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Vasile Ene, Institute of Mathematics, str. Academiei 14, 70109
Bucharest, Romania

ON SOME QUESTIONS RAISED BY J. FORAN

In a comprehensive survey article [7], J. Foran has raised several interesting questions related to some classes of continuous functions. In the following, we are dealing with three of these questions. As a part of our approach, we will settle in the negative two of Foran's conjectures.

Let $\mathcal{C} = \{F : F \text{ is continuous}\}$; $L = \{F : F \text{ is Lipschitz}\}$; $H = \{h : [a,b] \longrightarrow [c,d] : h \text{ is a homeomorphism}\}$; $\overline{H} = \{h \in H : h \in AC\}$. Banach's conditions T_1, T_2, S, Lusin's condition N and conditions VB, VBG, VB_*, AC, ACG are defined in [13]; $A(N)$, $B(N)$, $\mathfrak{F}$, $\mathfrak{B}$ in [9].

Definition 1. [8]. A function $F : [0,1] \longrightarrow R$ satisfies Foran's condition M (resp. M_*) on $E = \overline{E} \subset [0,1]$ if F is AC on each closed subset of E on which F is $VB \cap \mathcal{C}$ (resp. $VB_* \cap \mathcal{C}$).

Definition 2. [12]. Let $F : [0,1] \longrightarrow R$, $E^{\infty} = \{x : F'(x) = \pm\infty\}$; $N^{\infty} = \{F : |F(E^{\infty})| = 0\}$.

Definition 3. [7]. A continuous function f on a closed interval is B_2 provided $\{y : f^{-1}(y) \text{ is finite}\} \cap J$ is uncountable, where J is any open interval in the range of f.

Definition 4. [7]. A continuous function f on a closed interval satisfies condition S' provided to each open interval J in the range of f corresponds a number ε_J such that $|E| \geq \varepsilon_J$, whenever E is a measurable set for which $F(E) \supset J$.

Definition 5. Let $P = \overline{P} \subset [0,1]$. A function $f: P \to R$ is S^* (resp. T_1^*, S'^*, B_2^*) on P if f_P is S (resp. T_1, S', B_2) on $[a,b]$, where $a = \inf(P)$, $b = \sup(P)$, $\{(a_n,b_n)\}_n$ are the intervals contiguous to P and $f_P: [a,b] \to R$ is defined as follows: $f_P(x) = f(x)$, $x \in P$ and $f_P(x) = \frac{f(b_n)-f(a_n)}{b_n-a_n}\cdot(x-a_n) + f(a_n)$, $x \in [a_n,b_n]$.

Definition 6. Given a natural number N, let $\mathcal{F}(N)$ (resp. $\mathcal{B}(N)$) be the class of all continuous functions F defined on a closed interval I for which there exist a sequence of sets $\{E_n\}$ and a sequence of natural numbers $\{N_n\}$ such that $\sup\{N_n\} = N$, $I = \bigcup E_n$ and F is $A(N_n)$ (resp. $B(N_n)$) on E_n. If we drop the condition $\sup\{N_n\} < \infty$ we obtain Foran's class $\mathcal{F}$ (resp. $\mathcal{B}$). If the sets E_n are supposed to be closed we obtain conditions $[\mathcal{F}(N)]$, $[\mathcal{B}(N)]$, $[\mathcal{F}]$, $[\mathcal{B}]$.

Definition 7. [11] (p.416). For a function f satisfying property P on sets we say that f is generalized P on E, writing $f \in GP$ on E (resp. $f \in [GP]$ on E) if E can be written as the union of countably many sets (resp. closed sets) on each of which f is P. Thus we have properties like GS^*, GS'^*, GB_2^*, GT_1^*, GS, GT_1 (resp. $[GS^*]$, $[GS'^*]$, $[GB_2^*]$, $[GT_1^*]$, $[GS]$, $[GT_1]$).

J. Foran asks for a characterization of each of the following classes of continuous functions: a) $H \circ VBG$, b) $\overline{H} \circ ACG$, c) $\overline{H} \circ VBG$. With respect to the class a) we prove that it is contained in the class $[GB_2^*]$ and our conjecture is that the converse inclusion is also true. With respect to the class b) we show that it is contained in the class $[GS^*]$ and our conjecture is that the converse inclusion is also true. At the same time, we show that the class $[GS^*]$ is strictly contained in the Lusin class N. In this way, we settle in the negative Foran's conjecture asserting that the class $\overline{H} \circ ACG$ is identical to the class N. With respect to the class c), we prove that it is contained in the class $[GT_1^*]$ and we conjecture that the converse inclusion is also true. Moreover, we show that $[GT_1^*]$ is strictly contained in the Banach class T_2; this settles in the negative Foran's conjecture asserting the identity $\overline{H}\circ VBG = T_2$.

In what follows we need the following results:

Lemma 1. Let $f:P\to R$, $P = \overline{P} \subset [0,1]$, $f \in \mathcal{C}$ and let $s:f(P)\to \overline{R}_+$, $s(y)$ is the number (finite or infinite) of points of $f^{-1}(y)$. Then $s(y)$ is Borel measurable.

Proof. The proof is similar with that of [13] (Theorem 6.4, p. 280). Indeed, let $a = \inf(P)$, $b = \sup(P)$ and let $s_k^{(n)}$ be the characteristic function of the set $f(I_k^{(n)} \cap P)$, where $I_k^{(n)}$ are defined as in [13]. Clearly $s_k^{(n)}$ are Borel measurable and following [13], $s(y)$ is Borel measurable.

Lemma 2. $S = N \cap T_1$ for continuous functions on each closed subset of $[0,1]$.

Proof. The proof is identical with that of [13] (p.284–285) if we use Lemma 1 instead Theorem 6.4, p.280 of [13].

Theorem A. (Theorem 7.4, p.284 of [13] and the Corollary of p. 131 of [12]). $S = N \cap T_1 = N^{\infty} \cap T_1$ for continuous functions on a closed interval.

Lemma 3. (Krzyzewski-lemma, see [10]). If F'_{ap} exists at every point of a set E and $|F(E)| = 0$ then $F'_{ap}(x) = 0$ at almost all points $x \in E$.

We will need the symmetric perfect sets and functions defined on these sets which are given in the following construction:

Let $\alpha = \{a_k\}_k$, $k \geq 0$, be a sequence of positive numbers such that $a_0 = 1$, $a_{k-1} \geq 2a_k > 0$ and let $c_k = a_{k-1} - a_k$. Let $C(\alpha) = \{x :$ There exists $e_i(x)$ taking on 0 or 1 and $x = \sum e_i(x)c_i\}$. If $\alpha = \{1/3^k\}_k$ then $C(\alpha) = C$ (C = the Cantor ternary set) and if $\alpha = \{1/2^k\}_k$ then $C(\alpha) = [0,1]$. The open intervals deleted in the s-step of the construction of $C(\alpha)$ are $O_{e_1 \ldots e_{s-1}}(\alpha) = (\sum_{i=1}^{s-1} e_i c_i + a_s , \sum_{i=1}^{s-1} e_i c_i + c_s)$, $(e_1, \ldots, e_{s-1}) \in \{0,1\}^{s-1} = \{0,1\} \times \ldots \times \{0,1\}$ (s-1) times and the remaining intervals of the s-step are

$$R_{e_1 \ldots e_s}(\alpha) = [\sum_{i=1}^{s} e_i c_i , \sum_{i=1}^{s} e_i c_i + a_s], \text{ where } (e_1, \ldots, e_s) \in \{0,1\}^s.$$

Then $C(\alpha) \subset \bigcup_{(e_1, \ldots, e_s) \in \{0,1\}^s} R_{e_1 \ldots e_s}(\alpha)$, hence $|C(\alpha)| = \lim_{s \to \infty} 2^s a_s$. We say that α is of type (*) if $a_{k-1} > 2a_k$, $k = 1,2,\ldots$. Then each $x \in C(\alpha)$ is uniquely represented and $C(\alpha)$ is a symmetric perfect nowhere dense subset of $[0,1]$, $0,1 \in C(\alpha)$. We say that is of type (**) if $a_{k-1} \geq 2a_k$, $k = 1,2,\ldots$. We say that α is of type (***) if $c_{2k-1} = 2 \cdot c_{2k}$, $k = 1,2,\ldots$. Let $\alpha = \{a_k\}_k$, $\alpha' = \{a'_k\}_k$, $k \geq 0$, be two sequences of type (*), $c_k = a_{k-1} - a_k$, $c'_k = a'_{k-1} - a'_k$, $k \geq 1$. Let $\alpha'' = \{a''_k\}_k$, $k \geq 0$, be a sequence of type (**), $c''_k = a''_{k-1} - a''_k$, $k \geq 1$. Let $I^{\alpha,\alpha''} : C(\alpha) \to C(\alpha'')$, $I^{\alpha,\alpha''}(x) = I^{\alpha,\alpha''}(\sum_{i=1}^{\infty} e_i(x)c_i) = \sum_{i=1}^{\infty} e_i(x)c''_i$; $G^{\alpha,\alpha''} : C(\alpha) \to C(\alpha'')$, $G^{\alpha,\alpha''}(x) = G^{\alpha,\alpha''}(\sum_{i=1}^{\infty} e_i(x)c_i) = \sum_{i=1}^{\infty} (e_{2i-1}(x)c''_{2i} + e_{2i}(x)c''_{2i-1})$; $G^{\alpha',\alpha''} : C(\alpha') \to C(\alpha'')$, $G^{\alpha',\alpha''}(x) = G^{\alpha',\alpha''}(\sum_{i=1}^{\infty} e_i(x)c'_i) = \sum_{i=1}^{\infty} (e_{2i-1}(x)c''_{2i} +$

