Real Analysis Exchange Vol 15 (1989-90)

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EXTENSIONS OF DARBOUX FUNCTIONS

In 1875 in paper [2], the first example of a discontinuous Darboux function was given. Since that time, there have appeared many papers devoted to the study of the properties of those functions. It has turned out that the family of Darboux functions contains many other important classes of functions such as, for instance, derivatives ([2]), functions being approximate derivatives ([4]), and even certain subfamilies of the classes of derivatives and approximate derivatives that can take infinite values ([6],[14]).

The proving of a number of interesting properties for real Darboux functions of a real variable accounted for seeking a generalization of the notion of a Darboux function to the case of transformations defined and taking their values in more abstract spaces.

An essential difficulty in finding a generalization preserving the properties of real Darboux functions of a real variable as accurately as possible is the fact that those function possesses a number of interesting properties of the topological nature as well as many interesting properties connected with measure theory (cf. e.g.[15]). Analyzing one of better known definitions of the Darboux property, saying that a function $f : R \to R$ is a Darboux function if the image of each closed segment is connected, it can be noticed without difficulty that, while generalizing this notion, one should use the kind of sets whose topological nature as well as properties connected with measure theory are close to the properties of a closed segment of the line. Arcs seem to constitute such a family. In this context, throughout the paper, we shall adopt the following definition.

Definition ([8], [9], [10], [11]). We say that $f : X \to Y$, where X, Y are topological spaces, is a Darboux transformation (or possesses the Darboux property) if the image of any arc $L \subset X$ is a connected set.

The main aim of papers [8], [9] and [11] was to show that the adoption of such a definition of a Darboux function allows one to obtain, for transformations defined and taking their values in more general spaces, results analogous to those for a Darboux function defined on the line as well as creates new possibilities and problems which, many a time, have no analogue in the case of real functions of a real variable.

In many mathematical problems a rather essential role is played by the possibility of extending a given transformation defined on a subset of some space to a function defined on the whole space with the preservation of certain properties of the initial function. Many authors considered these problems for the function "approaching" Darboux functions (see [1], [12], [13]). The present paper is devoted to the investigation of the possibility of finding extensions of Darboux functions defined on some subsets of the plane and taking their values in R^2.

Throughout the paper, we apply the classical symbols and notations. However, in order to avoid any ambiguities, we shall now present those symbols used in the paper whose meanings are not explained in the main text. By the letter R we denote the set of real numbers (with the natural topology), whereas R^2 stands for the plane. The letters N, Q, I denote, respectively, the set of positive integers, the set of rational numbers and the interval $[0,1]$. The symbols (a,b), $(a,b]$, etc... denote open intervals, those open at the endpoint a, etc... in the spaces R or R^2.

Let $f : X \to Y$; then C_f (D_f) stands for the set of all points of continuity (discontinuity) of the function f. The symbol $\mathrm{const}_{x_0}^{X,Y}$ denotes a constant function mapping X into Y, such that the image of X in this transformation is a set $\{x_0\}$. The symbols $f \nabla g$ and $\underset{t\in T}{\nabla} f_t$ stand for combinations of transformations.

The image and preimage of an open (left-hand open etc...) interval is denoted by $f(a,b)$, $f^{-1}(a,b)$ $(f(a,b]$, $f^{-1}(a,b]$, etc...), and the dispensable double brackets are omitted.

By $\mathrm{Int}_A(B)$, $\mathrm{Fr}_A(B)$ we shall denote the interior and boundary in the subspace A of R^2. If $A = R^2$ then we write $\mathrm{Int}\, B$ and $\mathrm{Fr}\, B$, respectively. The closure of a set A we denote by $\bar{A}$.

We denote by the symbol $\overset{\to}{\measuredangle}([a,b], [a,c])$ an angle between the segments $[a,b]$ and $[a,c]$, positively oriented on the plane. Perpendicularity and parallelism are denoted by $\perp$ and $\parallel$, respectively. The symbol $\mathrm{proj}_L(A)$ means the projection of the set A on the line L.

By an arc $Ł$ we mean a subset of the plane R^2 which is homeomorphic (as a subspace) to I. If $h : I \xrightarrow{onto} Ł$ is a homeomorphism, then $h(0)$ and $h(1)$ are called endpoints of the arc $Ł$. The notation $L(a,b)$ is understood as: an arc with endpoints a and b. Let $Ł \subset R^2$ be an arbitrary arc, $h : I \xrightarrow{onto} Ł$ a homeomorphism, and let $c,d \in Ł$. Then there exists exactly one arc $Ł' \subset Ł$ such that $Ł' = L(c,d)$. The arc $Ł'$ will be denoted by $L_Ł(c,d)$.

