Real Analysis Exchange Vol 15 (1989-90)

ON ω-LIMIT SETS FOR VARIOUS CLASSES OF FUNCTIONS

by

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Introduction

In the study of the iterative behaviour of a self-map f of a closed interval, say $[0,1]$, one is often interested in the ω-limit sets which arise, that is, the sets $\omega(x,f)$ which are the cluster sets of the sequences of iterates $\{f^n(x)\}_{n=0}^{\infty}$. There are, of course, many possibilities for such sets. For example, the function $f(x) = kx(1-x)$ has only the set $\{0\}$ as an ω-limit set if $0 < k < 1$, but it has a wide variety of ω-limit sets when $k = 4$, including $[0,1]$ itself [*cf. D*].

On the other hand, in [ABCP] a continuous function $f : [0,1] \to [0,1]$ was constructed having a rich system of ω-limit sets in the sense that to each nowhere dense closed set M there corresponds an x such that $\omega(x,f) \cap \left[\frac{1}{3},\frac{2}{3}\right]$ is homeomorphic to M.

This result and others suggest the following questions:

(1) What sets can be ω-limit sets for continuous functions?

(2) What families of closed sets can constitute the family of all ω-limit sets for a single continuous function?

(3) What are the answers to the corresponding questions for other families of functions?

Question (1) has been answered in [ABCP] by the following:

Theorem 1. *A nonvoid subset E of $[0,1]$ is an ω-limit set for a continuous function $f : [0,1] \to [0,1]$ if and only if E is either a closed, nowhere dense set, or it is a union of finitely many nondegenerate closed intervals.*

We have found very little work in the literature related to Question (2). There are some fragmented results, but not enough to suggest any promising conjectures.

Question (3) does not seem to have been posed or studied before. In fact, the vast literature on iterative behaviour dealing with self-maps of a closed interval focuses almost exclusively on continuous functions, often assuming a great deal of additional regularity. Nevertheless, some applications arise naturally in which the continuity restriction may be inappropriate. For example, the familiar Newton's method for locating zeros leads to the iteration

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

If $g(x) = x - (f(x)/f'(x))$ and g is defined on $[0,1]$ and maps $[0,1]$ into $[0,1]$, we are led to the kind of problem under consideration. The natural condition for the application of Newton's method is to assume no more than just differentiability for f. If the resultant function g maps $[0,1]$ into $[0,1]$, then g will be Darboux Baire 1 but not necessarily continuous. Therefore it seems appropriate to study the iterative behaviour of Darboux Baire 1 functions. We know of no natural smaller class of functions containing all functions g of the form $g(x) = x - (f(x)/f'(x))$ mapping $[0,1]$ into $[0,1]$.

The main purpose of this article is to study Question (3) for several classes $\mathcal{E}$ of functions larger than $\mathcal{C}$, the class of continuous functions. In particular, we answer the analogue of Question (1) for any class containing $\mathcal{DB}_1$ and we obtain partial answers to Question (2) for certain classes containing $\mathcal{DB}_1$. We get a complete answer to the analogue of Question (2) for the class of measurable functions having the Darboux property.

Notation and Terminology

Throughout the sequel I will denote the interval $[0,1]$. Unless otherwise specified, all functions considered will be from I to I. For $f : I \to I$ and $x \in I$ we define $f^0(x) = x$ and $f^{n+1}(x) = f(f^n(x))$. We put $\gamma(x,f) = \{f^n(x) : n \in \omega_0\}$ and $\omega(x,f)$, the ω-limit set for f at x, is defined to be the cluster set (i.e., the set of all subsequential limit points) of the sequence $\{f^n(x)\}_{n=0}^{\infty}$. By $\Lambda(f)$ we mean the set of all ω-limit sets of f. We use $cl\, A$ or $\overline{A}$ to denote the closure of a set A.