$e_{2i}(x)c''_{2i-1})$; $F_1^{\alpha,\alpha''} : C(\alpha) \longrightarrow [0,1)$, $F_2^{\alpha,\alpha''} : C(\alpha) \longrightarrow [0,1)$, $F_1^{\alpha,\alpha''}(x) = F_1^{\alpha,\alpha''}(\sum_{i=1}^{\infty} e_i(x)c_i) = \sum_{i=1}^{\infty} e_{2i-1}(x)c''_{2i-1}$; $F_2^{\alpha,\alpha''}(x) = F_2^{\alpha,\alpha''}(\sum_{i=1}^{\infty} e_i(x)c_i) = \sum_{i=1}^{\infty} e_{2i}(x)c''_{2i}$. Extending $I^{\alpha,\alpha''}$, $G^{\alpha,\alpha''}$, $F_1^{\alpha,\alpha''}$, $F_2^{\alpha,\alpha''}$ (resp. $G^{\alpha',\alpha''}$) by linearity on the intervals contiguous to $C(\alpha)$ (resp. $C(\alpha')$), we have these functions defined and continuous on $[0,1]$. Clearly $I^{\alpha,\alpha}(x) = x$ on $[0,1]$. We have

(1) $$I^{\alpha,\alpha''}(x) = F_1^{\alpha,\alpha''}(x) + F_2^{\alpha,\alpha''}(x) ;$$

(2) $$G^{\alpha,\alpha''}(R_{e_1 \ldots e_{2k}}(\alpha) \cap C(\alpha)) = R_{e_2 e_1 \ldots e_{2k} e_{2k-1}}(\alpha'') \cap C(\alpha'')$$

and $G^{\alpha,\alpha''}(R_{e_1 \ldots e_{2k}}(\alpha)) = R_{e_2 e_1 \ldots e_{2k} e_{2k-1}}(\alpha'')$, $k=0,1,\ldots$.

(3) $$G^{\alpha',\alpha''} \circ G^{\alpha,\alpha'} = I^{\alpha,\alpha''} \quad \text{on } C(\alpha).$$

If in addition α'' is of type (***) (i.e., $c''_{2i-1} = 2c''_{2i}$) then

(4) $$2G^{\alpha,\alpha''}(x) = 3F_1^{\alpha,\alpha''}(x) + I^{\alpha,\alpha''}(x) = 4I^{\alpha,\alpha''}(x) - 3F_1^{\alpha,\alpha''}(x)$$
$$= 4F_2^{\alpha,\alpha''}(x) + F_1^{\alpha,\alpha''}(x).$$

Remark 1. a) If α is of type (*) then α is of type (**).
b) If $a \in [0,1)$ then there exists a sequence $\alpha = \{a_i\}_i$, $i \geq 0$, of type (*) but not of type (***) such that $|C(\alpha)| = a$. Put for example $a_i = a/2^i + (1-a)/4^i$.
c) If $a \in [0,1)$ then there exists a sequence $\alpha = \{a_i\}_i$, $i \geq 0$, of type (*) and of type (***) such that $|C(\alpha)| = a$. Put for example: $a_{2i-1} = 2/4^i - (3 \cdot 2^{i+1} - 10)a/(3 \cdot 8^i)$, $i = 1,2,\ldots$, $a_{2i} = 1/4^i - (2^i - 1)a/(8^i)$, $i=0,1,\ldots$. Then $c_{2i} = 1/4^i - (3 \cdot 2^i - 7)a/(3 \cdot 8^i)$;

$c_{2i-1} = 2/4^i - (3\cdot 2^{i+1} - 14)a/(3\cdot 8^i)$.

d) If $\alpha = \{1/2^i\}_i$, $i \geq 0$, then α is of types (**) and (***) but not of type (*) and $C(\alpha) = [0,1]$.

e) If $\alpha = \{1/3^i\}_i$, $i \geq 0$ and $\alpha'' = \{1/2^i\}_i$, $i \geq 0$, then $I^{\alpha,\alpha''} = \varphi$, where φ is the Cantor ternary function.

Lemma 4. Let N be a natural number and let $f,h:[0,1] \longrightarrow R$, h - increasing and AC. Let E be a closed subset of $[0,1]$. If there exists $\eta > 0$ such that for each $c,d \in E$, with $0 < d-c < \eta$, $\lambda_N(f([c,d] \cap E)) < h(d) - h(c)$ then $f \in A(N)$ on E. ($\lambda_N(X) = \inf\{\sum_{i=1}^{N} |I_i| : \{I_i\}_{i=1}^{N}$ is a sequence of N open intervals which covers the set $X\}$, see [11], p.404).

Proof. For $\varepsilon > 0$ let δ_ε be the δ given by the fact that h is AC. Let $\delta_\varepsilon^1 = \min\{\delta_\varepsilon, \eta\}$. By the definition it follows that $f \in A(N)$ on E.

The proofs of the following theorems 1, 2 and 3 will be deferred until the end of the paper.

Theorem 1. With the above notations we have:

a) $|F_1^{\alpha,\alpha''}(C(\alpha))| = |F_2^{\alpha,\alpha''}(C(\alpha))| = 0$ and $G^{\alpha,\alpha''}(C(\alpha)) = C(\alpha'')$; (hence $F_1^{\alpha,\alpha''}$ and $F_2^{\alpha,\alpha''}$ belong to $S = N \cap T_1$ on $[0,1]$;)

b) If $|C(\alpha)| \neq 0$ and $|C(\alpha'')| \neq 0$ then $F_1^{\alpha,\alpha''}$, $F_2^{\alpha,\alpha''}$, $G^{\alpha,\alpha''}$ belong to $\mathcal{T}(2) - \mathcal{B}(1)$ and the sets of points of $C(\alpha)$ at which $F_1^{\alpha,\alpha''}$, $F_2^{\alpha,\alpha''}$, $G^{\alpha,\alpha''}$ are approximately differentiable are null sets. $G^{\alpha,\alpha''}$ has finite or infinite derivative at no point of $C(\alpha)$.

c) If $|C(\alpha)| \neq 0$ and $|C(\alpha'')| \neq 0$ then $F_1^{\alpha,\alpha''}$, $F_2^{\alpha,\alpha''} \in S$ (hence $F_1^{\alpha,\alpha''}$, $F_2^{\alpha,\alpha''} \in [GS^*]$) on $[0,1]$, but $G^{\alpha,\alpha''} \notin [GS^*]$ on $[0,1]$.

d) If α'' is of type (*) then $G^{\alpha,\alpha''}$ is bijective on $C(\alpha)$;

e) If α'' is of type (**) but not of type (*) then $\|(G^{\alpha,\alpha''})^{-1}(y) \cap C(\alpha)\| = 1$ (resp. $\|(G^{\alpha,\alpha''})^{-1}(y) \cap C(\alpha)\| = 2$) if y has an unique representation (resp. two representations); ($\|X\|$ = the cardinal of the set X.)

f) $G^{\alpha,\alpha''}$ is monotone on no portion of $C(\alpha)$;

g) If there exists $M>0$ such that $c_{2i}^{\alpha''}/\min\{a_{2i}-2a_{2i+1},\ a_{2i+1}-2a_{2i+2}\} < M$, $i=0,1,\ldots$ then $G^{\alpha,\alpha''} \in L$. (In particular this holds when $C(\alpha) = C(\alpha'') = C$).

h) If $|C(\alpha)| = 0$ and $|C(\alpha'')| \neq 0$ then $G^{\alpha,\alpha''} \in (M \cap T_2) - N$ on $[0,1]$ and at least one of the functions $F_1^{\alpha,\alpha''}$ and $F_2^{\alpha,\alpha''}$ does not belong to $\mathcal{F}$ on $[0,1]$.

i) $M \circ L \supsetneqq M$; j) $\mathcal{F}(2) \circ M \supsetneqq M$.