By ρ we denote the natural metric in R^2 and ρ_A (or $\rho(x,A)$) denote the distance from the set A.

Let $X \subset R^2$ and let $f : X \to R^2$. A function $f^* : R^2 \to R^2$ is called an ε-extension of the function f $(\varepsilon \geq 0)$ if f^* is an extension of the function f (i.e. $f^*_{|X} = f$ and, for each $\alpha \in f^*(R^2)$, there exists $\beta \in f(X)$ such that $\rho(\alpha,\beta) \leq \varepsilon$.

Let Ψ be any family of sets. We shall denote by $\Psi_\sigma(\Psi_\delta)$ the family of all sets being countable unions (intersections) of sets from the class Ψ. So, let us define by transfinite induction the sequences $\{F_\alpha(\Psi)\}_{\alpha<\Omega}$ and $\{G_\alpha(\Psi)\}_{\alpha<\Omega}$(with Ω denoting the smallest uncountable ordinal number):

$$F_0(\Psi) = \Psi = G_0(\Psi),$$

$$F_\alpha(\Psi) = \begin{cases} \left(\bigcup\limits_{\xi<\alpha} F_\xi(\Psi)\right)_\sigma & \text{when } \alpha \text{ odd,} \\ \left(\bigcup\limits_{\xi<\alpha} F_\xi(\Psi)\right)_\delta & \text{when } \alpha \text{ even,} \end{cases}$$

$$G_\alpha(\Psi) = \begin{cases} \left(\bigcup\limits_{\xi<\alpha} G_\xi(\Psi)\right)_\delta & \text{when } \alpha \text{ odd,} \\ \left(\bigcup\limits_{\xi<\alpha} G_\xi(\Psi)\right)_\sigma & \text{when } \alpha \text{ even.} \end{cases}$$

Let F denote a family of closed sets. Then we shall write F_α in place of $F_\alpha(F)$ $(\alpha < \Omega)$. Let G stand for a family of open sets. Then we shall write G_α instead of $G_\alpha(G)$ $(\alpha < \Omega)$.

Let Ξ_α be the family of all functions $f : X \to Y$ where X,Y are some topological spaces, such that, for every open set $V \subset Y$,

$$f^{-1}(V) \in \begin{cases} F_\alpha & \text{in the case of } \alpha \text{ odd,} \\ G_\alpha & \text{in the case of } \alpha \text{ even.} \end{cases}$$

Then by the Baire class α $(\alpha < \Omega)$ we mean the family of functions:

$$B_\alpha(X,Y) = \begin{cases} \Xi_\alpha & \text{when } \alpha \text{ is a finite ordinal number,} \\ \Xi_{\alpha+1} & \text{when } \alpha \text{ is an infinite ordinal number.} \end{cases}$$

Throughout the paper, we assume the continuum hypothesis.

Before we prove the fundamental theorem of this paper we give 3 lemmas.

Let us adopt the following notation: if $F \subset R^2$, then let

$$A_0^F = \mathrm{Fr}\, F \text{ and } A_\alpha^F = \{x \in R^2 \colon \rho_F(x) = \alpha\} \text{ for } \alpha > 0.$$

LEMMA 1. Let $F \subset R^2$ be a closed convex set and let $p \in \mathrm{Int}\, F$. Then, if H_p denotes a half-line with the initial point p, then, for every $\alpha \geq 0$, $H_p \cap A_\alpha^F = \emptyset$ or $H_p \cap A_\alpha^F$ is a singleton.

From this lemma one can easily deduce:

LEMMA 2. Let $F \subset R^2$ be a closed convex set and let $p \in \mathrm{Int}\, F$; let H_p denote a half-line with the initial point p. Then, if there exists a real number $\alpha_0 \geq 0$ such that $H_p \cap A_{\alpha_0}^F \neq \emptyset$, then, for any $\alpha \geq 0$, the intersection $H_p \cap A_\alpha^F$ is a singleton.

LEMMA 3. Let F be a closed convex set such that $\mathrm{Int}\, F \neq \emptyset$ and let $p \in \mathrm{Int}\, F$; besides, let $\mathfrak{L} = L(a,b)$ be an arc such that $\mathfrak{L} \cap F = \{a,b\}$. Finally, let M_a (M_b) be a half-line with the initial point p, passing through a (b). Then one of the angles Γ, formed by these half-lines, possesses the following property: for each element $x \in \Gamma \cap \mathrm{Fr}\, F$ and each closed half-line H_x with the initial point x, such that $H_x \cap F = \{x\}$, the inequality $H_x \cap \mathfrak{L} \neq \emptyset$ is satisfied.