A function $f : I \to I$ is Darboux if it maps intervals onto intervals. A function f is in Baire (or Borel) class 1 (resp., 2) if and only if f is a pointwise limit of a sequence of continuous functions (resp., functions of Baire class 1). This is equivalent to the condition that $f^{-1}(G)$ is an F_σ- set (resp., $G_{\delta\sigma}$- set) for each open set G. We denote these three classes by $\mathcal{D}$, $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. It is well known that $f \in \mathcal{B}_1$ if and only if the restriction of f to each perfect set has a point of continuity. Moreover, the set of points of continuity of a Baire 1 function is a residual G_δ - set. For facts about $\mathcal{D}\mathcal{B}_1$ functions see the survey paper [CP]. In particular, the class $\mathcal{D}\mathcal{B}_1$ contains all derivatives and is closed under multiplication by, and addition of, continuous functions.

Darboux Baire 1 Functions

The example from [ABCP] cited in the introduction can be improved for functions in $\mathcal{D}\mathcal{B}_1$.

Example 1. There exists a Darboux, Baire 1 function f such that $\Lambda(f)$ contains a homeomorph of each nonempty, nowhere dense closed set.

Proof. We will use the ternary representation for points in $[\frac{1}{3},\frac{2}{3}]$. Thus, $[\frac{1}{3},\frac{2}{3}]$ is $[.1,.2]$.

Let S consist of $[.1,.2]$ minus the open interval $(.11\bar{0},.12\bar{0})$ and all open intervals of the form $(.a_1a_2\ldots a_n11\bar{0},\ .a_1a_2\ldots a_n12\bar{0})$. Then S is closed and, with the exception of the left-hand endpoints of the deleted intervals, it consists of all points in $[.1,.2]$ whose representation lacks two consecutive 1's. If $x \in S$, $x = .x_1 x_2 \ldots$ and $y_n = .x_1 x_2 \ldots x_n \bar{1}$, then $y_n \to x$ and $y_n \notin S$. Hence S is nowhere dense.

Let $\mathcal{Q}$ consist of all nonempty blocks of 0's and 2's, finite or infinite. If $A \in \mathcal{Q}$, let $\rho(A)$ denote the length of A. For each $x \in S$ there exist blocks $A_i \in \mathcal{Q}$ such that either

$$x = .1A_1 1A_2 \ldots 1A_n \ldots$$

or

$$x = .1A_1 1A_2 \ldots 1A_m .$$

Now let $S_o = \{x : x = .1A_i \text{ for some } A_i \in \mathcal{Q}\}$. Then $S_o \subseteq S$. Also, S_o is $\left[\frac{1}{3}, \frac{2}{3}\right]$ minus all open intervals of the form $\left(.1a_2 \ldots a_n 0\bar{2}, .1a_2 \ldots a_n 2\bar{0}\right)$. Hence S_o is closed. Moreover, S_o is nowhere dense relative to S. To see this, suppose $x = .1x_2 x_3 \ldots$ belongs to S_o. Let $y_n = .1x_2 \ldots x_n 101 x_{n+4} \ldots$. Then $y_n \to x$ but $y_n \in S - S_o$.

Next we define a function $g : S \to I$ by

$$g(x) = \begin{cases} .1A_2 1A_3 \ldots & \text{if } x = .1A_1 1A_2 1A_3 \ldots \\ .1A_2 \ldots 1A_m & \text{if } x = .1A_1 \ldots 1A_m \\ .1 & \text{if } x = .1A_1 \end{cases} .$$

Suppose $x \in S - S_o$. Then $x = .1A_1 \ldots 1A_m$ or $x = .1A_1 1A_2 \ldots$, where $m > 1$. Put $k = 1 + \rho(A_1) + \rho(A_2)$. Suppose $y \in S$ and $|y - x| < 3^{-(m+1)}$ and $m \geq k$. Then x and y agree in the first m ternary positions and consequently $g(x)$ and $g(y)$ agree in the first $m - (1 + \rho(A_1))$ ternary positions. Hence

$$|g(x) - g(y)| < 3^{-(m+1-1-\rho(A_1))} = 3^{-m+\rho(A_1)} .$$

So, given $\epsilon > 0$ choose $m \geq k$ such that $3^{-m+\rho(A_1)} < \epsilon$ and put $\delta = 3^{-(m+1)}$. Then $|x - y| < \delta$ implies $|g(x) - g(y)| < \epsilon$, i.e., g is continuous at each point of $S - S_o$. Since $g|(S - S_o)$ is continuous and g is constant on S_o, it follows that g is Baire 1 on S.

