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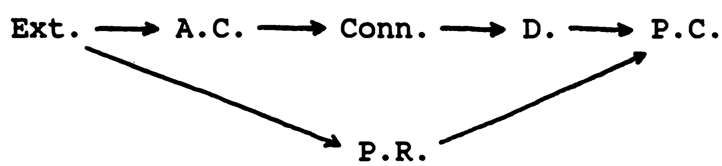
Darboux Functions with a Perfect Road

Following theorem 1.1, Chapter II, in his book [2], A.M. Bruckner made the remark that in the implication, Conn. + Baire class 1 $\to$ perfect road, the assumption that the function be a Baire class 1 function is not needed. The function need only be a Borel function, since uncountable Borel sets contain perfect sets. Recently Brown, Humke, and Laczkovich [1], gave an improvement of that implication by noting that for Borel measurable functions, if $f:[a,b] \to R$ is a Darboux function (D.), then f has a perfect road (P.R.). In [3] property B was defined and it was announced that if f is a Darboux function having property B, then f has a perfect road. It was also announced in [3] that there are 2^c many Darboux functions with property B. These results extend the results of Brown, Humke, and Laczkovich to a class of functions that contains 2^c-many.

The weak Cantor intermediate value property (WCIVP) was defined in [5] and it was proved that if $f:I \to I$ is an extendable function (Ext.), then f has the WCIVP where $I = [0,1]$. In the same paper an almost continuous function (A.C.) $f:I \to I$ was constructed that has neither the WCIVP nor a P.R.. This gave a negative answer to a question (Does A.C. $\to$ Ext.?) implied by Stallings in [11].

In [6] it was proved that if $f:I \to I$ is an extendable function, then f has a perfect road. For the class of arbitrary functions $[a,b] \to R$ only the following implications hold, [1].



The purpose of this paper is to prove that if $f:[a,b] \to R$ is a Darboux function, then the statements

(1) f has property B,

(2) f has the WCIVP, and

(3) f has a perfect road

are equivalent. Examples will be given that show that in general these properties are not equivalent. Also an example will be given that shows that the sum of two D. + P.R. functions is not a D. + P.R. function. Some questions will be discussed at the end. We now define these classes of functions and properties.

The statement that a set A is between two real numbers a and b means that A is a subset of the open interval with end-points a and b.

Let X and Y be topological spaces and let $f:X \to Y$. Then:

D. : f is a Darboux function if $f(C)$ is connected whenever C is connected in X.

Conn. : f is a connectivity function if the graph of f restricted to C, denoted $f|C$, is connected in $X \times Y$ whenever C is connected in X.

A.C. : f is an almost continuous function provided that if $U \subset X \times Y$ is any open set containing the graph of f, then U contains the graph of a continuous function $g: X \to Y$.

Ext. : f is an extendable function if there exists a connectivity function $g: X \times I \to Y$ such that $f(x) = g(x,0)$ for any $x \in X$.

P.C. : f is peripherally continuous if for each $x \in X$ and for each pair of open sets U and V such that $x \in U$ and $f(x) \in V$ there exists an open set W such that $x \in W \subset U$ and $f(bd(W) \subset V$ where bd = boundary.

Let $f:[a,b] \to R$ be a function. Then:

P.R. : f has a perfect road if for each x in $[a,b]$ there exists a perfect set P having x as a bilateral limit point such that $f|P$ is continuous at x. If x is an endpoint, then the bilateral condition is replaced with a unilateral condition.

WCIVP : f has the WCIVP if for p and q in $[a,b]$ such that $p \neq q$ and $f(p) \neq f(q)$ implies there exists a Cantor set C between p and q such that $f(C)$ is between $f(p)$ and $f(q)$.

B. : f has property B provided that for any pair of open intervals (p,q) and E if $(p,q) \cap f^{-1}(E)$ is uncountable, then $(p,q) \cap f^{-1}(E)$ contains a perfect set.

We now prove the main result which will be followed by the examples.

Theorem: Let $f:[a,b] \to R$ be a Darboux function. Then the following three statements are equivalent.

(1) f has property B.