P r o o f. Suppose to the contrary that such an angle Γ does not exist. So, denote by Γ_1 and Γ_2 two angles formed by M_a and M_b. Then there exist points $x_i \in \Gamma_i \cap \operatorname{Fr} F$ $(i=1,2)$ and closed half-line H_{x_i} with the initial point x_i, such that $H_{x_i} \cap F = \{x_i\}$ and $H_{x_i} \cap Ł = \emptyset$. Of course, $x_1, x_2 \notin \{a,b\}$.

In that case, let M_i $(i=1,2)$ stand for a half-line with the initial point x_i, lying on a line passing through p, such that $M_i \cap F = \{x_i\}$. Denote $M_o = M_1 \cup M_2$ and let V_o (U_o) denote an open angle between M_1 and M_2 which contains a (b). Let A_o be a closed convex angle between M_1 and H_{x_1} (if $H_{x_1} \subset M_1$, we adopt $A_o = \emptyset$). Note that $A_o \cap F = \{x_1\}$. Let then

$$V^o = \begin{cases} (V_o \cup A_o) \setminus H_{x_1} & \text{when } H_{x_1} \subset U_o, \\ V_o \setminus A_o & \text{when } H_{x_1} \subset V_o, \end{cases}$$

$$U^o = \begin{cases} U_o \setminus A_o & \text{when } H_{x_1} \subset U_o, \\ (U_o \cup A_o) \setminus H_{x_1} & \text{when } H_{x_1} \subset V_o. \end{cases}$$

Similarly, let A_* be a closed convex angle between M_2 and H_{x_2} (if $H_{x_2} \subset M_2$, let us adopt $A_* = \emptyset$). Note that, analogously as above, $A_* \cap F = \{x_1\}$. Let then

$$M^* = H_{x_1} \cup [x_1,p] \cup [x_2,p] \cup H_{x_2},$$

$$V^* = \begin{cases} (V^o \cup A_*) \setminus H_{x_2} & \text{when } H_{x_2} \subset U_o, \\ V^o \setminus A_* & \text{when } H_{x_2} \subset V_o \end{cases}$$

and

$$U^* = \begin{cases} U_o \setminus A_* & \text{when } H_{x_2} \subset U_o, \\ (U_o \cup A_*) \setminus H_{x_2} & \text{when } H_{x_2} \subset V_o. \end{cases}$$

It is easily noticed that $M^* \cap Ł = \emptyset$ and M^* cuts R^2 into the sets V^* and U^* between the sets $\{a\}$ and $\{b\}$, which contradicts the connectedness of $Ł$.

THEOREM 4. Let $F \subset R^2$ be a closed and convex set and let $f : F \to R^2$ be a Darboux function. Then, for every $\varepsilon > 0$, there exists a Darboux function $f^* : R^2 \to R^2$ being an ε-extension of the function f and such that $C_{f^*} = C_f$.

REMARK. This theorem would not be true if we demanded that the function f^* be the 0-extension of the function f.

P r o o f of theorem 4. Evidently, the theorem is true if F is identical with R^2. So, in the sequel, we shall always assume that $F \neq R^2$.

Let $\varepsilon > 0$.

Assume now that $\operatorname{Int} F \neq \emptyset$ and F contains no half-plane. Then there exist a point $p \in \operatorname{Int} F$ and lines M^1 and M_1 such that $M^1 \cap M_1 = \{p\}$ and either of these lines intersects Fr F at two distinct points. Besides, let M^* denote a perpendicular to M^1 passing through p.

The line M^1 cuts R^2 into two open half-planes. Let $r > 0$ be a number such that $K(p,r) \subset F$. Let $\{m_1^1, m_2^1\} = M^1 \cap \mathrm{Fr}\, K(p,r)$. Denote by the symbol L_i $(i=1,2)$ a half-line with the initial point p, passing through m_i^1. Then either of the intersections $L_i \cap \mathrm{Fr}\, F$ is a one-element set and, in consequence, by lemma 2, either of the intersections $L_i \cap A_1^F$ is a one-element set, too. So, let $\{n_i^1\} = L_i \cap A_1^F$ $(i=1,2)$.

Let further x_1, x_2 be elements of the line M^*, such that $\rho(x_k,p) = \frac{r}{2}$ $(k=1,2)$. Then $x_1, x_2 \in \mathrm{Int}\, F$. Consequently, let K_i^k be a half-line with the initial point x_k, passing through n_i^1.