Remark 2. A continuous, bijective function on C and monotone on no portion of C was constructed before in [5]. There exists a function $f: C \to C$, $f \in \mathcal{C} \cap L$ and bijective such that f is monotone on no portion of C and $f \circ f(x) = x$ on C. Put for example $C(\alpha) = C(\alpha') = C(\alpha'') = C$ and $f = G^{\alpha,\alpha''}$ (see Theorem 1,d),f),g) and (3)).

Lemma 5. Let $f: P \to R$, $P = \overline{P} \subset [0,1]$ and let $H_1 = \{x : f'_{|P}(x) = 0\}$. Then $|f(H_1)| = 0$.

Proof. Let $H = \{x \in H_1 : x$ is a billateral point of accumulation of $P\}$. Then $f'_P(x) = 0$ at each $x \in H$ and $H_1 - H$ is at most denumerable. By Theorem 4.5,p.271 of [13], it follows that $|f_P(H)| = 0$, hence $|f(H_1)| = 0$.

Proposition 1. Let $f: P \to R$, $f \in \mathcal{C}$, $P = \overline{P} \subset [0,1]$ and let $E = \{x \in P : f'(x)$ does not exist with respect to $P\}$. If $|f(E)| = 0$ then $f \in T_1$ on P.

Proof. Using Lemma 5 instead of Theorem 4.5,p.271 of [13], the proof is similar to the proof of Theorem 6.2,p.278,2° of [13].

Remark 3. The converse of Proposition 1 is true only if $P = [0,1]$. (See Theorem 1,a),b) and Theorem 6.2,p.278 of [13] or Theorem 1,p.130 of [12].)

Remark 4. For continuous functions defined on $[0,1]$ *we have:* a) $H\circ L = H\circ AC = H\circ S = H\circ S' = S'$ (see [7]); b) $H\circ VB = H\circ T_1 = B_2$ (see [7]); c) $\overline{H}\circ AC = \overline{H}\circ S = S$ (see [7]); d) $\overline{H}\circ VB = \overline{H}\circ T_1 = T_1$ (see [7]); e) $L\circ H = AC\circ H = VB\circ H = VB$ (see [7]); f) $S\circ S = S$ (see [13],p.289); g) $S'\circ S = S'$ (since by a), $S'\circ S = H\circ S\circ S = H\circ S = S'$). This follows also by definitions. Indeed, let $f,g:[0,1]\longrightarrow R$ and let $G = f\circ g$, $f\in S'$, $g\in S$. Let $E\subset[0,1]$ and let J be an interval such that $J\subset G(E) = f(g(E))$. Then there exists $\varepsilon_1>0$ such that $|g(E)|>\varepsilon_1$. Since $g\in S$ there exists $\varepsilon>0$ such that $|E|>\varepsilon$. h) $T_1 = S\circ H$ (since by d),e),c), $T_1 = \overline{H}\circ VB = \overline{H}\circ AC\circ H = S\circ H$); i) $T_1 = S\circ T_1$ (since by f),b), $S\circ T_1 = S\circ S\circ H = S\circ H = T_1$); j) $B_2 = S'\circ H$ (since by b),e), a), $B_2 = H\circ VB = H\circ AC\circ H = S'\circ H$); k) $B_2 = S'\circ T_1$ (since by j),b),g), $S'\circ T_1 = S'\circ S\circ H = S'\circ H = B_2$); l) $H\circ B_2 = B_2$ (since by b), $H\circ B_2 = H\circ H\circ VB = H\circ VB = B_2$); m) $B_2\circ H = B_2$ (since by j), $B_2\circ H = S'\circ H\circ H = S'\circ H = B_2$).

Theorem 2. For continuous functions on $[0,1]$ *we have:*

a) $S = N\cap T_1 = M\cap T_1 = N^{\infty}\cap T_1 = M_{*}\cap T_1$ (*hence* $G^{\alpha,\alpha''}\notin T_1$, $G^{\alpha,\alpha''}$ *from Theorem 1,h)*);

b) $ACG = [ACG] = [\mathcal{F}(1)] \subsetneqq [GS^{*}]$ *and* $S \subsetneqq [GS^{*}]$;

c) $[GS^{*}] = [GT_1^{*}]\cap N = [GT_1^{*}]\cap M \subsetneqq [GT_1^{*}]\cap N^{\infty} = [GT_1^{*}]\cap M_{*}$; $ACG - B_2 \neq \emptyset$, hence $ACG - S' \neq \emptyset$;

d) $[GS] = [GT_1]\cap N \subsetneqq [GT_1]\cap M \subsetneqq [GT_1]\cap N^{\infty} = [GT_1]\cap M_{*}$;

e) $\mathcal{F} = [\mathcal{F}] \subsetneqq [GS] \subsetneqq GS \subset N$ *and* $[GS^{*}] \subsetneqq [GS]$;

f) $VBG = [VBG] = [\mathfrak{B}(1)] \subsetneq [GT_1^*]$; $\mathfrak{B} = [\mathfrak{B}] \subsetneq [GT_1]$ $(T_1 - \mathfrak{B} \neq \emptyset)$; $[GT_1^*] \subsetneq [GT_1] \subsetneq T_2$.

Remark 5. That $ACG - S' \neq \emptyset$ was shown in [6].

Proposition 2. For functions defined on a bounded real set E we have: a) $S \circ S = S$; b) $(N \cap T_1) \circ T_1 = T_1$.

Proof. Let $g: E \to K$, $f: K \to R$, $G = f \circ g$.

a) Suppose $f, g \in S$. Let $\varepsilon > 0$ and let δ_ε be the δ given by the fact that $f \in S$. For δ_ε let $\eta > 0$ be the δ given by the fact that $g \in S$. Let $E_1 \subset E$, $|E_1| < \eta$. Then $|G(E_1)| < \varepsilon$.

b) Suppose $f \in T_1 \cap N$, $g \in T_1$. Let $A = \{y : G^{-1}(y)$ is infinite$\}$ and $A_1 = \{y : f^{-1}(y)$ is infinite$\}$. Then $|A_1| = 0$. Let $B_1 = \{z \in K : g^{-1}(z)$ is infinite$\}$. Then $|B_1| = 0$. Since $f \in N$, $|f(B_1)| = 0$. We have $A \subset A_1 \cup f(B_1)$. Indeed, let $y \in A$ then $G^{-1}(y) = g^{-1}(f^{-1}(y))$ is infinite. It follows that $f^{-1}(y)$ is infinite, hence $y \in A_1$ or there exists $z \in f^{-1}(y)$ such that $g^{-1}(z)$ is infinite, hence $z \in B_1$, so $y = f(z) \in f(B_1)$. It follows now that $|A| = 0$.

Lemma 6. Let $g: [a,b] \to [c,d]$, $f: [c,d] \to R$, $F: [a,b] \to R$, $F = f \circ g$, $f, g \in \mathcal{C}$. Let $P = \overline{P} \subset [a,b]$. Then $F_P = (f_{g(P)} \circ g_P)_P$. Moreover, if $g \in H$ then $F_P = f_{g(P)} \circ g_P$.

Proof. The first part of Lemma 6 is evident. We prove the second part. Clearly $F_P(x) = f_{g(P)} \circ g_P(x)$ for $x \in P$. Let $I_n = (a_n, b_n)$, $n \geq 1$, be the intervals contiguous to P with respect to $[\inf(P), \sup(P)]$. Since $g \in H$ the intervals contiguous to $g(P)$ are exactly $J_n = g(I_n)$. Hence $f_{g(P)} \circ g_P$ is linear on each $[a_n, b_n]$. Since $F(a_n) = F_P(a_n)$, $F(b_n) = F_P(b_n)$, it follows that $F_P = f_{g(P)} \circ g_P$

Remark 6. That $g \in H$ is essential in Lemma 6. Indeed, let $f = g$, $f: [0,1] \to [0,1]$, $f(x) = 1/3 - x$, $x \in [0, 1/3]$; $f(x) = x$,

$x \in [2/3, 1]$, $f(x) = 2x - 2/3$, $x \in (1/3, 2/3)$. Let $P = [0, 1/3] \cup [2/3, 1]$ and $F = f \circ g$. Then $f_P = f$ on $[0,1]$, $F_P(x) = x$ on $[0,1]$, but $F(x) \neq x$ on $(1/3, 2/3)$, and $F(1/2) = f(1/3) = 0$. (See also the function f of Remark 2.)

