Real Analysis Exchange Vol 15 (1989-90)
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On Functions Discontinuous on Countable Sets

If f is a real valued function on $(0,1)$ such that each $x \in (0,1)$ is a left and right accumulation point of the set $f^{-1}(f(x))$, then (by a familiar argument) f is constant on any interval on which it is continuous. The question we pose here is this: must f be constant on any interval on which f is continuous at all but (possibly) countably many points? The answer is no. We will construct a bounded function F on $(0,1)$ such that F is continuous at each irrational point, discontinuous at each rational point in $(0,1)$, and such that each $x \in (0,1)$ is a left and right condensation point of $F^{-1}(F(x))$. To wit, each set $(0,x) \cap F^{-1}(F(x))$ and $(x,1) \cap F^{-1}(F(x))$ will meet any neighborhood of x in continuum many points.

By a <u>nondifferentiable</u> function on $(0,1)$ we mean a function that has no derivative, finite or infinite, at any point in $(0,1)$. By a <u>nowhere</u> <u>monotone</u> function on $(0,1)$ we mean a function that is not monotone on any subinterval of $(0,1)$. In [1] it is shown that a continuous function f, not constant on any interval, is nowhere monotone on $(0,1)$ if and only if the set

$$P_0 = \{y \in (\inf f, \sup f) : f^{-1}(y) \text{ is a perfect set}\}$$

is a residual subset of the interval $(\inf f, \sup f)$. In [2] it is shown that if f is a continuous nondifferentiable function on $[0,1]$ then the set $(\inf f, \sup f) \setminus P_0$ has measure zero. These results inspire our theorems:

Theorem 1. Let f be a function conditions on $(0,1)$ at all except possibly countably many points, and constant on no interval. Let f have an infinite unilateral derivative at no point. Put

$$P = \{y \in (\inf f, \sup f) : f^{-1}(y) \text{ equals a perfect set minus a countable set}\}.$$

Then the following are equivalent.

(1) $(\inf f, \sup f) \setminus P$ is a first category set,

(2) P is dense in the interval $(\inf f, \sup f)$,

(3) f is nowhere monotone on $(0,1)$.

Theorem 2. Let f be a function continuous on $(0,1)$ at all except possibly countably many points, and let f have an infinite unilateral derivative at no point. Let P be as in Theorem 1. Then the following are equivalent.

(4) The set of all points where f has a nonzero derivative has measure zero.

(5) The set $(\inf f, \sup f) \setminus P$ has measure zero.

It will follow that our example is a function F satisfying the hypotheses of Theorems 1 and 2 as well as the properties (1), (2), (3), (4) and (5). In this case, $P = (\inf F, \sup F)$.

For other examples, consult [0, Theorem 4].

Let g_1 be a function satisfying the hypotheses of Theorem 1 and also the properties (1), (2) and (3) of this theorem, such that $\inf g_1(0, \frac{1}{2}] = 0$, $\sup g_1(0, \frac{1}{2}] = 1$, and let g_2 be an increasing function on $(\frac{1}{2}, 1)$, discontinuous at a dense set of points in $(\frac{1}{2}, 1)$, such that $\inf g_2(\frac{1}{2}, 1) = 0$, and $\sup g_2(\frac{1}{2}, 1) = 1$. Let $g(x) = g_1(x)$ for $0 < x \le \frac{1}{2}$ and $g(x) = g_2(x)$ for $\frac{1}{2} < x < 1$. Then g satisfies properties (1) and (2), but not (3) of Theorem 1. Moreover, g satisfies all the hypotheses of Theorem 1 save one; g has an infinite unilateral derivative at each point of discontinuity of g_2.

We need some preliminary lemmas.

Lemma 1. If E is a second category subset of an interval I, then there exists a subinterval J of I such that $J \cap E$ is an uncountable and dense subset of J.

Proof. This is straight-forward, so we leave it. $\square$

Lemma 2. Let f be a function on $(0,1)$, continuous at all but possibly countably many points of $(0,1)$. Let $y \in f(0,1)$. Then $f^{-1}(y)$ equals a perfect set minus a countable set if and only if the set $f^{-1}(y)$ contains no isolated point.