Adopt further notations. Let H_i^k $(i,k=1,2)$ be a closed half--line contained in K_i^k, with the initial point n_i^1 and let H_i^* $(i=1,2)$ be an open half-line contained in L_i, with the initial point n_i^1. Note that

(*) for each $\alpha > 1$ and each $i,k \in \{1,2\}$, there exist points $p_i^k \in H_i^k$ and $p_i^* \in H_i^*$ such that $p_i^k, p_i^* \in A_\alpha^F$.

Let now P_1' denote a closed convex set determined by the half-lines H_1^2, H_2^2 and the segment $[n_1^1, n_2^1]$ and, similarly, let Q_1' denote a closed convex set determined by the half-lines H_1^1, H_2^1 and the segment $[n_1^1, n_2^1]$. Let further P_1'' be an open convex angle formed by the half-lines H_1^2 and H_1^1, and let Q_1'' be an open convex angle formed by the half-lines H_2^1 and H_2^2.

In view of lemma 2, it is easy to see that

$$P_1'' \cup Q_1'' \subset \bigcup_{\alpha>1} A_\alpha^F.$$

Let us arrange all points from H_1^* in a transfinite sequence $\{y_\beta\}_{\beta<\Omega}$ and, similarly, points of the half-line H_2^* - in a transfinite sequence $\{z_\beta\}_{\beta<\Omega}$. Let then $K_{i,\beta}$ $(i=1,2)$ be a closed half-line with the initial point y_β, parallel to H_1^i, and let $K_\beta = K_{1,\beta} \cup K_{2,\beta}$ $(\beta < \Omega)$. In an analogous way we define the sets H_β $(\beta < \Omega)$ for the sequence $\{z_\beta\}_{\beta<\Omega}$.

Denote $F_1 = F \cap P_1'$ and $F_2 = F \cap Q_1'$. Then F_1 and F_2 are convex sets, thus $f(F_1)$ and $f(F_2)$ are connected sets. Adopt $F_1' = \bigcup_{x\in f(F_1)} K(x,\varepsilon)$ and $F_2' = \bigcup_{x\in f(F_2)} K(x,\varepsilon)$, and let $F' = F_1' \cup F_2'$. Then the sets F_1', F_2' and F' are connected.

In the family $\mathfrak{a}_1^1 = \{A_\alpha^F \cap P_1' : \alpha > 1\}$ let us introduce the equivalence relation $*$ defined in the following manner:

$$(1) \qquad (A_{\alpha'}^F \cap P_1') * (A_{\alpha''}^F \cap P_1') \Longleftrightarrow \alpha' - \alpha'' \in Q.$$

This relation decomposes the family $\mathfrak{a}_1^1$ into disjoint equivalence classes. Denote the collection of these classes by $\hat{\mathfrak{a}}_1^1$. Since F_1' is a set of cardinality continuum, there exists a one-to-one function $h_1 : \hat{\mathfrak{a}}_1^1 \xrightarrow{\text{onto}} F_1'$.

Similarly, in the family $\mathfrak{a}_2^1 = \{A_\alpha^F \cap Q_1' : \alpha > 1\}$ let us introduce the equivalence relation defined by the formula analogous to (1) and let $\hat{\mathfrak{a}}_2^1$ stand for the collection of the equivalence classes of this relation. Then there exists a one-to-one function

$h_2 : \hat{\mathfrak{a}}_2^1 \xrightarrow{\text{onto}} F_2'$.

Let us now define in the family $\mathfrak{a}_3^1 = \{K_\beta : \beta < \Omega\}$ the equivalence relation $\circ$ defined in the following way:

$$(1') \qquad K_{\beta'} \circ K_{\beta''} \Longleftrightarrow \rho(y_{\beta'}, y_{\beta''}) \in Q.$$

Let $\hat{\mathfrak{a}}_3^1$ denote the collection of all equivalence classes of this relation and let h_3 denote a bijective transformation of the collection $\hat{\mathfrak{a}}_3^1$ onto F'.

Similarly, in the family $\mathfrak{a}_4^1 = \{H_\beta : \beta < \Omega\}$ we define the equivalence relation analogously as (1') and assume that h_4 is a bijective mapping of the collection $\hat{\mathfrak{a}}_4^1$ of all equivalence classes of this relation onto F'.