Now let

$$h(x) = \begin{cases} g(x) & \text{if } x \in S - \{.2\} \\ .2 & \text{if } x = .2 \\ .1 & \text{if } x \in [.1,.2] - S \end{cases} .$$

Then h is Baire 1. By Proposition 1 of [BCK] there exists a function f in $\mathcal{DB}_1$ such that $f : [.1,.2] \to [.1,.2]$ and $f = h$ on S. (Note that both .1 and .2 are fixed points for f, a fact we shall need in Example 2.)

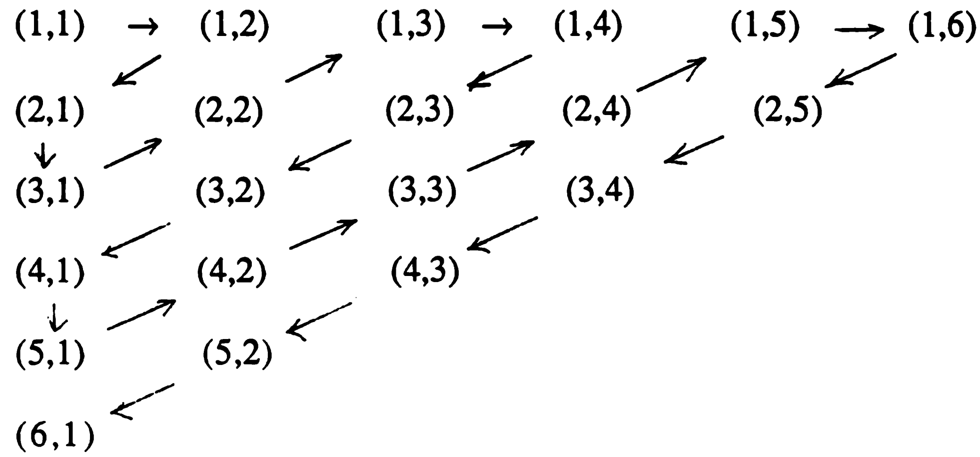
Now suppose M is any closed, nowhere dense subset of I. Then there exists a subset M_o of the Cantor set S_o such that M_o is homeomorphic to M. We can find a countable subfamily $\mathcal{K}$ of $\mathcal{Q}$ such that the closure of the set $\{.1B\,\overline{0} : B \in \mathcal{K}\}$ is exactly M_o; in addition we may specify that $\rho(B) < \infty$ for each $B \in \mathcal{K}$. Enumerate $\mathcal{K}$ as $\{B_n\}_{n=1}^{\infty}$ where $\rho(B_n) \le \rho(B_{n+1})$ for each n, and put $x = .1B_1 1B_2 \ldots 1B_n \ldots$. Then $x \in S - S_o$ and $f^{n+1}(x) = .1B_n 1B_{n+1} \ldots$ is also in $S - S_o$. Since $\left|f^{n+1}(x) - .1B_n\overline{0}\right| \le 3^{-(1+\rho(B_n))}$ and $\rho(B_n) \to \infty$, it follows that $\{f^n(x)\}_{n=0}^{\infty}$ has M_o as its cluster set. That is, $\omega(x,f) = M_o$.

We have been unable to determine whether a continuous function exists satisfying the condition of Example 1.

According to Theorem 1 an ω-limit set for a continuous function is either nowhere dense and closed, or it is a finite union of closed intervals; it cannot be a combination of these two types of sets. However, for Darboux Baire 1 functions such combinations are possible, as shown by the next theorem. First, we need a lemma.

<u>Lemma 1</u>. *There exists a sequence $\{a_n\}_{n=1}^{\infty}$ of positive integers such that (1) each positive integer is cofinal in $\{a_n\}_{n=1}^{\infty}$, and (2) any pair k,m of positive integers appears at most two times as consecutive terms in $\{a_n\}_{n=1}^{\infty}$.*

<u>Proof</u>. Let N denote the set of positive integers. Well-order $N \times N$ as indicated in the following array.