Proof. Let x_0 be an isolated point of $f^{-1}(y)$ and let $f^{-1}(y) = A \setminus B$ where A is a perfect set and B is a countable set. Then $x_0 \in A$, so x_0 is a condensation point of A and likewise of $f^{-1}(y)$. But this is impossible.

Now let $f^{-1}(y)$ contain no isolated points. Then its closure $f^{-1}(y)^-$ is a perfect set. The only points in $f^{-1}(y)^- \setminus f^{-1}(y)$ are points at which f is discontinuous. By hypothesis $f^{-1}(y)^- \setminus f^{-1}(y)$ is a countable set. It follows that $f^{-1}(y)$ equals a perfect set $f^{-1}(y)^-$ minus a countable set $f^{-1}(y)^- \setminus f^{-1}(y)$. $\square$

We say that a function g is Darboux continuous on an interval I if whenever $a, b \in I$, $g(a) \neq g(b)$ and y lies between $g(a)$ and $g(b)$, there is an x between a and b such that $g(x) = y$.

Lemma 3. Let g be a function on the interval I such that g is continuous at all but (possibly) countably many points of I, and g has an infinite unilateral derivative at no point of I. Then g is Darboux continuous on I.

Proof. Let $a, b \in I$, $a < b$, $g(a) \neq g(b)$, and let y be a number between $g(a)$ and $g(b)$. Say for definiteness, $g(a) < y < g(b)$. The proof for the reverse inequalities is analogous. Suppose there is no $x \in (a, b)$ with $g(x) = y$. The sets $C = [a, b] \cap g^{-1}(-\infty, y)$ and $D = [a, b] \cap g^{-1}(y, \infty)$ are nonvoid. Moreover $[a, b] = C \cup D$, $C \cap D = \emptyset$, and because $[a, b]$ is connected, $\bar{C} \cap \bar{D} \neq \emptyset$ where $\bar{C}$ is the closure of C and $\bar{D}$ is the closure of D.

Let $u \in C \cap \bar{D}$ be a right (left) accumulation point of D. Then any neighborhood of u contains points in C and D to the right (left) of u because g does not have an infinite right (left) derivative at u; by the argument in the preceding paragraph, any neighborhood of u contains points of $\bar{C} \cap \bar{D}$ to the right (left) of u. So any point of $C \cap \bar{D}$ is an accumulation point of $\bar{C} \cap \bar{D}$. Likewise any point of $\bar{C} \cap D$ is an accumulation point of $\bar{C} \cap \bar{D}$. But $\bar{C} \cap \bar{D} = (C \cap \bar{D}) \cup (\bar{C} \cap D)$ because $C \cup D = [a, b]$. It follows that $\bar{C} \cap \bar{D}$ is a nonvoid perfect set. Moreover, g must be discontinous at each $x \in \bar{C} \cap \bar{D}$ because $g(x) \neq y$. Finally, $\bar{C} \cap \bar{D}$ can contain at most countably many points, and this is impossible. □

Lemma 4. Let g be a function on the interval $(0, 1)$, continuous at all points of $(0, 1)$ except (possibly) countably many. Let each $x \in (0, 1)$ be a left and right accumulation point of both the sets $g^{-1}[g(x), \infty)$ and $g^{-1}(-\infty, g(x)]$. Then each neighborhood of any $x \in (0, 1)$ contains continuum many points of the set $(0, x) \cap g^{-1}(g(x))$ and of the set $(x, 1) \cap g^{-1}(g(x))$.

Proof. We will prove only for $(x, 1) \cap g^{-1}(g(x))$. The proof for $(0, x) \cap g^{-1}(g(x))$ is analogous.

Let c be any number $> x$. Then (x, c) contains points of $g^{-1}(-\infty, g(x)]$ and of $g^{-1}[g(x), \infty)$. Moreover it follows from the hypothesis that g has an infinite unilateral derivative at no point. The interval (x, c) must contain points of $g^{-1}(g(x))$ by Lemma 3. Put

$$X = \{u \in (x, c) : g(u) = g(x)\}.$$

Then $X \neq \emptyset$.