Let $g_1 : \bigcup_{\alpha>1} A_\alpha^F \to R^2$ be a function defined by the formula

$$g_1(x) = \begin{cases} h_1([A_1^1(x)]) & \text{when} \quad x \in P_1' \cap \bigcup_{\alpha>1} A_\alpha^F, \\ h_2([A_2^1(x)]) & \text{when} \quad x \in Q_1' \cap \bigcup_{\alpha>1} A_\alpha^F, \\ h_3([A_3^1(x)]) & \text{when} \quad x \in P_1'', \\ h_4([A_4^1(x)]) & \text{when} \quad x \in Q_1'', \end{cases}$$

where, for $x \in \bigcup_{\alpha>1} A_\alpha^F$, $A_i^1(x)$ $(i=1,2,3,4)$ denotes the set of the damily $\mathfrak{a}_i^1$ to which x belongs, while the symbol $[\cdot]$ denotes the equivalence class determined by one of the above relations.

Suppose that we have already defined the function $g_1, g_2, \ldots, g_n$ and the lines $M^1, M^2, \ldots, M^{2^{n-1}}$ corresponding to them (M^1 is the line considered in the construction of the function g_1). These lines divide the circle $\mathrm{Fr}\, K(p,r)$ into 2^n parts, and the plane - - into 2^n convex angles (with vertices at the point p). If $n>1$, then, for each of those angles, let us draw a line being the bisector of this angle. Denote the lines by $M^{2^{n-1}+1},\dots,M^{2^n}$. Whereas if $n=1$, then let $M^2 = M_1$. The lines $M^1,\dots,M^{2^n}$ divide the plane into 2^{n+1} convex angles (with vertices at the point p) and intersect $\mathrm{Fr}\, F$ at the points $k_1,k_2,\dots,k_s$ where $s \le 2^{n+1}$.

About the points $k_1,\dots,k_s$ we may additionally assume that if $s' < s''$, then $\vec{\measuredangle}([p,k_1],[p,k_{s'}]) \subset \vec{\measuredangle}([p,k_1],[p,k_{s''}])$. So, for each $t \in \{1,2,\dots,s\}$, let H_t denote a half-line with the initial point p, passing through k_t, and let

$$\{m^t_{n+1}\} = A^F_{\frac{1}{n+1}} \cap H_t \quad \text{and} \quad \{m^t_n\} = A^F_{\frac{1}{n}} \cap H_t.$$

In view of lemma 2, it is easily noticed that

$$\rho(k_t,m^t_{n+1}) < \rho(k_t,m^t_n) \qquad \text{for} \quad t \in \{1,2,\dots,s\}.$$

In order to simplify the notation in the further part of the proof, let us accept the following convention:

the index 0 is equivalent to the index s,

the index s+1 is equivalent to the index 1

(that is, e.g. H_o stands for H_s and $H_{s+1} = H_1$ and the thing is similar when other notations are introduced, for instance, $F_o = F_s$, $F_{s+1} = F_1$, etc.).

It is easy to observe that none of the angles $\vec{\measuredangle}([p,k_i],[p,k_{i+1}])$ (for $i=1,\dots,s$) under consideration is flat.

Let further, for every $t \in \{1,2,\dots,s\}$, H_t',H_t'' denote two distinct half-lines with the initial point m_{n+1}^t, such that:

1. H_t' (H_t'') is contained in a convex angle formed by the half-lines H_t and H_{t-1} (H_{t+1}).
2. If L is a line containing H_t' or H_t'', then $L \cap K(p,r) \neq \emptyset$.
3. $H_t' \cap H_{t-1}'' = \emptyset$ and $H_t'' \cap H_{t+1}' = \emptyset$.

Let us now adopt some further notations. For any $t\in\{1,\dots,s\}$, let P_t denote a closed convex set determined by the half-lines H_t'', H_{t+1}' and the segment $[m_{n+1}^t,p],[m_{n+1}^{t+1},p]$, and let P_t' denote open convex angles formed by H_t' and H_t''. Finally, let F_t stand for a convex set formed by the intersection of the set F; a closed half-plane determined by a line passing through the points k_t and k_{t+1}, to which p does not beong; and a closed angle formed by the half-lines H_t and H_{t+1}. It is easy to check that $f(F_t)$ ($t \in \{1,\dots,s\}$) is a connected set. So, let

$$F_t' = \bigcup_{x\in f(F_t)} K(x, \frac{\varepsilon}{n+1}) \quad \text{for} \quad t \in \{1,\dots,s\}.$$

Then F_t' ($t \in \{1,\dots,s\}$) is also a connected set and card $F_t' = \mathfrak{c}$ and, similarly, $F_{t-1}' \cup F_t'$ is a connected set of cardinality continuum (for $t \in \{1,\dots,s\}$).