Let $\{a_n\}_{n=1}^{\infty}$ be the sequence obtained by lining up the coordinates of the above pairs in the order given. Thus, $\{a_n\}_{n=1}^{\infty}$ will be 1,1,1,2,2,1, 3,1,2,2,1,3,1,4,2,3,3,2,4,1,... . It is easily verified that $\{a_n\}_{n=1}^{\infty}$ has the desired properties.

Theorem 2. *Any nonempty, closed subset of I is an $\omega-\text{limit}$ set for some Darboux, Baire 1 function.*

Proof. Let F be a closed, non-empty subset of I. If F is nowhere dense then by Theorem 1 F is an $\omega-\text{limit}$ set for some continuous function. So we may assume that F has nonempty interior.

Let $c \in (a,b) \subseteq F$. Choose monotonic sequences $\{a_n\}_{n=1}^{\infty}$ in (a,c) and $\{b_n\}_{n=1}^{\infty}$ in (c,b) such that $a_n \to c$ and $b_n \to c$. Let $J_1 = [\inf F, a_1)$, $J_2 = (b_1, \sup F]$, $J_{2n+1} = (a_n, a_{n+1})$ and $J_{2n} = (b_{n+1}, b_n)$ for each $n > 1$. For each n let $\{d_{m,n}\}_{m=1}^{\infty}$ be a sequence in J_n whose cluster set is $F \cap \overline{J_n}$. Let D be the set of all possible $d_{m,n}$.

Now let $\{a_n\}_{n=1}^{\infty}$ be a sequence as specified in Lemma 1. Define b_n to be $1+\text{card}\{k : k < n,\ a_k = a_n\}$. Then we may enumerate D as $\{d_{a_n, b_n}\}$ where $d_{a_n, b_n} \neq d_{a_m, b_m}$ whenever $m \neq n$. Define g as follows:

$$g(x)=\begin{cases} d_{a_1,b_1}=d_{11} & \text{if } x=c \\ d_{a_{n+1},b_{n+1}} & \text{if } x=d_{a_n,b_n} \\ c & \text{if } x\notin D\cup\{c\}\ . \end{cases}$$

It is easily seen that $F=\omega(x,g)$ for each x in D. In fact, $g^n(d_{11})=d_{a_n,b_n}$ for each n. From property (2) of the sequence $\{a_n\}_{n=1}^{\infty}$, for any m and k there can be at most two pairs of consecutive iterates of the form $d_{m,\alpha}, d_{k,\beta}$. Therefore, for any m and k the graph of g intersects $J_m \times J_k$ at most twice. From this it is easy to see that for an open set G, (1) $g^{-1}(G)$ is finite or a sequence converging to c whenever G misses c and (2) $g^{-1}(G)$ is the complement of a finite set or a sequence converging to c when $c\in G$. In either case $g^{-1}(G)$ is an F_σ-set, so g is Baire 1.

By Proposition 1 of [BCK] there exists a Darboux Baire 1 function f such that $f=g$ except on a null, meager set which misses D. Then clearly $\omega(d_{11},f)=F$. This completes the proof.

It is natural to ask whether the results of Example 1 and Theorem 2 can be combined. Specifically, does there exist a Darboux Baire 1 function f such that $\Lambda(f)$ contains a homeomorphic copy of every nonempty closed set? We do not have a complete answer to this question. However, Example 2, below, shows that functions in $\mathcal{DB}_1$ can produce copies of all the ω-limit sets for continuous functions (see Theorem 1).

<u>*Example 2*</u>. *There exists a Darboux Baire 1 function f such that $\Lambda(f)$ contains homeomorphic copies of all possible ω-limit sets for continuous functions.*

<u>Proof</u>. Let $I_o=\left[\frac{1}{2},1\right]$ and $I_k=\left[\frac{1}{k+2},\frac{1}{k+1}\right]$ for $k\geq 1$. From Example 1 there exists a $\mathcal{DB}_1$ function f_o such that $f_o\left(\frac{1}{2}\right)=\frac{1}{2}$, $f_o(1)=1$ and $\Lambda(f_o)$ contains a homeomorph of each closed nowhere dense set.