Lemma 7. Let $f : [0,1] \to R$, be a continuous function which is T_1 (resp. S; B_2) on $[0,1]$. If $P = \overline{P} \subset [0,1]$ then f_P is T_1 (resp. S; B_2) on $[a,b] = [\inf(P), \sup(P)]$.

Proof. Let $\{I_n\}$ be the intervals contiguous to P with respect to $[a,b]$. Suppose $f \in T_1$ on $[0,1]$. We prove that $f_P \in T_1$ on $[a,b]$. Let $A = \{y : f^{-1}(y) \cap [a,b]$ is infinite$\}$. By the definition of T_1 it follows that $|A| = 0$. Let $A' = \{y : f_P^{-1}(y) \cap [a,b]$ is infinite$\}$; $A'' = \{y : f_P^{-1}(y) \supset I_n$ for some natural number $n\}$. We show that $A' - A'' \subset A$. Let $y \in A' - A''$ such that $f_P^{-1}(y) \cap P$ is infinite. Then $f^{-1}(y) \cap P$ is infinite, hence $y \in A$. Let $y \in A' - A''$ such that $f_P^{-1}(y) \cap P$ is finite. It follows that there exists a sequence $\{n_i(y)\}$, $i \geq 1$, such that $\|f_P^{-1}(y) \cap I_{n_i(y)}\| = 1$. Since $f \in \mathcal{C}$ it follows that $\|f^{-1}(y) \cap I_{n_i(y)}\| \geq 1$, $i = 1,2,\ldots$, hence $y \in A$. Since A'' is denumerable, it follows that $|A'| = 0$, hence $f_P \in T_1$ on $[a,b]$. Suppose that f is S on $[0,1]$. We prove that f_P is S on $[a,b]$. By Banach's theorem (Theorem 7.4, p. 284 of [13]), $\mathcal{C} \cap S = \mathcal{C} \cap T_1 \cap N$ on $[a,b]$. Since $f \in S = T_1 \cap N$, $f_P \in N$ on $[a,b]$ and by the first part of this lemma $f_P \in T_1$ on $[a,b]$, hence $f_P \in N \cap T_1 = S$ on $[a,b]$. Suppose that f is B_2 on $[0,1]$. We prove that $f_P \in B_2$ on $[a,b]$. Let $J \subset \mathrm{rng}(f_P)$ be a nondegenerate interval. (Here $\mathrm{rng}(f)$ denoted the range of the function f.) Clearly $J \subset \mathrm{rng}(f)$. Let $A_J = \{y \in J : f^{-1}(y) \cap [a,b]$ is finite$\}$. By the definition of B_2 it follows that A_J is uncountable. Let $A_J' = \{y \in J : f_P^{-1}(y) \cap [a,b]$ is finite$\}$ and $A_J'' = \{y \in J :$ there exists at least one natural number n_y such that

$f_P^{-1}(y) \supset I_{n_y}\}$. To prove that $f_P \in B_2$ it is sufficient to show that $A_J - A_J'' \subset A_J'$ (since A_J'' is countable). Let $y \in A_J - A_J''$ then $f^{-1}(y) \cap P$ is either finite or empty, $B_y = \{n : f^{-1}(y) \cap I_n \neq \emptyset\}$ is finite and $\|f_P^{-1}(y) \cap I_n\| = 1$, for each $n \in B_y$. Hence $y \in A_J'$.

Some Open Questions. a) Is the converse of Lemma 4 true for continuous functions on $[0,1]$?

b) Let $f:[0,1] \to R$ and let E be a subset of $[0,1]$. Let N be a natural number. Then f is said to be $L(N)$ on E if there exists $L>0$ such that for each $a,b \in E$, $a<b$, $\lambda_N(f([a,b] \cap E)) < L$. If in Definition 6 condition $A(N)$ is replaced by $L(N_n)$ we obtain the classes $\mathfrak{L}$ and $\mathfrak{L}(N)$. We conjecture that: 1) $\mathfrak{L} \circ H = \mathfrak{F} \circ H = \mathfrak{B}$; 2) $\mathfrak{L} \circ \overline{H} = \mathfrak{F} \circ \overline{H} = \mathfrak{F}$; 3) $\mathfrak{L}(k) \circ H = \mathfrak{F}(k) \circ H = \mathfrak{B}(k)$; 4) $\mathfrak{L}(k) \circ \overline{H} = \mathfrak{F}(k) \circ \overline{H} = \mathfrak{F}(k)$, $k \geq 2$; 5) $[GS^*] \circ H = [GT_1^*]$ and $[GS'^*] \circ H = [GB_2^*]$; 6) $H \circ ACG = [GS'^*]$ (see Question 3 of [7]) and $H \circ [GS'^*] = [GS'^*]$.

c) How can the following classes of continuous functions on closed intervals be characterized: $\overline{H} \circ \mathfrak{F}(k)$; $\overline{H} \circ \mathfrak{F}$; $H \circ \mathfrak{F}(k)$; $H \circ \mathfrak{F}$? The same question if $\mathfrak{F}$ is replaced by $\mathfrak{L}$ and $\mathfrak{B}$.

d) Does Lemma 7 remain true if S is replaced by S' ?

Proof of Theorem 1. a) $F_2^{\alpha,\alpha''}(C(\alpha)) \subset \bigcup_{j_1,\dots,j_n \in \{0,1\}^n} [\sum_{i=1}^n j_i c_{2i}'', \sum_{i=1}^n j_i c_{2i}'' + \sum_{i=n+1}^{\infty} c_{2i}'']$. Clearly $a_i'' \leq 1/2^i$, $i = 1,2,\dots$. It follows that $\sum_{i=1}^{\infty} c_{2i}'' < 1/4^n$, hence $|F_2^{\alpha,\alpha''}(C(\alpha))| \leq \lim_{n\to\infty} 2^n(1/4^n) = 0$. Similarly $|F_1^{\alpha,\alpha''}(C(\alpha))| = 0$. If $k = 0$, by (2), $G^{\alpha,\alpha''}(C(\alpha)) = C(\alpha'')$. That $F_1^{\alpha,\alpha''}$ and $F_2^{\alpha,\alpha''}$ belong to $S = N \cap T_1$ on $[0,1]$ follows by [13] (Theorem 6.2, p.278) and Theorem A.

b) Let $|C(\alpha)| = a$ and $|C(\alpha'')| = b$. By hypothesis $a \neq 0$ and $b \neq 0$. First we shall prove that $(I^{\alpha,\alpha''})'(x) = b/a$ a.e. on $C(\alpha)$ and $I^{\alpha,\alpha''}$

is AC on $[0,1]$. Let $A = \{x \in C(\alpha) : I^{\alpha,\alpha''}$ is derivable at $x\}$. Let $x_0 \in A$, $x_0 = \sum_{i=1}^{\infty} e_i c_i$, $x_n = \sum_{\substack{i=1 \\ i \neq n+1}}^{\infty} e_i c_i + (1-e_{n+1})c_{n+1}$. It follows that $(I^{\alpha,\alpha''}(x_n)-I^{\alpha,\alpha''}(x_0))/(x_n-x_0) = (a''_n - a''_{n+1})/(a_n - a_{n+1}) = (2^n a''_n - 2^n a''_{n+1})/(2^n a_n - 2^n a_{n+1}) \to b/a$, hence $(I^{\alpha,\alpha''})'(x) = b/a$ if $x \in A$. Observing that $I^{\alpha,\alpha''}$ is increasing on $[0,1]$, it follows that $|C(\alpha) - A| = 0$. Also $\int_0^1 (I^{\alpha,\alpha''})'(x)dx = \int_{C(\alpha)} (b/a)dx + \int_{[0,1]-C(\alpha)} (I^{\alpha,\alpha''})'(x)dx = 1$, hence $I^{\alpha,\alpha''} \in AC$ on $[0,1]$. We shall prove that $F_1^{\alpha,\alpha''}$, $F_2^{\alpha,\alpha''} \in A(2)$ on $C(\alpha)$. By (1), since $A(1) + A(2) = A(2)$, it is sufficient to prove that $F_2^{\alpha,\alpha''} \in A(2)$ on $C(\alpha)$. By [3] it follows that if $u,v \in C(\alpha)$ then there exists J_1 and J_2 such that $F_2^{\alpha,\alpha''}([u,v] \cap C(\alpha)) \subset J_1 \cup J_2$ and $|J_1| + |J_2| \leq I^{\alpha,\alpha''}(v) - I^{\alpha,\alpha''}(u)$. Since $I^{\alpha,\alpha''} \in AC$, by Lemma 4, it follows that $F_2^{\alpha,\alpha''} \in A(2)$ on $C(\alpha)$. We shall prove that the sets of points of $C(\alpha)$ at which $F_1^{\alpha,\alpha''}$ and $F_2^{\alpha,\alpha''}$ are approximately differentiable, are null sets. Let $B = \{x \in A : F_2^{\alpha,\alpha''}$ is approximately differentiable at $x\}$. By (1), $B = \{x \in A : F_1^{\alpha,\alpha''}$ is approximately differentiable at $x\}$. By Lemma 3 together with $|F_1^{\alpha,\alpha''}(B)| = |F_2^{\alpha,\alpha''}(B)| = 0$, it follows that $(F_1^{\alpha,\alpha''})'_{ap}(x) = (F_2^{\alpha,\alpha''})'_{ap}(x) = 0$ a.e. on B. By (1), since $(I^{\alpha,\alpha''})'(x) = b/a$ on A, it follows that $|B| = 0$. By [13](p.222-223) it follows that $F_1^{\alpha,\alpha''}$, $F_2^{\alpha,\alpha''} \notin \mathfrak{B}(1)$. If α'' satisfies condition (***) the assertion for $G^{\alpha,\alpha''}$ follows easily by (4). We shall prove without condition (***) that $G^{\alpha,\alpha''} \in A(2)$ on $C(\alpha)$. Let $0 < v-u$, $u,v \in C(\alpha)$. Let s be the first natural number such that $[u,v]$ contains an open interval $O_{e_1 \ldots e_{s-1}}(\alpha) = (u_1,v_1)$, from the step s. Then $[u,v] \subset$