Note that

$$P_t \cap A^F_\alpha \neq \emptyset$$

and

$$(m^t_{n+1}, m^t_n] \cap A^F_\alpha \neq \emptyset \text{ for } \alpha \in (\tfrac{1}{n+1}, \tfrac{1}{n}] \quad \text{and } t \in \{1,\ldots,s\},$$

and, moreover,

$$(m^t_{n+1}, m^t_n] \subset \bigcup_{\frac{1}{n+1} < \alpha \leq \frac{1}{n}} A^F_\alpha.$$

Now, fix some number $t \in \{1,\ldots,s\}$. Let then $\gamma^t_i \in L^t_i \cap K(p,r)$ $(i=1,2)$ where L^t_1 and L^t_2 are lines containing H'_t and H''_t, respectively. Let, for any $w \in [0, \frac{1}{2n(n+1)})$, $\{\hat{m}^t_w\} = H_t \cap A^F_{\frac{2n+1}{2n(n+1)} - w}$. In virtue of the continuity of the function ρ_F and lemma 2, it is easy to observe that, for each $w \in [0, \frac{1}{2n(n+1)})$, $\hat{m}^t_w \in (m^t_{n+1}, \hat{m}^t_0]$. So, for any $w \in (0, \frac{1}{2n(n+1)})$, let $L'_{t,w}$ $(L''_{t,w})$ denote a half-line with the initial point γ^t_1 (γ^t_2), passing through $\hat{m}^t_w$. Denote by the symbol Q^t_w a closed convex angle formed by these half-lines and let $\{q'_{t,w}\} = L'_{t,w} \cap A^F_{\frac{2n+1}{2n(n+1)}+w}$ and $\{q''_{t,w}\} = L''_{t,w} \cap A^F_{\frac{2n+1}{2n(n+1)}+w}$. Finally, assume that for each $w \in (0, \frac{1}{2n(n+1)})$, $K^t_w = [q'_{t,w}, \hat{m}^t_w] \cup [\hat{m}^t_w, q''_{t,w}] \cup (A^F_{\frac{2n+1}{2n(n+1)}+w} \cap Q^t_w)$ and $K^t_0 = \{\hat{m}^t_0\}$.

Note that

$$K^t_w \cap K^t_{w'} = \emptyset \qquad \text{if} \qquad w \neq w'.$$

We shall now show that:

$$(**)\quad \begin{array}{l} \text{for any } x \in P_t' \cap \bigcup_{\frac{1}{n+1}<\alpha<\frac{1}{n}} A_\alpha^F, \text{ there exists exactly} \\ \text{one element } w \in [0, \frac{1}{2n(n+1)}) \text{ such that } x \in K_w^t. \end{array}$$

Relationship (**) is, of course, true if $x = \hat{m}_o^t$. Consequently, in the sequel, we shall always assume that $x \neq \hat{m}_o^t$. So, let $x \in$ $\in P_t' \cap A_{\alpha_x}^F$.

Assume first that $\alpha_x > \frac{2n+1}{2n(n+1)}$. Denote $w_x = \alpha_x - \frac{2n+1}{2n(n+1)}$. Consider two cases:

1^o $x \in Q_{w_x}^t$. Then, as can easily be seen, $x \in K_{w_x}^t$.

2^o $x \notin Q_{w_x}^t$. Then, let us draw lines T_x^1, T_x^2 passing through the point x and through the points γ_1^t and γ_2^t, respectively. The line containing H_t cuts R^2 between $\{x\}$ and $\{\gamma_1^t\}$ or between $\{x\}$ and $\{\gamma_2^t\}$. Without loss of generality, suppose that the first situation takes place. Then T_x^1 intersects the segment $[\hat{m}_o^t, m_{n+1}^t]$ at some point $\hat{m}_{w^*}^t$ where $w^* > w_x$. Note that then $x \in [q'_{t,w^*}, \hat{m}_{w^*}^t]$ and, in consequence, $x \in K_{w^*}^t$.

So, assume that $\alpha_x \leq \frac{2n+1}{2n(n+1)}$ and adopt $w_x = \frac{2n+1}{2n(n+1)} - \alpha_x$. Consider the following cases:

1^{oo} $x \in H_t$. Then, as can easily be noticed, $x \in K_{w_x}^t$.

2^{oo} $x \notin H_t$. In this case, reasoning analogously as in 2^o, we shall get that $x \in K_{w^*}^t$ for some $w^* > w_x$.

In view of the disjointness of the sets K_w^t, the considerations we have carried out allow us to infer that relation (**) is really true.