For each k choose F_k to be a subset of the interior of I_k which is a union of k disjoint nondegenerate closed intervals. By Lemma 9 of [ABCP] it follows that there exists a continuous function $f_k: I_k \to I_k$ such that $F_k\in\Lambda(f_k)$ and the endpoints of I_k are fixed points of f_k.

Now put $f(0)=0$ and $f(x)=f_n(x)$ for $x \in I_n$. It is readily checked that $f \in \mathcal{DB}_1$ and $\bigcup_{n=0}^{\infty} \Lambda(f_n) = \Lambda(f)$. By Theorem 1, f has the desired properties.

Baire 1 and Darboux Baire 2 functions

By relaxing the requirements for membership in $\mathcal{DB}_1$ we arrive naturally at the classes $\mathcal{DB}_2$ and $\mathcal{B}_1$. Since each contains $\mathcal{DB}_1$ we know from Theorem 2 that each nonvoid closed set is realizable as an ω-limit set for functions in each of these classes. This raises the question of determining the possible families $\Lambda(f)$ for f in these classes. We have some partial results which show that various possibilities exist for these classes which do not exist for functions in $\mathcal{DB}_1$.

Our results show, in particular, that *any* nonvoid closed set can be the *only* ω-limit set for a function in $\mathcal{DB}_2$ or in $\mathcal{B}_1$. Since a function in $\mathcal{DB}_1$ has a fixed point this situation is possible for a $\mathcal{DB}_1$ function only when the given closed set is a singleton.

Theorem 3. If $\mathcal{Q}$ is a nonvoid countable family of nonvoid closed sets, then there exists a Darboux Baire 2 function f such that $\Lambda(f) = \mathcal{Q}$.

Proof: We give a proof for $\mathcal{Q}$ infinite. The modifications necessary for the finite case will be apparent. Let $\{A_n\}_{n=1}^{\infty}$ be an enumeration of $\mathcal{Q}$.

First we may pick a sequence of mutually disjoint countably infinite sets $\{D_m\}_{m=1}^{\infty}$ such that $D_m' = A_m$ for each m. (See the proof of Theorem 6 for the details.) Put $D = \bigcup_{m=1}^{\infty} D_m$ and enumerate each D_m as $\{d_{m,k}\}_{k=1}^{\infty}$.

Since each uncountable Borel set contains a Cantor set we can find a sequence $\{C_k\}_{k=1}^{\infty}$ of mutually disjoint Cantor sets such that $C_k \subseteq I_k - D$ for each k, where $\{I_k\}_{k=1}^{\infty}$ is the set of all open rational intervals. Let $C = \bigcup_{k=1}^{\infty} C_k$. Let g_k be a continuous function from C_k onto I. Define h_k by $h_k(x) = x$ for $x \in I - \bigcup_{j=k}^{\infty} C_j$ and $h_k(x) = c$ for $x \in \bigcup_{j=k}^{\infty} C_j$, where $c \notin C$. Put $f_k = h_k \circ g_k$. Then f_k is a Baire 2 function mapping C_k onto $I - \bigcup_{j=k}^{\infty} C_j$.

Now define f as follows:

$$f(x) = \begin{cases} d_{m,k+1} & \text{if } x = d_{mk} \\ f_k(x) & \text{if } x \in C_k \\ d_{11} & \text{if } x \in I - C \,. \end{cases}$$

Then it is easily checked that f is Baire 2. Since f maps each interval onto I f will also be Darboux. Moreover, $\Lambda(f) = \mathcal{Q}$.