Each point of X is a right accumulation point of X by the argument in the preceding paragraph. The closure of X, then, is a perfect set and has the power of the continuum. At each point $v \in \bar{X}$, where g is continuous, $g(v) = g(x)$. It follows that $g(v) = g(x)$ for continuum many points $v \in \bar{X}$. But $X \subset (x, c)$, so $\bar{X} \subset [x, c]$. Thus the interval $[x, c]$ contains continuum many points v for which $g(v) = g(x)$. □

Proof of Theorem 1. We will prove (1) ⇒ (2) ⇒ (3) ⇒ (1).

(3) ⇒ (1). Let $(\inf f, \sup f) \setminus P$ be a second category set. By Lemma 2, the set $f^{-1}(y)$ contains an isolated point if $y \in (\inf f, \sup f) \setminus P$. For each $y \in (\inf f, \sup f) \setminus P$ select an isolated point $x(y)$ in $f^{-1}(y)$. Let $X = \{x(y) : y \in (\inf f, \sup f) \setminus P\}$. Then $f(X)$ is a second category set.

It follows that there is a $c > 0$ such that $f(W)$ is a second category set where $W = \{x \in X :$ the distance from$\{x\}$ to $f^{-1}(f(x)) \setminus \{x\} > c\}$. Now $(0,1)$ is the union of finitely many intervals of length $< c$, so there exist $a, b \in (0,1)$ $(a < b)$ such that $b - a < c$ and $f((a,b) \cap W)$ is a second category set. Note that if $x \in (a,b) \cap W$, then

$$(*) \qquad \{x\} = (a,b) \cap f^{-1}(f(x)).$$

By Lemma 1, there is an open interval J such that $f((a,b) \cap W) \cap J$ is an uncountable dense subset of J. Choose an $x_0 \in (a,b) \cap W$ such that f is continuous at x_0 and $f(x_0) \in J$. Let I be an open subinterval of (a,b) containing x_0 such that $f(I) \subset J$.

We claim that there exist no points $r, s, t (r < s < t)$ in I such that either $f(s) < \min(f(r), f(t))$ or $f(s) > \max(f(r), f(t))$. For otherwise there is a $y \in f((a,b) \cap W) \cap J$ such that $y \in (f(s), \min(f(r), f(t)))$ or $y \in (\max(f(r), f(t)), f(s))$; in either case, by Lemma 3, there exist $x_1 \in (r,s), x_2 \in (s,t)$ such that $y = f(x_1) = f(x_2)$, contrary to $(*)$. It follows that f is monotone on the interval I.

(1) ⇒ (2). Clear.

(2) ⇒ (3). Let f be monotone on the open subinterval I of $(0,1)$. Then f must be nonconstant on I by hypothesis, and f must be continuous on I; for if f is discontinuous at any point $x \in I$, then f has an infinite unilateral derivative at x. Each point $u \in I$ is an isolated point of the set $f^{-1}(f(u))$, so $P \cap f(I) = \emptyset$. Thus P is not dense in the interval $(\inf f, \sup f)$. □

Let f be continuous on the compact interval $[0,1]$. Then $f^{-1}(y)$ is a closed set for any y in the range of f. If $f^{-1}(y) = A \setminus B$ where A is perfect and B is countable, then any point in $f^{-1}(y)$ is a condensation point of A and of $f^{-1}(y)$;

it follows that $f^{-1}(y)$ is a perfect set. Moreover, f is Darboux continuous so our Lemma 3 is not needed here. A review of our arguments with these points in mind shows that we can restate some result in [1] as follows.

Corollary 1 (K.M. Garg). Let f be continuous on $[0,1]$ and constant on no interval. Let

$$P_0 = \{y \in (\inf f, \sup f) : f^{-1}(y) \text{ is a perfect set}\}.$$

Then the following are equivalent.

(1′) $(\inf f, \sup f) \setminus P_0$ is a first category set.

(2′) P_0 is dense in the interval $(\inf f, \sup f)$.

(3′) f is nowhere monotone.

Proof of Theorem 2.