Let, in the sequel, $t \in \{1,\dots,s\}$ and let $\hat{P}'_t$ $(\tilde{P}_t)$ denote an open convex angle contained in P'_t, formed by the lines H'_t and H_t (H''_t and H_t). Besides, adopt $A'_t = A^F_{\frac{1}{n}} \cap \hat{P}'_t$, $A''_t = A^F_{\frac{1}{n}} \cap (\tilde{P}_t \cup P_t)$, and further, for each $x \in A'_t$, let π'_x denote a measure of a convex angle contained in P'_t formed by the half-line H_t and a half-line with the initial point m^t_{n+1}, passing through the point x, and, for any $x \in A''_t$, let π''_x denote a measure of a convex angle formed by the half-line H_t and a half-line with the initial point p, passing through the point x. Let π'_t, π''_t stand for measures of angles between H_t and H'_t and H_t and H^*_{t+1}, respectively, where H^*_{t+1} denotes a half-line with the initial point p, passing through the point being the intersection $H'_{t+1} \cap A^F_{\frac{1}{n}}$. Then, as can be seen:

> for each $a \in (0,\pi'_t)$, there exists exactly one element $x \in A'_t$ such that $a = \pi'_x$ and, for each $x \in A'_t$, $0 < \pi'_x < \pi'_t$,

and

> for each $a \in (0,\pi''_t]$, there exists exactly one element $x \in A''_t$ such that $a = \pi''_x$ and, for each $x \in A''_t$, $0 < \pi''_x \le \pi''_t$.

So, let $\mathfrak{a}^t = \{A^F_\alpha \cap P_t : \frac{1}{n+1} < \alpha < \frac{1}{n}\}$ $\quad (t \in \{1,\dots,s\})$. Then, in each of these families, one can define the equivalence relation * in the following manner:

$$(A_{\alpha'} \cap P_t) * (A_{\alpha''} \cap P_t) \Longleftrightarrow \alpha' - \alpha'' \in Q.$$

This relation decomposes each of these families into disjoint equivalence classes. The collection of those equivalence classes is denoted by $\hat{\mathfrak{a}}^t$. Then there exists a one-to-one function $h^t : \hat{\mathfrak{a}}^t \xrightarrow{\text{onto}} F'_{t-1} \cup F'_t$.

At present, let us introduce in the set A'_t (A''_t), for $t \in \{1,\dots,s\}$, the relation Δ defined by the formula

$$x \,\Delta\, y \iff \pi'_x - \pi'_y \in Q \qquad (\pi''_x - \pi''_y \in Q).$$

The collections of equivalence classes determined by these relations are denoted by $\hat{\mathfrak{a}}'_t$ and $\hat{\mathfrak{a}}''_t$, respectively. Then there exist one-to-one functions

$$h'_t : \hat{\mathfrak{a}}'_t \xrightarrow{\text{onto}} F'_{t-1} \quad \text{and} \quad h''_t : \hat{\mathfrak{a}}''_t \xrightarrow{\text{onto}} F'_t$$

$(t \in \{1,\dots,s\})$.

Let, finally, $\mathfrak{a}_t = \{K^t_w : 0 \le w < \frac{1}{2n(n+1)}\}$ $(t \in \{1,\dots,s\})$. In each of these families one can define the equivalence relation $\circ$ in the following way:

$$K^t_w \circ K^t_{w'} \iff w - w' \in Q.$$

The collection of equivalence classes of this relation is denoted by $\hat{\mathfrak{a}}_t$ $(t \in \{1,\dots,s\})$. Let h_t be a bijective mapping of $\hat{\mathfrak{a}}_t$ onto $F'_{t-1} \cup F'_t$.

So, let $g_{n+1} : \bigcup_{\frac{1}{n+1}<\alpha\leq\frac{1}{n}} A_\alpha^F \to R^2$ be a function defined by the formula

$$g_{n+1}(x) = \begin{cases} h^t([A^t(x)]) & \text{when } x \in P_t \cap \bigcup_{\frac{1}{n+1}<\alpha<\frac{1}{n}} A_\alpha^F \text{ for } 1 \leq t \leq s, \\ h_t([A_t(x)]) & \text{when } x \in P_t' \cap \bigcup_{\frac{1}{n+1}<\alpha<\frac{1}{n}} A_\alpha^F \text{ for } 1 \leq t \leq s, \\ h_t'([x]) & \text{when } x \in A_t' \text{ for } 1 \leq t \leq s, \\ h_t''([x]) & \text{when } x \in A_t'' \text{ for } 1 \leq t \leq s, \\ f(k_t) & \text{when } x = m_n^t \text{ for } 1 \leq t \leq s, \end{cases}$$

where, for $x \in \bigcup_{\frac{1}{n+1}<\alpha\leq\frac{1}{n}} A_\alpha^F$, the symbol $A^t(x)$ $(A_t(x))$ denotes the collection of the family $\mathfrak{a}^t$ $(\mathfrak{a}_t)$, to which x belongs, while the symbol $[\cdot]$ - the equivalence class determined by one of the above relations.