(2) f has the WCIVP.

(3) f has a P.R..

Proof. (1) $\to$ (2). Select any $p < q$ such that $f(p) \neq f(q)$. Let E be the open interval with endpoints $f(p)$ and $f(q)$. Since f is Darboux, $(p,q) \cap f^{-1}(E)$ is uncountable. Since f has property B, $(p,q) \cap f^{-1}(E)$ contains a perfect set and hence a Cantor set C. So $C \subset (p,q)$ and $f(C)$ is between $f(p)$ and $f(q)$. Hence f has the WCIVP.

(2) $\to$ (3). Select any x that is not an endpoint. We first construct a perfect set A to the left of x having x as a limit point such that $f|A$ is continuous at x.

Suppose there exists an $\varepsilon > 0$ such that f is constant on $[x-\varepsilon,x]$. Then let $A = [x-\varepsilon,x]$ and we are done. Now assume that for each $\varepsilon > 0$, f is not constant on $[x-\varepsilon,x]$.

Let $\varepsilon_1 > 0$. Then there exists an $a_1 \in (x-\varepsilon_1,x)$ such that $f(a_1) \neq f(x)$. We may assume that $|f(a_1) - f(x)| < \varepsilon_1$. For suppose to the contrary that $|f(a_1) - f(x)| \geq \varepsilon_1$. Let y be between $f(a_1)$ and $f(x)$ such that $|y - f(x)| < \varepsilon_1$. Then there exists $z \in (a_1,x)$ such that $f(z) = y$. Now rename $a_1 = z$. Since f has the WCIVP, there exists a Cantor set C_1 between a_1 and x such that $f(C_1)$ is between $f(a_1)$ and $f(x)$. So the diameter of $f(C_1) \cup \{f(x)\}$ is less than ε_1.

Repeating the above argument we can construct sequences $\{\varepsilon_n\}$, $\{a_n\}$, and $\{C_n\}$ such that

(1) $0 < \varepsilon_n < (1/2)d(C_{n-1},x)$ for $n = 2,3,\ldots;$

(2) $a_n \in (x - \varepsilon_n, x)$ and $|f(a_n) - f(x)| < \varepsilon_n$; and

(3) $C_n \subset (a_n,x)$ is a Cantor set and $f(C_n)$ is between $f(a_n)$ and $f(x)$.

Since ε_n converges to 0, $a_n \to x$, $f(a_n) \to f(x)$, $x \in \overline{\bigcup_{n=1}^{\infty} C_n}$, and $A = (\bigcup_{n=1}^{\infty} C_n) \cup \{x\}$ is a perfect set. It follows that $f|A$ is continuous at x.

Repeat this argument for the right side and then take the union of the two sets to get the perfect road for f at x. At the end-points we will have only a unilateral condition.

(3) $\to$ (1) Suppose $(p,q) \cap f^{-1}(E)$ is uncountable. Select any $x \in (p,q) \cap f^{-1}(E)$. Then $f(x) \in E$. Since f has a perfect road, there exists a perfect set P having x as a bilateral limit point such that $f|P$ is continuous at x. By continuity we can select a perfect set $Q \subset P$ such that $Q \subset (p,q)$ and $f(Q) \subset E$. Hence it follows that $Q \subset (p,q) \cap f^{-1}(E)$ and we are done.

In the proof of (3) $\to$ (1) it should be noted that f need not be a Darboux function.

For arbitrary functions we have the following examples.

Example 1. Property B $\not\longrightarrow$ P.R.

Define $f:[a,b] \longrightarrow R$ as follows. If x is rational, let $f(x) = 0$; and, if x is irrational, let $f(x) = 1$. This example also shows that Property B $\not\longrightarrow$ WCIVP.

Example 2. P.R. $\not\longrightarrow$ WCIVP.

Define $f:I \longrightarrow I$ as follows. Let $C \subset I$ be the standard Cantor set and let E be the set of endpoints of the intervals in $I - C$. Let $C^{\circ} = C - E$. Let $f(x) = 0$, if x is in C°; and, let $f(x) = 1$, if x is in $I - C^{\circ}$.