(4) ⇒ (5). Assume that $(\inf f, \sup f) \setminus P$ does not have measure zero. Let Y be the set of all $y \in (\inf f, \sup f) \setminus P$ such that $f^{-1}(y)$ does not contain a point where f has a strict relative maximum or strict relative minimum. Then Y does not have measure zero. By Lemmas 2 and 3, we can (and do) choose for each $y \in Y$, a point $x(y) \in f^{-1}(y)$ such that $f(t) - f(x(y))$ changes sign as t passes through $x(y)$. Thus either the bilateral lower derivate of f is nonnegative at $x(y)$, or the bilateral upper derivate of f is nonpositive at $x(y)$. Let $X = \{x(y) : y \in Y\}$. By a theorem of Denjoy-Young-Saks [3, Theorem 4.2, p. 270] f is differentiable almost everywhere on X. But $f(X) = Y$, and f' is not infinite at any point of X. It follows from [3, Theorem 4.5, p. 271] that f' is nonzero on a subset of X, not of measure zero.

(5) ⇒ (4). Assume that the set of points where f' exists and is nonzero does not have measure zero. Without loss of generality we assume that f' is positive on a set not of measure zero. Then there is a $c > 0$ such that the set $\{x : f'(x) > c\}$ is not of measure zero. It follows that $\{x : f'(x) > c\}$ has a subset E, not of measure zero, such that $f(x) - cx$ is an increasing function on E. (To see this, observe that there is a set $E_1 \subset \{x : f'(x) > c\}$, not of measure zero, and a number $d > 0$, such that $f(x_1) - f(x_2) > c(x_1 - x_2)$ for $x_1 \in E_1, x_2 \in E_1$ and $0 < x_1 - x_2 < d$; let E be a subset of E_1, not of measure zero, such that diameter $E < d$.) Then there exists an increasing function g on $(0,1)$ such that $g(x) - cx$ is increasing on E and $g(x) = f(x)$ for all $x \in E$. Hence $g' \geq c$ almost everywhere on E, so by [3, Lemma 9.4, p. 126], $g(E)$, and hence $f(E)$, do not have measure zero. By Lemma 2, $f(E) \cap P = \emptyset$ because each $x \in E$ is an isolated point of the set $f^{-1}(f(x))$. Hence the set $(\inf f, \sup f) \setminus P$ cannot have measure zero. □

Let f be a continuous function on the compact interval $[0,1]$. It is easy to show that $y \in P$ if and only if $f^{-1}(y)$ is a perfect set. Moreover, f is Darboux continuous so our Lemma 3 is not needed here. With these observations in mind, we can restate the results in [2] as follows.

Corollary 2 (S. Minakshisundaram). Let f be continuous on $[0,1]$, and let f have an infinite (bilateral) derivative at no point. Let

$$P_0 = \{y \in (\inf f, \sup f) : f^{-1}(y) \text{ is a perfect set}\}.$$

Then the following are equivalent.

(4′) The set of points where f has a nonzero derivative has measure zero.

(5′) The set $(\inf f, \sup f) \setminus P_0$ has measure zero.

In all that follows, E will be a countable dense subset of $(0,1)$. Let $e_1, e_2, e_3, \ldots, e_n, \ldots$ be an enumeration of E.

Definition. Let X be a closed set. We say that $u, v \in X$ $(u < v)$ are consecutive points of X if $(u,v) \cap X = \emptyset$.

Note that (u,v) is a component interval of the open set $R \setminus X$.

Now let $0 \le a < b \le 1$ and let f be a function defined on the doubleton set $\{a,b\}$ such that $f(a) \ne f(b)$. Let e be some element of E with $a < e < b$ and define $h(a) = f(a), h(b) = f(b), h(e) = \frac{1}{2}(f(a) + f(b))$. We call h an even extension of f on $\{a,b\}$.

Let $\{c_n\}_{-\infty}^{\infty} \subset E \cap (a,b)$ be elements of E such that $c_i < c_j$ if $i < j$ and such that $\lim_{n\to\infty} c_n = b$, $\lim_{n\to-\infty} c_n = a$. Define $k(a) = f(a), k(b) = f(b), k(c_n) = f(a)$ for n even and $k(c_n) = f(b)$ for n odd. We call k an odd extension of f on $\{a,b\}$.

More generally, let f be defined on a closed subset X of $[a,b]$, such that $f(u) \ne f(v)$ whenever $u, v \in X$ $(u < v)$ are consecutive points of X.

Definition. Let h be the common extension of f together with an even (odd) extension of f on $\{u,v\}$ for each pair of consecutive points u, v $(u < v)$ of X. We call h an even (odd) extension of f on X.

