbf{\Gamma}(t_1, t_2)$. This rectangle has base $t_2 - t_1$. Its height is equal to

$$v(t_1, t_2) = \sup\{\, |f(t') - f(t'')| \,,\, t',\, t'' \in [t_1, t_2] \,\} .$$

It is called the *oscillation of f on the interval* $[t_1, t_2]$. The convex set $\mathcal{K}(\mathbf{\Gamma}(t_1, t_2))$ touches all four sides of the rectangle. Its size is equivalent to the diagonal of $R(t_1, t_2)$, and its deviation is smaller than the two side–lengths; so

$$p(t_1, t_2) \simeq \sqrt{v(t_1, t_2)^2 + (t_1 - t_2)^2} \ , \ q(t_1, t_2) \preceq \min\{(t_2 - t_1, v(t_1, t_2)\} .$$

Using the inequality

$$\tau\, v(a, b) \preceq \int_a^b v(t - \tau, t + \tau)\, dt \ ,$$

we obtain that the average size is equivalent to the average oscillation, which is named *variation of the function* f in [**8**]:

$$\overline{p}(\tau) \simeq \frac{1}{b - a} \int_a^b v(t - \tau, t + \tau)\, dt \ .$$

Let us consider a typical case, where

$$q(t_1, t_2) \geq c(t_2 - t_1) \tag{38}$$

for some constant c. This inequality is verified when there exist t_3, t_4 in $[t_1, t_2]$ such that $|t_4 - t_3| \geq c(t_2 - t_1)$ and $f(t_3) = f(t_4)$. It is the case, in particular, for strictly self–affine graphs.

If (38) is true, then $\mathbf{\Gamma}$ has constant deviation of order 1. We take $\beta = 1$, and

$$\alpha = \liminf_{\tau \to 0} \frac{\log \overline{p}(\tau)}{\log \tau} .$$

Then, Theorem 5.7 implies

$$\Delta(\mathbf{\Gamma}) = 2 - \alpha , \tag{39}$$

a formula known as the *variation method* [**8**], [**2**], [**3**]. It corresponds to case b) of Remark (5.8). In the strictly self–affine case, the parameter α plays the role of Hölder coefficient H [**4**], characterizing the *non–differentiability property* of the function f. In the case of Weierstrass, or Weierstrass–Mandelbrot functions, $\alpha = H$ is directly related to the fractional derivability of the function [**9**]. Note that, even if (38) is not verified, (39) is still true. We conjecture that (39) is true for all curves such that

$$q(t_1, t_2) \preceq |t_1 - t_2| .$$

7. Self–similar curves

A strictly self–similar curve may be shown to verify the assumptions of Theorem (5.7); the proof is omitted. The similarity implies that the size of a subarc is equivalent to its deviation, which gives

$$\alpha = \beta , \tag{40}$$

an equality which corresponds to case c) of Remark (5.8). Then

$$\Delta(\mathbf{\Gamma}) = \frac{1}{\alpha} .$$

In return, Formula (40) could help to characterize the *statistical self–similarity*, at least in a weak sense:

7.1 Definition *We say that a simple, parametrized curve* $\mathbf{\Gamma}$ *is* statistically self–similar *if it possesses the following two properties:*

(i) $\mathbf{\Gamma}$ *is expansive.*

(ii) *There exists a parameter* α *such that*

$$p(t_1, t_2) \simeq q(t_1, t_2) \simeq |t_1 - t_2|^{\alpha} .$$

8. Methods for the fractal dimension evaluation

Formula (39) gives a method for evaluating the dimension of graphs that has been proven to give excellent results. Our *variable–step algorithm with constant deviation*, defined in (23), may be used for a much larger family of curves. One drawback of this method is that the extremity A of the considered curve plays a special role. A numerical method is always better when it gives every point of the curve the same importance. Accordingly, the evaluated length L_ϵ in (23) may be replaced by an average length corresponding to different starting points. This idea finally leads to a new *local convex hull method*, which is as follows:

For every ϵ, and for all $x \in \mathbf{\Gamma}$, we have defined a "next" point $u_\epsilon(x)$ on $\mathbf{\Gamma}$, such that

$$q(x^\frown u_\epsilon(x)) = \epsilon$$

if $u_\epsilon(x) \neq B$. Now a domain $S(\epsilon)$ enclosing $\mathbf{\Gamma}$ is constructed, by taking the union of the convex sets $\mathcal{K}(x^\frown u_\epsilon(x))$, for all $x \in \mathbf{\Gamma}$. This domain may also be defined as

$$S(\epsilon) = \bigcup \{ \text{ segments } xy \text{ such that } x,\, y \in \mathbf{\Gamma} \text{ and } q(x^\frown y) \leq \epsilon \} \,.$$

Since $\mathbf{\Gamma}$ is continuous, $S(\epsilon)$ is a simply connected domain, which is much more adapted to the structure of $\mathbf{\Gamma}$ than the Minkowski sausage. It may be used for the same purpose; indeed,

8.1 Proposition *If* $\mathbf{\Gamma}$ *is expansive:*

$$\Delta(\mathbf{\Gamma}) = \limsup_{\epsilon \to 0} \left(2 - \frac{\log \mathcal{A}(S(\epsilon))}{\log \epsilon} \right) \,. \tag{41}$$

Proof of Proposition (8.1) Let $\epsilon > 0$, and $\mathbf{\Gamma}_1, \ldots, \mathbf{\Gamma}_N$ be a covering of $\mathbf{\Gamma}$ as in Definition 3.1. We have proved, in Proposition 5.1, that

$$\mathcal{A}(\cup \mathcal{K}(\mathbf{\Gamma}_i)) \simeq \mathcal{A}(\cup_{x \in \mathbf{\Gamma}} B_\epsilon(x)) \,. \tag{42}$$

Since $q(\mathbf{\Gamma}_i) \leq c\,\epsilon$ for all i, we have

$$\cup_{i=1}^{N} \mathcal{K}(\mathbf{\Gamma}_i)) \subset S(c_1 \epsilon) \,, \tag{43}$$

for some constant c_1. Let x_1, x_2 be two points of $\mathbf{\Gamma}$ such that $q(x_1{}^\frown x_2) \leq \epsilon$. For some constant c_2, every point of $\mathcal{K}(x_1{}^\frown x_2)$ is at a distance $\leq c_2\, \epsilon$ from $x_1{}^\frown x_2$. Therefore,

$$\mathcal{K}(x_1{}^\frown x_2) \subset \cup_{x \in \mathbf{\Gamma}} B_{c_2 \epsilon}(x) \,.$$

This implies

$$\cup_{i=1}^{N} \mathcal{K}(\mathbf{\Gamma}_i) \subset S(c\,\epsilon) \subset \cup_{x \in \mathbf{\Gamma}} B_{c_1 c_2 \epsilon}(x) \,.$$

With (42), we finally obtain

$$\begin{aligned}\mathcal{A}(\cup_{x\in\Gamma} B_\epsilon(x)) &\simeq \mathcal{A}(\cup_{i=1}^N \mathcal{K}(\mathbf{\Gamma}_i))\\ &\leq \mathcal{A}(S(c_1\epsilon))\\ &\leq \mathcal{A}(\cup_{x\in\Gamma} B_{c_1c_2\epsilon}(x))\,.\end{aligned}$$

We deduce

$$\liminf_{\epsilon\to 0} \frac{\log \mathcal{A}(S(\epsilon))}{\log \epsilon} = \liminf_{\epsilon\to 0} \frac{\log \mathcal{A}(\cup_{x\in\Gamma} B_\epsilon(x))}{\log \epsilon}\,,$$

which terminates the proof.

I wish to thank one of the referees for pointing out that convex hulls have already been used for a local study of certain irregular planar curves (e.g. geographical coastlines): the interested reader is refered to [**6**].

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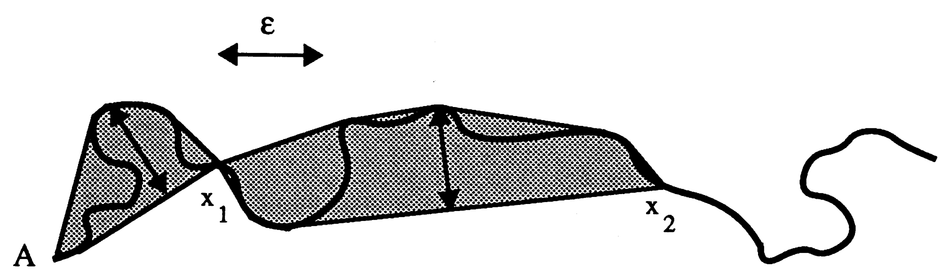