For Theorem 3 to hold for a family $\mathcal{E}$ of functions, $\mathcal{E}$ must have members whose graphs are dense in $I \times I$. To see this choose $\{a_n\}_{n=0}^{\infty}$ to be a sequence of distinct points in I such that $\{(a_{2n}, a_{2n+1}) : n \in \omega_o\}$ is dense in $I \times I$. Put $\mathcal{Q} = \{\{(a_{2n}, a_{2n+1})\} : n \in \omega_o\}$. Let J and H be any two open subintervals of I. Choose n so that $(a_{2n}, a_{2n+1}) \in J \times H$. If for some x and f, $\omega(x,f) = \{a_{2n}, a_{2n+1}\}$ for some n, then $\{f^k(x)\}_{k=0}^{\infty}$ is eventually in $J \cup H$ and frequently in each of J and H. This implies that the intersection of the graph of f with $J \times H$ is nonempty. Hence the graph of f is dense in $I \times I$. As a consequence f is discontinuous everywhere and hence, cannot be a Baire 1 function.

It follows that Theorem 3 cannot be valid when $\mathcal{DB}_2$ is replaced by $\mathcal{B}_1$ even in the case when $\mathcal{Q}$ is a disjoint family. But Theorem 3 is valid for Baire 1 functions when $\mathcal{Q}$ is a finite disjoint family.

<u>Theorem 4</u>. *If $\mathcal{Q}$ is a finite nonempty family of nonempty mutually disjoint closed sets, then there exists a Baire 1 function f such that $\Lambda(f) = \mathcal{Q}$.*

<u>Proof</u>: If $\mathcal{Q} = \{F\}$, then the function g constructed in the proof of Theorem 2 has the property that $F = \omega(x,g)$ for all $x \in I$. Suppose $\mathcal{Q} = \{F_1, \ldots, F_n\}$. Let c_n and g_n correspond to the special point c and function g constructed in Theorem 2, with F_n being the counterpart of F. Let $W_1, \ldots, W_n$ be disjoint open sets such that $F_n \subseteq W_n$. Define f as follows:

$$f(x)=\begin{cases} g_k(x) & \text{if } x\in W_k \\ c_1 & \text{if } x\notin \bigcup_{k=1}^{n} W_k \end{cases}$$

Then f is Baire 1 and $\Lambda(f) = \mathcal{Q}$.

We can prove a more restrictive countable version of Theorem 3 for Baire 1 functions.

<u>Theorem 5</u>. If $\mathcal{Q}$ is a nonempty countable family of nonempty closed sets such that $\cup\,\mathcal{Q}$ is nowhere dense, then there exists a Baire 1 function f such that $\Lambda(f) = \mathcal{Q}$.

<u>Proof</u>: For any set A let $B_n(A)=\{x : dist(x,A)<n^{-1}\}$. Enumerate $\mathcal{Q}$ as $\{K_n\}_{n=1}^{\infty}$. (If $\mathcal{Q}$ is finite the necessary modifications will be clear.)

Since $\bigcup_{n=1}^{\infty} K_n$ is nowhere dense we will be able to find a sequence $\{W_n\}_{n=1}^{\infty}$ of mutually disjoint open sets such that (1) $K_n\cap W_m=\phi$ for each n and m; (2) $K_n\subseteq\overline{W_n}$ for each n; (3) $W_n\subseteq B_n(K_n)$ for each n, and (4) $W_n\cap W_m=\phi$ for $n\neq m$.

To prove this we proceed as follows: First we construct a sequence $\{d_{1m}\}_{m=1}^{\infty}$ in $B_1(K_1)-cl\left(\bigcup_{n=1}^{\infty} K_n\right)$ whose cluster set is K_1, and such that $d_{1m}\neq d_{1j}$ when $m\neq j$. To accomplish this let E be a countable dense subset of K_1 and let $\{S_m\}_{m=1}^{\infty}$ be an enumeration of $\{B_n(x): x\in E,\ n\in\omega_o-\{0\}\}$. If $S_n=B_k(x)$, put $T_n=B_m(x)$ where $m=n+k$. Then $(\operatorname{diam} T_n)\to 0$. By induction pick $d_{11}\in T_1\cap B_1(K_1)-cl\left(\bigcup_{n=1}^{\infty} K_n\right)$. Having selected d_{1j} for $j<k$ pick d_{1k} in $T_k\cap B_1(K_1)-cl\left(\bigcup_{n=1}^{\infty} K_n\right)-\{d_{1j} : i<k\}$. Clearly $\{d_{1m}\}_{m=1}^{\infty}$ has K_1 as its cluster set.