$R_{e_1\ldots e_{s-1}}(\alpha)$. Let $u_2, v_2 \in C(\alpha)$ such that $G(u_2) = \inf_{x\in[u,u_1]\cap C(\alpha)} G(x)$, $G(v_2) = \sup_{x\in[v_1,v]\cap C(\alpha)} G(x)$, $u_2 = \sum_{i=1}^{s-1} e_i c_i + \sum_{i=1}^{\infty} e_i^! c_i$, $v_2 = v_1 + \sum_{i=s+1}^{\infty} e_i'' c_i$. Let $h^{\alpha,\alpha''}(x) = \sum_{i=1}^{\infty} e_i(x) c_{i-1}''$, $x \in C(\alpha)$, $c_0'' = 2$. Extending $h^{\alpha,\alpha''}$ linearly on each interval contiguous to $C(\alpha)$ we have $h^{\alpha,\alpha''}$ defined and continuous on $[0,1]$, $h^{\alpha,\alpha''}(0) = 0$, $h^{\alpha,\alpha''}(1) = 2$, $(h^{\alpha,\alpha''})'(x) = 2b/a$ a.e. on $C(\alpha)$ (see the proof for $I^{\alpha,\alpha''}$), $h(C(\alpha)) = C(\alpha'') + [1 + C(\alpha'')]$, h is strictly increasing on $[0,a_1] \cup [b_1,1]$ and constant on $[a_1,b_1]$, $h^{\alpha,\alpha''} \in AC$ (see the proof for $I^{\alpha,\alpha''}$) on $[0,1]$. We have $G^{\alpha,\alpha''}(u_1) - G^{\alpha,\alpha''}(u_2) = \sum_{2i-1\geq s}(1- e_{2i-1}^!)c_{2i}'' + \sum_{2i\geq s}(1- e_{2i}^!)c_{2i-1}'' < \sum_{2i-1\geq s}(1- e_{2i-1}^!)c_{2i-2}'' + \sum_{2i\geq s}(1- e_{2i}^!)c_{2i-1}'' = \sum_{j\geq s}(1- e_j^!)c_{j-1}'' = h^{\alpha,\alpha''}(u_1) - h^{\alpha,\alpha''}(u_2)$. Analogously, $G^{\alpha,\alpha''}(v_2) - G^{\alpha,\alpha''}(v_1) < h^{\alpha,\alpha''}(v_2) - h^{\alpha,\alpha''}(v_1)$. Hence $G^{\alpha,\alpha''}([u,v]\cap C(\alpha)) \subset [G^{\alpha,\alpha''}(u_2), G^{\alpha,\alpha''}(u_1)] \cup [G^{\alpha,\alpha''}(v_1), G^{\alpha,\alpha''}(v_2)]$ and $G^{\alpha,\alpha''}(u_1) - G^{\alpha,\alpha''}(u_2) + G^{\alpha,\alpha''}(v_2) - G^{\alpha,\alpha''}(v_1) \leq h^{\alpha,\alpha''}(v) - h^{\alpha,\alpha''}(u)$. By Lemma 4 it follows that $G^{\alpha,\alpha''} \in A(2)$ on $C(\alpha)$. We shall prove that $G^{\alpha,\alpha''}$ has finite or infinite derivative at no point of $C(\alpha)$ and $G^{\alpha,\alpha''}$ has not a finite approximate derivative a.e. on $C(\alpha)$. Let $x_0 = \sum_{i=1}^{\infty} e_i c_i$. For each natural number n we have four situations:

(I) Suppose $e_{2n-1} = e_{2n} = 0$. Let $x \in R_{e_1\ldots e_{2n-2}01}(\alpha)$, $y \in R_{e_1\ldots e_{2n-2}1000}(\alpha)$. Then $x_0 < x < y$; $G^{\alpha,\alpha''}(x) > G^{\alpha,\alpha''}(x_0)$; $G^{\alpha,\alpha''}(y) > G^{\alpha,\alpha''}(x_0)$, hence $(G^{\alpha,\alpha''}(x) - G^{\alpha,\alpha''}(x_0))/(x-x_0) - (G^{\alpha,\alpha''}(y) -$

$G^{\alpha,\alpha''}(x_0))/(y-x_0) > (G^{\alpha,\alpha''}(x) - G^{\alpha,\alpha''}(y))/(y-x_0) > 3a''_{2n+2}/a_{2n-2}$ $\rightarrow 3b/16a$. Let $x_n = x_0 + c_{2n-1}$ then $(G^{\alpha,\alpha''}(x_n) - G^{\alpha,\alpha''}(x_0))/(x_n-x_0) = (a''_{2n-1} - a''_{2n})/(a_{2n-2} - a_{2n-1}) \rightarrow b/2a$.

(II) Suppose $e_{2n-1} = 0$ and $e_{2n} = 1$. Let $R_{e_1\ldots e_{2n-2}00}(\alpha)$, $y \in R_{e_1\ldots e_{2n-2}10}(\alpha)$. Then $x < x_0 < y$; $G^{\alpha,\alpha''}(x) < G^{\alpha,\alpha''}(x_0)$; $G^{\alpha,\alpha''}(y) < G^{\alpha,\alpha''}(x_0)$; $(G^{\alpha,\alpha''}(x_0) - G^{\alpha,\alpha''}(x))/(x_0-x) > a''_{2n}/a_{2n-1} \rightarrow b/2a$ and $(G^{\alpha,\alpha''}(y) - G^{\alpha,\alpha''}(x_0))/(y-x_0) < 0$.

(III) Suppose $e_{2n-1} = 1$ and $e_{2n} = 0$. Let $x \in R_{e_1\ldots e_{2n-2}01}(\alpha)$, $y \in R_{e_1\ldots e_{2n-2}11}(\alpha)$. Then $x < x_0 < y$; $G^{\alpha,\alpha''}(x) > G^{\alpha,\alpha''}(x_0)$; $G^{\alpha,\alpha''}(y) > G^{\alpha,\alpha''}(x_0)$; $(G^{\alpha,\alpha''}(y) - G^{\alpha,\alpha''}(x_0))/(y-x_0) > a''_{2n}/a_{2n-1} \rightarrow b/2a$ and $(G^{\alpha,\alpha''}(x) - G^{\alpha,\alpha''}(x_0))/(x-x_0) < 0$.

(IV) Suppose $e_{2n-1} = e_{2n} = 1$. Let $x \in R_{e_1\ldots e_{2n-2}10}(\alpha)$, $y \in R_{e_1\ldots e_{2n-2}0111}(\alpha)$. Then $y < x < x_0$; $G^{\alpha,\alpha''}(x_0) > G^{\alpha,\alpha''}(x)$; $G^{\alpha,\alpha''}(x_0) > G^{\alpha,\alpha''}(y)$; $(G^{\alpha,\alpha''}(x_0) - G^{\alpha,\alpha''}(x))/(x_0-x) - (G^{\alpha,\alpha''}(x_0) - G^{\alpha,\alpha''}(y))/(x_0-y) > (G^{\alpha,\alpha''}(y) - G^{\alpha,\alpha''}(x))/(x_0-y) > 3a''_{2n+2}/a_{2n-2} \rightarrow 3b/16a$. Let $x_n = x_0 - a''_{2n-1}$ then $(G^{\alpha,\alpha''}(x_0) - G^{\alpha,\alpha''}(x_n))/(x_0-x_n) = a''_{2n}/a_{2n-1} \longrightarrow b/2a$.