Figure 1

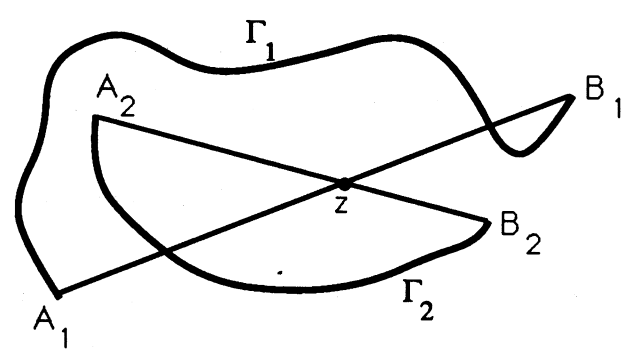

Figure 2

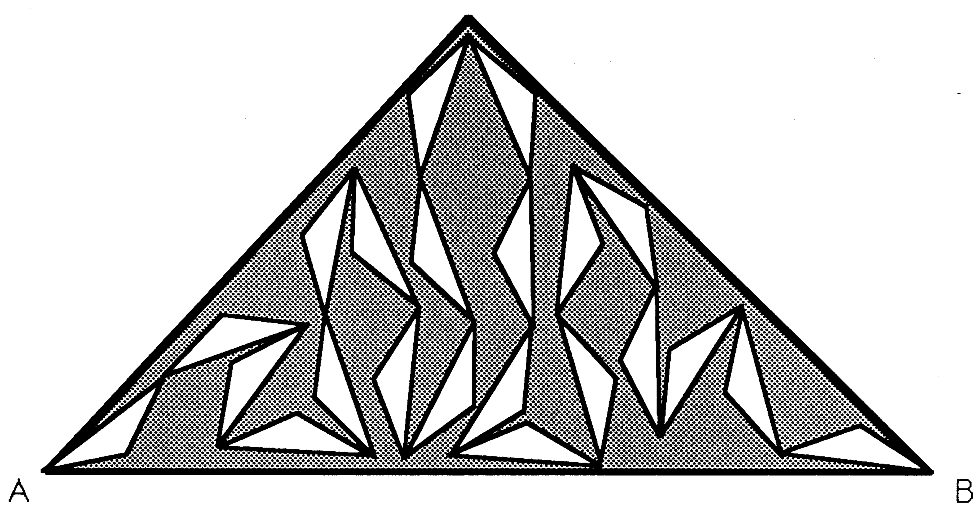

Figure 3

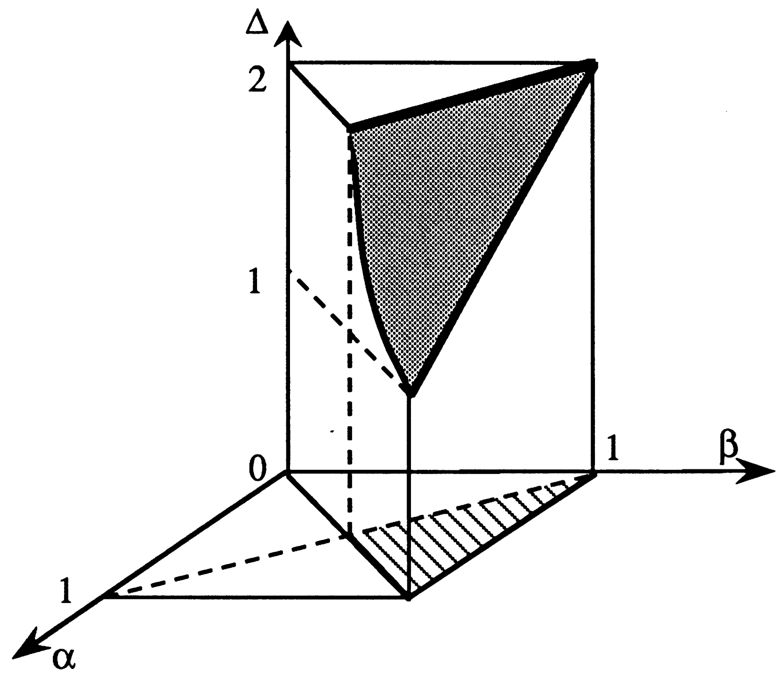

Figure 4