Likewise we may choose $\{d_{2m}\}_{m=1}^{\infty}$ to be a sequence in $B_2(K_2)-cl\left(\bigcup_{n=1}^{\infty} K_n\right)-\{d_{1m} : m\geq 1\}$ whose cluster set is K_2 and such that $d_{2m}\neq d_{2j}$ for $j\neq m$. We continue in this way obtaining for each k

a sequence $\{d_{km}\}_{m=1}^{\infty}$ in $B_k(K_k) - cl\left(\bigcup_{n=1}^{\infty} K_n\right) - \{d_{jm} : j < k,\ m \geq 1\}$. Let $D = \{d_{mk} : m \geq 1,\ k \geq 1\}$. Then each d_{mk} is isolated in D since any limit point of D is contained in $cl\left(\bigcup_{n=1}^{\infty} K_n\right)$ which misses D.

It follows that for each m and k there exists an open interval $W_{mk} \subseteq B_m(K_m)$ centered at d_{mk} with radius less than $\frac{1}{2}\,\mathrm{dist}(d_{mk}, D - \{d_{mk}\})$. Now put $W_m = \bigcup_{k=1}^{\infty} W_{mk}$. It is clear that conditions (1) through (4) are satisfied.

By Lemma 1 and Theorem 2 of [ABCP] we can find a sequence of points $\{y_n\}_{n=1}^{\infty}$ and a sequence of continuous functions $\{g_n\}_{n=1}^{\infty}$ such that $\omega(y_n, g_n) = K_n$ and $\gamma(y_n, g_n) \subseteq W_n$.

Define f as follows:

$$f(x) = \begin{cases} g_m(x) & \text{if } x \in \gamma(y_m, g_m) \\ y_1 & \text{if } x \notin \bigcup_{m=1}^{\infty} \gamma(y_m, g_m)\ . \end{cases}$$

We will show f is Baire 1 by showing each of its countably many level sets is the difference of two closed sets, and hence, both an F_σ- and G_δ-set. First since $\gamma(y_n, g_n) \subseteq W_n$, $\omega(y_n, g_n) \subseteq I - W_n$ and $cl(\gamma(y_n, g_n) = \omega(y_n, g_n) \cup \gamma(y_n, g_n))$ we have $\gamma(y_n, g_n) = cl(\gamma(y_n, g_n)) - \omega(y_n, g_n)$. Secondly we show $\bigcup_{m=1}^{\infty} \gamma(y_m, g_m) = cl\left(\bigcup_{m=1}^{\infty} \gamma(y_m, g_m)\right) - cl\left(\bigcup_{n=1}^{\infty} K_n\right)$. Clearly the left hand side is a subset of the right hand side. Let x belong to the right hand side. Then there is an open neighborhood V of x missing $\bigcup_{n=1}^{\infty} K_n$ such that V hits only finitely many of the sets $B_n(K_n)$ and hence finitely many of the sets $\gamma(y_n, g_n)$. Hence, x belongs to the left hand side.

Neither Theorem 4 nor Theorem 5 is valid in general for continuous functions. For example, if $\mathcal{Q} = \left\{K, \{\frac{2}{3}\}\right\}$ where K is any closed subset of $\left[0, \frac{1}{3}\right]$ consisting of more than one point, then the continuity of f would force a fixed point in $\left[0, \frac{1}{3}\right]$ giving an additional member of $\mathcal{Q}$.

However, Theorem 5 and hence, Theorem 4, is valid for a continuous function in case $\mathcal{Q}$ consists of singletons and $\cup\, \mathcal{Q}$ is closed. We may define a continuous function f giving $\Lambda(f) = \mathcal{Q}$ as follows: let f be the identity on $\cup \mathcal{Q}$ and for (a,b) an open component of $I - \cup \mathcal{Q}$ select f to be a piecewise linear function connecting (a,a) to $(\frac{a+b}{2}, \frac{a+2b}{3})$ to (b,b).