By (I), (II), (III) and (IV) it follows that $G^{\alpha,\alpha''}$ has finite or infinite derivative at no point of $C(\alpha)$. Also $G^{\alpha,\alpha''}$ has a finite approximative derivative at no point $x_0 \in C(\alpha)$, x_0 a point of density of $C(\alpha)$. Clearly $G^{\alpha,\alpha''} \notin \mathfrak{B}(1)$ (see [13], p.222-223).

c) That $F_1^{\alpha,\alpha''}$ and $F_2^{\alpha,\alpha''}$ belong to S follows by a). Suppose that $G^{\alpha,\alpha''} \in [GS^*]$ on $[0,1]$ then it follows that there exists (u,v)

such that $(u,v)\cap C(\alpha) \neq \emptyset$ and $G^{\alpha,\alpha''}$ is S^* on $(u,v)\cap C(\alpha)$. There exists $R_{e_1\dots e_{2k}}(\alpha) \subset (u,v)$ such that $G^{\alpha,\alpha''}$ is S, hence T_1 on $R_{e_1\dots e_{2k}}(\alpha)$. By (2), b) and Theorem 6.2,p.278 of [13], it follows that $G^{\alpha,\alpha''} \notin T_1$ on $R_{e_1\dots e_{2k}}(\alpha)$, a contradiction.

d) Since α'' is of type (*), each $y \in C(\alpha'')$ has an unique representation $y = \sum_{i=1}^{\infty} e_i(y)c_i''$ and $(G^{\alpha,\alpha''})^{-1}(y)\cap C(\alpha) =$

$\{\sum_{i=1}^{\infty} e_{2i-1}(y)c_{2i} + e_{2i}(y)c_{2i-1})\}$, hence $G^{\alpha,\alpha''}$ is bijective on $C(\alpha)$.

e) If α'' is of type (**), but not of type (*) and $y \in C(\alpha'')$ has two representations, $y = \sum e_i(y)c_i'' = \sum e_i'(y)c_i''$, then

$$(G^{\alpha,\alpha''})^{-1}(y)\cap C(\alpha) = \{\sum_{i=1}^{\infty} e_{2i-1}(y)c_{2i} + e_{2i}(y)c_{2i-1}) ;$$

$$\sum_{i=1}^{\infty}(e_{2i-1}'(y)c_{2i} + e_{2i}'(y)c_{2i-1})\}.$$

f) Let $(u,v)\cap C(\alpha)$ be a portion of $C(\alpha)$. Then there exists a $R_{e_1\dots e_{2k}}(\alpha) \subset (u,v)$. Let $x_1 = \sum_{i=1}^{2k} e_i c_i + a_{2k+1}$, $x_2 = \sum_{i=1}^{2k} e_i c_i + c_{2k+1}$, $x_3 = \sum_{i=1}^{2k} e_i c_i + a_{2k}$ then $x_1 < x_2 < x_3$ belong to $R_{e_1\dots e_{2k}}(\alpha)$ and $G^{\alpha,\alpha''}(x_1) > G^{\alpha,\alpha''}(x_2) < G^{\alpha,\alpha''}(x_3)$.

g) Let $x<y$ belong to $C(\alpha)$ and let k be the first natural number such that (x,y) contains an open interval $O_{e_1\dots e_{k-1}}$ from the step k, with $2i < k \leq 2i+2$ for some natural number i. Then $[x,y]\subset R_{e_1\dots e_{2i}}(\alpha)$, hence $y-x > a_{k-1} - 2a_k > \min\{a_{2i+1} - 2a_{2i+2}, a_{2i} - 2a_{2i+1}\}$ and $|G^{\alpha,\alpha''}(y) - G^{\alpha,\alpha''}(x)| < a_{2i}''$. It follows that $G^{\alpha,\alpha''}$ satisfies condition L with constant M.

h) By c) and a), clearly $G^{\alpha,\alpha''} \in T_2 - N$. We prove that $G^{\alpha,\alpha''} \in M$.

The proof is based on an ideea of J. Foran of [8](p.85). In order to show that $G^{\alpha,\alpha''}$ satisfies Foran's condition M, by Theorem 1 of [8], it suffices to show that if $A\subset C(\alpha)$ and $G^{\alpha,\alpha''}$ is monotone on A then $G^{\alpha,\alpha''}$ satisfies Lusin's condition N on A. Suppose that $G^{\alpha,\alpha''}$ is increasing on $A\subset C(\alpha)$. Clearly $R_{e_1\dots e_{2k}}(\alpha'')$ are nonoverlapping intervals and $|R_{e_1\dots e_{2k}}(\alpha'')| = a''_{2k} \le 1/4^k$. Let $C_1 x C_2 x\dots x C_k = \{(e_1,\dots,e_{2k}) : A\cap R_{e_1\dots e_{2k}}(\alpha) \neq \emptyset\}$. By (2) it follows that $\|C_1 x C_2 x\dots x C_k\| \le 3^k$, hence $|G^{\alpha,\alpha''}(A)| < (3/4)^k \to 0$. Since $\mathcal{F} + \mathcal{F} = \mathcal{F} \subset N$ and $G^{\alpha,\alpha''} \notin N$ it follows that at least one of the functions $F_1^{\alpha,\alpha''}$ and $F_2^{\alpha,\alpha''}$ does not belong to $\mathcal{F}$ on $[0,1]$.

i) Let $C(\alpha) = C(\alpha') = C$ then $G^{\alpha,\alpha'} \in L$ (see g)) and $G^{\alpha',\alpha''} \in M\cap T_2$. If $|C(\alpha'')| > 0$ then $I^{\alpha,\alpha''}$ is increasing on $C(\alpha)$ and $I^{\alpha,\alpha''} \notin M$ on $C(\alpha)$. Since $M\circ L \supset M$, by (3) it follows that $M\circ L \supsetneq M$.

j) If $|C(\alpha'')| > 0$ and $|C(\alpha')| > 0$ then $G^{\alpha',\alpha''} \in \mathcal{F}(2)$ (see b)). If $|C(\alpha)| = 0$ then by b), $G^{\alpha,\alpha'} \in (M\cap T_2) - N$, but $I^{\alpha,\alpha''} \notin M$ on $C(\alpha)$. Since $\mathcal{F}(2)\circ M \supset M$, by (3) it follows that $\mathcal{F}(2)\circ M \supsetneq M$.

<u>Proof of Theorem 2.</u> a) By Theorem A, $S = N\cap T_1 = N^\infty \cap T_1$. Since $N\subset M \subset N^\infty = M_*$, by [4](Remark 1,e), Theorem 6 and Theorem 1, g)), it follows that $S = N\cap T_1 = M\cap T_1 = N^\infty\cap T_1 = M_*\cap T_1$.

b) $\mathcal{F}(1)\subset [GS^*]$. Indeed, let $f:[0,1]\to R$, $f\in ACG\cap\mathcal{C}$. It follows that there exist $P_n = \overline{P}_n$ such that $[0,1] = \bigcup_n P_n$ and $f_{P_n} \in AC\subset S$, hence $f\in [GS^*]$. By Theorem 1,a),b), $S - ACG \neq \emptyset$, hence $ACG \subsetneq [GS^*]$ on $[0,1]$. By [13](p.279), $ACG - S \neq \emptyset$, hence $S\subsetneq [GS^*]$ on $[0,1]$.

c) By a) and the definitions of $[GS^*]$ and $[GT_1^*]$ it follows that $[GS^*]\subset [GT_1^*]\cap N \subset [GT_1^*]\cap M$. Let $f\in \mathcal{C}\cap[GT_1^*]\cap M$ on $[0,1]$. It follows that there exist $P_n = \overline{P}_n$ such that $f_{P_n} \in \mathcal{C}\cap T_1\cap M = \mathcal{C}\cap S$ (see a)), hence $f\in [GS^*]$ on $[0,1]$. Then $[GS^*] = [GT_1^*]\cap N =$