Note that the domain of h is also closed. Moreover, if u, v $(u < v)$ are consecutive points of the domain of h, then $h(u) \ne h(v)$.

Let f_0 be the function defined on the doubleton set $\{0,1\} = E_0$ such that $f_0(0) = 0$, $f_0(1) = 1$. Let f_1 be an odd extension of f_0 such that the domain E_1 of f_1 contains e_1. Let f_2 be an even extension of f_1 such that the domain E_2 of f_2 contains e_2. Let f_3 be an odd extension of f_2 such that the domain E_3 of f_3 contains e_3. Let f_4 be an even extension of f_3 such that the domain E_4 of f_4 contains e_4. And so forth. In general, f_n is an even or odd extension of f_{n-1} depending on whether n is even or odd. In either case, the domain E_n if f_n contains e_n.

Let f be the greatest common extension of all the function f_n $(n = 1,2,3,\ldots)$ on $E \cup \{0,1\} = \bigcup_{n=0}^{\infty} E_n$. Let L_n denote the value $|f_n(u) - f_n(v)|$ where $u, v \in E_n$ $(u < v)$ are consecutive points of E_n. By construction, $L_n = L_{n-1}$ if n is odd, and $L_n = \frac{1}{2}L_{n-1}$ if n is even.

The domain of f is dense in $[0,1]$. Let $x \in (0,1) \setminus E$. Let u_n, v_n $(u_n < v_n)$ be the consecutive points of E_n for which $u_n < x < v_n$. Then

$$\max\{f(v_n), f(u_n)\} - \min\{f(v_n), f(u_n)\} = L_n$$

and $\lim L_n = 0$. By construction $f((u_n, v_n) \cap E)$ is a subset of the closed interval joining $f(u_n)$ to $f(v_n)$. It follows that the limit $\lim_{e\in E, e\to x} f(e)$ exists. Define $F(x)$ to be this limit. For $t \in E \cup \{0,1\}$ put $F(t) = f(t)$. Thus F is defined on $[0,1]$ and $0 \leq F \leq 1$. It also follows that F is continuous at each point of $(0,1) \setminus E$. Because

$$\min(F(u_n), F(v_n)) \leq F(x) \leq \max(F(u_n), F(v_n)),$$

it follows that if n is odd, there exist $t_1, t_2 \in (u_n, x)$ and $t_3, t_4 \in (x, v_n)$ such that $F(t_1) \leq F(x) \leq F(t_2)$ and $F(t_3) \leq F(x) \leq F(t_4)$. Hence x is a left and right accumulation point of each set $F^{-1}[F(x), \infty)$ and $F^{-1}(-\infty, F(x)]$.

Now let $s \in E$. Say $s \in E_n \setminus E_{n-1}$. It is clear that F is left and right discontinuous at s; consider f_{n+1} if n is even and f_{n+2} if n is odd. Thus F is discontinuous at the points in $E \cup \{0,1\}$ and at no other points. Moreover, f_{n+1} or f_{n+2} shows that s is a left and right accumulation point of the set $F^{-1}(F(s))$. Finally by Lemma 4, it follows that each neighborhood of any $x \in (0,1)$ contains continuum many points of the set $(0,x) \cup F^{-1}(F(x))$ and of the set $(x,1) \cap F^{-1}(F(x))$.

We note that F can have no nonzero unilateral derivative at any point in $(0,1)$, and F satisfies the hypotheses of Theorems 1 and 2 as well as the conditions (1), (2), (3), (4), (5) of these Theorems.

We see that at each point of $(0,1)$, the upper Dini derivates D^-F and D^+F are nonnegative and the lower Dini derivates D_-F and D_+F are nonpositive.

It is easy to show that F attains its maximum and minimum values on any subinterval I of $(0,1)$, closed or not. To prove this observe that $\sup F(I) =$

$\sup F(E \cap I)$ and $\inf F(I) = \inf F(E \cap I)$, and examine carefully the functions f_n. We will leave the argument.

In conclusion, a necessary condition that g be nondecreasing on $(0, 1)$ is that g be continuous at all but at most countably many points and the upper Dini derivates of g be everywhere nonnegative. But this condition is not sufficient for g to be nondecreasing on $(0, 1)$, as our example F showed.

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Received 19 September, 1989