Measurable Functions

When all the members of a class of functions $\mathcal{E}$ are well-behaved, the problem of characterizing the sets $\Lambda(f)$ for $f \in \mathcal{E}$ becomes difficult for at least two reasons. Firstly, the existence of one type of ω-limit set may require the existence of others. For example, if some $\omega(x,f)$ for a continuous function contains an interval, then f must have infinitely many pairwise disjoint ω-limit sets. (See [BCR].) Secondly, certain combinations of ω-limit sets are ruled out by continuity considerations. For example, we saw in the previous section that certain countable families of sets are not realizable as ω-limit sets for a single $f \in \mathcal{B}_1$ even though any finite subcollection can constitute the entire family $\Lambda(f)$ for some $f \in \mathcal{B}_1$

One might expect that dropping regularity conditions on $\mathcal{E}$ will allow a large variety of combinations of sets to serve as ω-limit sets for a single function. But it may not be obvious to what extent one can select exactly which sets are to be ω-limit sets for some functions and which sets are not. The final result settles this.

Theorem 6. If $\mathcal{Q}$ is any nonempty family of nonempty closed subsets of I, then there exists a function f which is Lebesgue measurable and Darboux such that $\Lambda(f) = \mathcal{Q}$.

Proof: Let Q be a null, meager F_σ subset of I which is c-dense in I. Let $\mathcal{Q} = \{A_\alpha\}_{\alpha<\xi}$ where ξ is some ordinal not greater than c, the power of the continuum. Let $\{z_\alpha : \alpha < \beta\}$ be a well-ordering of $\xi \times (\omega_o - \{o\})$ where $\beta = card(\xi \times \omega_o)$ (a cardinal is an ordinal not bijective with any smaller ordinal). Let $\{J_n\}_{n=1}^{\infty}$ be a countable base of open intervals for I such that diam $J_n \to 0$.

By induction on β we pick a set $\{w_\alpha : \alpha < \beta\}$ as follows: If $z_o = (\gamma, n)$ choose $w_o \in Q \cap J_n$. Having chosen w_η for each $\eta < \alpha$ suppose $z_\alpha = (\mu, m)$; pick $w_\alpha \in Q \cap J_m - \{w_\eta : \eta < \alpha\}$. For each $\gamma < \xi$ put $D_\gamma = \{w_\alpha : z_\alpha = (\gamma, n),\ J_n \cap A_\gamma \neq \phi,\ \text{for some } n\}$. Then $D_\gamma \cap D_\eta = \phi$ whenever $\gamma \neq \eta$. It is easy to see that $D'_\gamma = A_\gamma$. Since each D_γ is countably infinite we can enumerate D_γ as $\{d_{\gamma k}\}_{k=0}^{\infty}$. Finally put $D = \cup\{D_\gamma : \gamma < \xi\}$ and note that $D \subseteq Q$.

Since each uncountable Borel set contains a Cantor set we can find a sequence $\{C_k\}_{k=1}^{\infty}$ of mutually disjoint Cantor sets such that $C_k \subseteq I_k - Q \subseteq I_k - D$ for each k, where $\{I_k\}_{k=1}^{\infty}$ is an enumeration of all open rational intervals. Put $C = \bigcup_{k=1}^{\infty} C_k$.

As in the proof of Theorem 3 let f_k be a Baire 2 function mapping C_k onto $I - \bigcup_{j=k}^{\infty} C_j$. Then we define

$$f(x) = \begin{cases} d_{\gamma k+1} & \text{if } x = d_{\gamma k} \\ f_k(x) & \text{if } x \in C_k \\ d_{\infty} & \text{if } x \in I - C . \end{cases}$$

As in Theorem 3, f is Darboux because it maps each I_k onto I. The function f restricted to the G_δ- set $I - Q$ is a Baire 2 function. Since Q has measure zero $f\,|\,Q$ is Lebesgue measurable and hence, f itself is Lebesgue measurable. It is obvious that $\Lambda(f) = Q$.

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Received June 29, 1989