Example 3. WCIVP $\not\longrightarrow$ Property B.

Define $f:I \longrightarrow I$ as follows. Let $C \subset I$ be the standard Cantor set. Let C° be defined as in example 2. Let $D \subset C^{\circ}$ be an uncountable set that contains no Cantor set. If $x \in D$, let $f(x) = 0$; and, if $x \in C - D$, let $f(x) = 1$. Now assume $x \in I - C$ and let $J = (p,q)$ be the open interval in $I - C$ containing x. Let $m \in J$ be the midpoint of J. In I^2 draw the line segments L_1 from $(p, 1/4)$ to $(m,1/2)$ and L_2 from $(m,1/2)$ to $(q,1/4)$. Define f on J to be continuous so that the graph of $f|J$ contains no points below $L_1 \cup L_2$, $\{y:\text{there exists } x_n \in J \text{ such that } x_n \longrightarrow p \text{ and } f(x_n) \longrightarrow y\} = [1/4,1]$, and $\{y:\text{there exists } x_n \in J \text{ such that } x_n \longrightarrow q \text{ and } f(x_n) \longrightarrow y\} = [1/4,1]$.

Example 4. Define $f(x) = \sin 1/x$, if $0 < x < 1$; and $f(x) = 0$, if $x = 0$. Define $g(x) = -\sin 1/x$, if $0 < x < 1$; and $g(x) = 1$, if

$x = 0$. Now f and g are Ext. functions and hence are D. + P.R. functions. But $f(x) + g(x) = 0$, if $0 < x < 1$; and $f(x) + g(x) = 1$, if $x = 0$. So $f + g$ is not D. + P.R.

We now raise the following questions.

Question 1. Does A.C. + P.R. $\longrightarrow$ Ext.? In a paper under preparation, Rosen, Roush, and I have shown that the answer is no.

Question 2. Does Conn. + P.R. $\longrightarrow$ A.C.? Harvey Rosen [10] has shown that if $f: I \longrightarrow R$ is a Darboux function with a G_δ-graph, then f has property B and hence has a P.R.. Jones and Thomas [7] constructed a connectivity function with a G_δ-graph that is not almost continuous. Since Conn. $\longrightarrow$ D., the answer to this question is no.

Question 3. Does D. + P.R. $\longrightarrow$ Conn.? M.H. Miller, Jr., [9] constructed a Darboux function $f: I \longrightarrow I$ with a G_δ-graph that has no fixed point. Since every connectivity function $I \longrightarrow I$ has a fixed point, f is not a connectivity function. From Rosen's result in [10] , the answer to this question is no.

Question 4. Does A.C. + G_δ-graph $\longrightarrow$ Ext.? Unknown. K.R. Kellum and D.B. Garrett [8] constructed an almost continuous function $I \longrightarrow I$ having a G_δ-graph that is not of Baire class 1 but is of Baire class 2. Is this function extendable?

The negative answers to Questions 2 and 3 also follows from theorem 2 of the paper by Brown, Humke, and Laczkovich, [1].

Remark. In [4] Roush and I defined the CIVP. It is an easy exercise to show that if $f:I \rightarrow I$ is a Darboux function that is also Borel measurable, then f has the CIVP. Since a Borel measurable function is Marczewski measurable, it follows that f has the SCIVP.

Theorem 2, page 489, Kuratowski, Topology, Volume 1 says that "If f is a function defined on an analytic set A such that the graph of f is analytic, then f is Borel measurable." Thus the examples of the functions $I \rightarrow I$ discussed in Questions 2, 3, and 4 are each Borel measurable and hence each satisfies the SCIVP.

A function $f:[a,b,] \rightarrow R$ has the CIVP [SCIVP] if for each pair of points p and q in $[a,b]$ such that $p < q$ and $f(p) \neq f(q)$ and for each Cantor set K between $f(p)$ and $f(q)$ there exists a Cantor set $C \subset (p,q)$ such that $f(C) \subset K$ [and $f|C$ is continous].

Clearly SCIVP $\rightarrow$ CIVP $\rightarrow$ WCIVP.

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