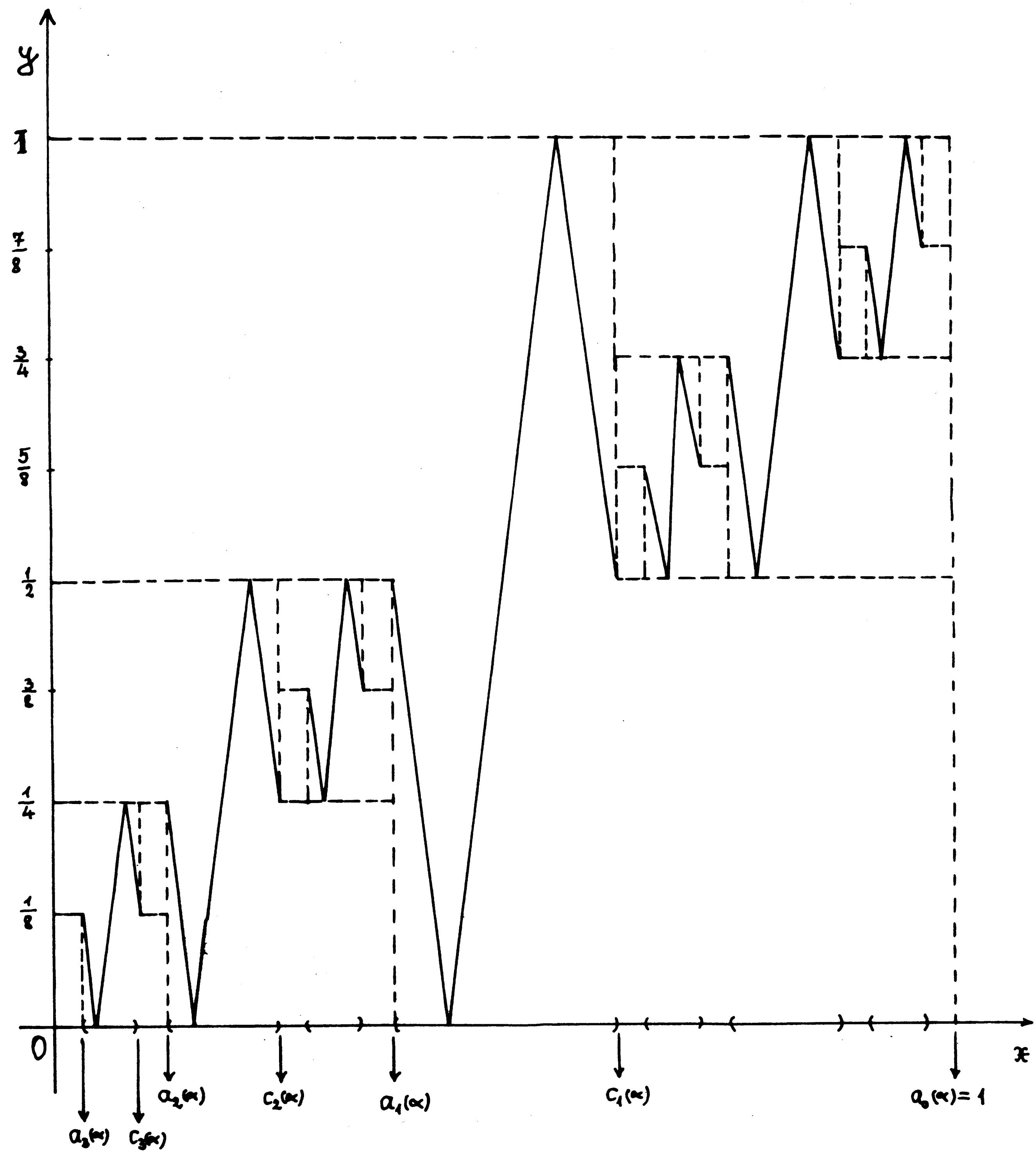

fig 1

there exist $P_n = \overline{P}_n$ and natural numbers N_n such that $f \in A(N_n)$ on P_n. By definitions, $f \in S$ on P_n, hence $\mathcal{F} = [\mathcal{F}] \subset [GS]$ on $[0,1]$. By Theorem 1,b),c) it follows that $\mathcal{F} - [GS^*] \neq \emptyset$, hence $[GS^*] \subsetneq [GS]$ on $[0,1]$. By Theorem 1,a),b) it follows that $S - \mathcal{F} \neq \emptyset$, hence $\mathcal{F} \subsetneq [GS]$. Clearly $[GS] \subset GS \subset N$. To prove that $[GS] \subsetneq GS$ we shall construct the following example. At first we construct a continuous function $g:[0,1] \longrightarrow [0,1]$, using the notations of c). We suppose that $|C(\alpha)| > 0$. Let $g(x) = I^{\alpha}(x)$, $x \in C(\alpha)$; $g(x) = g(a_i^s) + (i/2^s) \cdot I^{\alpha}((x-a_i^s)/(c_i^s-a_i^s))$, $x \in [a_i^s, c_i^s]$; $g(x) = g(c_i^s) - (1/2^{s-1}) \cdot I^{\alpha}((x-c_i^s)/(d_i^s-c_i^s))$, $x \in [c_i^s, d_i^s]$; $g(x) = g(d_i^s) + (1/2^s) \cdot I^{\alpha}((x-d_i^s)/(b_i^s-d_i^s))$, $x \in [d_i^s, b_i^s]$. Let $P_1 = C(\alpha) \cup (\bigcup_{s=1}^{\infty} \bigcup_{i=1}^{2^{s-1}} \{[(c_i^s-a_i^s) \cdot C(\alpha) + a_i^s] \cup [(d_i^s-c_i^s)C(\alpha)+c_i^s] \cup [(b_i^s-d_i^s)C(\alpha)+d_i^s]\})$. We show that $g(0) = 0$; $g(1) = 1$; g is constant on each interval contiguous to P_1; g is ACG on $[0,1]$; $g^{-1}(y)$ is infinite for each $y \in [0,1]$. Using the function g, we can construct a continuous function f_1: $[0,1] \longrightarrow [0,1]$ and a nowhere dense, perfect subset Q_1 of $[0,1]$ with positive measure, such that $f_1(0) = f_1(1) = 0$; $\inf(Q_1) = 0$, $\sup(Q_1) = 1$; f_1 is constant on each interval contiguous to Q_1; $f_1 \in ACG$; $f^{-1}(y)$ is infinite. Let $\{I_n^1\}_n = \{(u_n^1, v_n^1)\}_n$ be the intervals contiguous to Q_1. Let $Q_k = Q_{k-1} \cup (\bigcup_{n=1}^{\infty} (u_n^{k-1} + (v_n^{k-1} - u_n^{k-1}) Q_1))$, $k = 2,3,\ldots$, where (u_n^k, v_n^k), $n = 1,2,\ldots$ are the intervals contiguous to Q_k. Let $f_{k+1}(x) = 0$, $x \in Q_k$; $f_{k+1}(x) = (1/2^{n+k+1}) \cdot f_1((x-a_n^k)/(b_n^k-a_n^k))$, $x \in [a_n^k, b_n^k]$, $k = 1,2,\ldots$. Let $F(x) = \sum_{k=1}^{\infty} f_k(x)$. Let $H = [0,1] - \bigcup_{m=1}^{\infty} Q_m$. $F \in ACG$ on $\cup Q_n$ and $|F(H)| = 0$, hence

$F \in GS \subset N$. But $F \notin [GT_1]$ because F is not T_1 on any interval, hence by d) $F \notin [GS]$; but clearly $F \in GS$.

f) Since VB = B(1) on a set E it follows that VBG = $\mathfrak{B}(1)$. Let $f:[0,1] \to R$, $f \in \mathcal{C} \cap VBG$. Then there exist $P_n = \bar{P}_n$ such that $f_{P_n} \in VB \subset T_1$ ([13],p.279). It follows that for continuous functions on [0,1], $VBG = [VBG] = [\mathfrak{B}(1)] \subset [GT_1^*]$. Since $[VBG] \cap N \cap \mathcal{C} = [ACG] \cap \mathcal{C}$ ([13], Theorem 6.8,p.228) and $[GT_1^*] \cap N = [GS^*]$ (see c)) on [0,1], by b), it follows that $[\mathfrak{B}(1)] \subsetneq [GT_1^*]$. By [9]((ii),p.360), $\mathfrak{B} = [\mathfrak{B}]$. By [9]((iv),p.360) and [13](p.279), it follows that $[\mathfrak{B}] \subset [GT_1]$. Each of the functions F_q, defined in the proof of Theorem 2 of [2], belongs to $T_1 - [\mathfrak{B}]$, hence $[\mathfrak{B}] \subsetneq [GT_1]$. Clearly $[GT_1^*] \subset [GT_1] \subset T_2$. By c),d) and e) it follows that $[GT_1^*] \subsetneq [GT_1]$. Let F be the function defined in e). Then $F \in N - [GS]$, hence $F \in T_2$ (see [13], Theorem 7.3,p.284). But $F \notin [GT_1]$, hence $[GT_1] \subsetneq T_2$.

<u>Theorem 3. For continuous functions defined on closed intervals we have</u>: a) $H \circ VBG \subset H \circ [GT_1^*] \subset H \circ [GB_2^*] = [GB_2^*]$(<u>see</u> [7], <u>Question 4</u>); b) $\bar{H} \circ ACG \subset \bar{H} \circ [GS^*] = [GS^*]$(<u>see</u> [7], <u>Question 8</u>) ; <u>Moreover</u> $[GS^*] \circ [GS^*] = [GS^*]$; c) $\bar{H} \circ VBG \subset \bar{H} \circ [GT_1^*] \subset [GS^*] \circ [GT_1^*]$ (<u>see</u> [7], <u>Question 9</u>); <u>Moreover</u> $[GS^*] \circ [GT_1^*] = [GT_1^*]$; d) $[GS] \circ [GS] = [GS]$ <u>and</u> $GS \circ GS = GS$; e) $[GS^*] \circ H \subset [GS^*] \circ [GT_1^*] = [GT_1^*]$ <u>and</u> $[GS] \circ H \subset [GS] \circ [GT_1] = [GT_1]$; f) $[GS'^*] \circ H \subset [GS'^*] \circ [GT_1^*] \subset [GB_2^*] = [GB_2^*] \circ H$; g) $GL \circ H = ACG \circ H = VBG \circ H = VBG$ <u>and</u> $GL \circ \bar{H} = ACG \circ \bar{H} = ACG$.

<u>Proof.</u> Let $f:[a,b] \to R$, $g:[c,d] \to R$, $g([c,d]) \subset [a,b]$ and let $F = f \circ g$, $f,g \in \mathcal{C}$.

a) The two inclusions are evident. We shall prove that $H \circ [GB_2^*] = [GB_2^*]$. It suffices to show that $H \circ [GB_2^*] \subset [GB_2^*]$. Suppose that $f \in H$, $g \in [GB_2^*]$. Then there exist $E_n = \bar{E}_n$ such that $[c,d] = \bigcup E_n$ and

$g_{E_n} \in B_2^*$. Clearly $f_{g(E_n)} \in H$. By Remark 4,1), $f_{g(E_n)} \circ g_{E_n} \in B_2$. By Lemma 6 and Lemma 7, $F_{E_n} \in B_2$.

b) Clearly $\overline{H} \circ ACG \subset \overline{H} \circ [GS^*]$. To prove that $[GS^*] = \overline{H} \circ [GS^*] = [GS^*] \circ [GS^*]$, it suffices to show that $[GS^*] \circ [GS^*] \subset [GS^*]$. Suppose that $f, g \in [GS^*]$. Then there exist $E_n = \overline{E}_n$ such that $[a,b] = \bigcup E_n$ and $f_{E_n} \in S$. Let $T_n = g^{-1}(E_n)$. Then T_n is closed, $[c,d] = \bigcup T_n$ and there exists a sequence of closed sets $T_{n,k}$ such that $T_n = \bigcup T_{n,k}$ and $g_{T_{n,k}} \in S$. By Proposition 2 or Remark 4,f), it follows that $f_{g(T_{n,k})} \circ g_{T_{n,k}} \in S$. By Lemma 6 and Lemma 7, $F_{T_{n,k}} \in S$.

c) Clearly $\overline{H} \circ VBG \subset \overline{H} \circ [GT_1^*] \subset [GS^*] \circ [GT_1^*]$. To prove that $[GS^*] \circ [GT_1^*] = [GT_1^*]$, see the proof of b), Remark 4,i), Lemma 6 and Lemma 7.

d) See the proof of b) and Proposition 2,a).

e) The first part follows by c). To prove that $[GS] \circ [GT_1] = [GT_1]$, see the proof of b) and Remark 4,i).

f) The first inclusion is evident and for the second see the proof of b) and Remark 4,k). To prove that $[GB_2^*] \circ H = [GB_2^*]$, it suffices to show that $[GB_2^*] \circ H \subset [GB_2^*]$. Suppose that $f \in [GB_2^*]$ and $g \in H$. Then there exist $E_n = \overline{E}_n$ such that $[a,b] = \bigcup E_n$ and $f_{E_n} \in B_2$. Let $T_n = g^{-1}(E_n)$. Then $T_n = \overline{T}_n$, $[c,d] = \bigcup T_n$ and $g_{T_n} \in H$. By Lemma 6 $F_{T_n} = f_{E_n} \circ g_{T_n}$ and by Remark 4,m) it follows that $F_{T_n} \in B_2$.

g) Since $H \cap N = \overline{H}$ and $VBG \cap N = ACG$ we have to prove only that $GL \circ H = ACG \circ H = VBG \circ H = VBG$. Clearly $GL \circ H \subset ACG \circ H \subset VBG \circ H = VBG$, so it remains to prove that $VBG \subset GL \circ H$. Let $F : [0,1] \to R$, $F \in VBG \cap \mathcal{C}$. Then there exist $E_n = \overline{E}_n$ such that $\bigcup E_n = [0,1]$ and $F_{E_n \cup \{0,1\}}$ is VB on $[0,1]$. Let $h_n(x) = A_n(x)/L_n$, $x \in [0,1]$, where $A_n(x)$ is the total arc-length of the graph of $F_{E_n \cup \{0,1\}}$ from 0 to x and $L_n = A_n(1)$ ([1], p.125). Let $h : [0,1] \to [0,1]$, $h(x) = \sum_{n=1}^{\infty} h_n(x)/2^n$. Let

$[GT_1^*] \cap M$ on $[0,1]$. To prove that $[GT_1^*] \cap M \subsetneq [GT_1^*] \cap N^\infty$ we construct the following example: for $C(\alpha)$ let $J_i^s(\alpha) = (a_i^s, b_i^s)$, $i = 1,2,\dots,2^{s-1}$ be the open intervals from the step s, numbered from the left to the right. Let $c_i^s < d_i^s$ belong to $J_i^s(\alpha)$. Let $\alpha'' = \{1/2^k\}$, $k \geq 0$. Put $I^{\alpha',\alpha''} = I^\alpha$. Let $f:[0,1] \to [0,1]$ be defined as follows: $f(x) = I^\alpha(x)$, $x \in C(\alpha)$; $f(c_i^s) = (i-1)/2^{s-1}$, $f(d_i^s) = i/2^{s-1}$, $i = 1,2,\dots,2^{s-1}$, $s = 1,2,\dots$. Extending f **linearly** on each interval contiguous to $C(\alpha) \cup (\bigcup_{s=1}^{\infty} \bigcup_{i=1}^{2^{s-1}} \{c_i^s, d_i^s\})$ we have f defined and continuous on $[0,1]$ (see fig.1 for the representation of the first three steps in the construction of the graph of f). The continuity follows by the fact that $O(f;R_{e_1\dots e_s}(\alpha)) = 1/2^s$. Since $f(\bigcup_{i=1}^{2^{s-1}} J_i^s(\alpha)) = [0,1]$ and $f^{-1}(y) \cap C(\alpha)$ has at most two points, it follows that $f^{-1}(y)$ is denumerable for each $y \in [0,1]$. Moreover, the set $f^{-1}(y) \cap R_{e_1\dots e_s}(\alpha)$ is infinite for each s. For $x_0 \in C(\alpha)$ and for each s there exist $e_1,\dots e_s$ such that $x_0 \in R_{e_1\dots e_s}(\alpha)$. It follows that 0 is a derived number for f at x_0. Since $\overline{\lim}_{s\to\infty} O(f;R_{e_1\dots e_s}(\alpha))/|R_{e_1\dots e_s}(\alpha)| \geq 1$, f has a finite or infinite derivative at no point of $C(\alpha)$, hence $f \in N^\infty$. Clearly $f \in [GT_1^*] - M$ on $[0,1]$ if $|C(\alpha)| = 0$. If $|C(\alpha)| > 0$ then $f \in ACG - B_2$, hence by Remark 4,j) it follows that $f \in ACG - S'$.

d) By **Lemma** 2, $[GS] = [GT_1] \cap N$. By Theorem 1,h) it follows that $[GT_1] - N \neq \emptyset$. Since $[GT_1] \cap N \subset [GT_1] \cap M$, it follows that $[GT_1] \cap N \subsetneq [GT_1] \cap M$. By c), $([GT_1] \cap N^\infty) - M \neq \emptyset$, hence $[GT_1] \cap M \subsetneq [GT_1] \cap N^\infty$.

e) Clearly $[GS^*] \subset [GS]$. By [9]((ii),p.360), if $f \in \mathcal{F}$ on $[0,1]$ then

$a < b$, $a, b \in E_n$, then $2^n[F(b)-F(a)]/(h_n(b)-h_n(a)) < 2^n L_n$. Let $h(a) = c$ and $h(b) = d$. Then $[F \circ h^{-1}(d) - F \circ h^{-1}(c)]/(d-c) < 2^n \cdot L_n$. Thus $F \circ h^{-1}$ is a Lipschitz function with constant $2^n \cdot L_n$.

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