Going on like this, we shall define an infinite family $\{g_n\}_{n=1}^{\infty}$ of functions. Since the domains of transformations belonging to this family and the function f are disjoint, the functions $f, g_1, g_2, \ldots$ are compatible. Put

$$f^* = f \nabla \mathop{\nabla}_{n=1}^{\infty} g_n : R^2 \to R^2.$$

We shall demonstrate that f^* satisfies all conditions occuring in the assertion of the theorem.

It can be noticed without difficulty that f^* is an ε-extension of the function f.

We shall now prove that f^* is a Darboux function. For the purpose, it should be shown that:

(2) $\qquad f^*(Ł)$ is a connected set for any arc $Ł \subset R^2$.

We shall first demonstrate that:

(3) $\qquad f^*(Ł)$ is a connected set for any arc $Ł \subset \overline{R^2 \setminus F}$.

It is easy to see that if $Ł \subset \bigcup_{\alpha>1} A_\alpha^F$, then

$$f^*(Ł) = g_1(Ł) = \begin{cases} \{h_1([A_\alpha^F \cap P_1'])\} & \text{when } Ł \subset A_\alpha^F \cap P_1' \text{ for some } \alpha > 1, \\ \{h_2([A_\alpha^F \cap Q_1'])\} & \text{when } Ł \subset A_\alpha^F \cap Q_1' \text{ for some } \alpha > 1, \\ \{h_3([K_\beta])\} & \text{when } Ł \subset K_\beta \text{ for some } \beta < \Omega, \\ \{h_4([H_\beta])\} & \text{when } Ł \subset H_\beta \text{ for some } \beta < \Omega, \\ F_1' & \text{when } Ł \subset P_1' \cap \bigcup_{\alpha>1} A_\alpha^F, \text{ and} \\ & Ł \not\subset A_\alpha^F \cap P_1' \text{ for any } \alpha > 1, \\ F_2' & \text{when } Ł \subset Q_1' \cap \bigcup_{\alpha>1} A_\alpha^F, \text{ and} \\ & Ł \not\subset A_\alpha^F \cap P_1' \text{ for any } \alpha > 1, \\ F' & \text{otherwise,} \end{cases}$$

and, thus, $f^*(Ł)$ is then a connected set.

It can be proved that if $Ł \subset \bigcup_{\frac{1}{n+1}<\alpha\le\frac{1}{n}} A_\alpha^F$ $(n = 1,2,\ldots)$, then $f^*(Ł) = g_{n+1}(Ł)$ is a one-element set or a union of some number of sets F_t' with successive indices, thus it is a connected set, too.

So, let $Ł$ be any arc contained in $\overline{R^2 \setminus F}$, such that, for any positive integer n, $Ł \not\subset \bigcup_{\frac{1}{n+1}<\alpha\le\frac{1}{n}} A_\alpha^F$ and $Ł \not\subset A_o^F$ (if $Ł \subset A_o^F$, then $Ł \subset F$ and, thereby, $f^*(Ł) = f(Ł)$).

Let M denote any half-line with the initial point p, non--disjoint from $Ł$. Then, for any real number φ, let M^φ (M_φ) denote a half-line with the initial point p, such that the measure of the angle between the lines M and M^φ (M_φ) is equal to φ and this angle, when M is its initial side, has a positive (negative) orientation. Let us adopt:

$$\varphi_1 = \sup \{\varphi_o \in [0,2\pi] : M^\varphi \cap Ł \neq \emptyset \text{ for any } \varphi \le \varphi_o\},$$

$$\varphi_2 = \sup \{\varphi_o \in [0,2\pi] : M_\varphi \cap Ł \neq \emptyset \text{ for any } \varphi \le \varphi_o\}.$$

Evidently, $M^{\varphi_1} \cap Ł \neq \emptyset \neq M_{\varphi_2} \cap Ł$. The closed angle between M and M^{φ_1} (which is positively oriented) will be denoted by M^+, whereas that between M and M_{φ_2} (negatively oriented) - by M^-. Besides, let us adopt $M_o = M^+ \cup M^-$. Let $F^* = \mathrm{Fr}\, F \cap M_o$. It is not difficult to notice that each half-line with the initial point p, contained in M_o, is non-disjoint from $Ł$.

In view of the connectedness of $Ł$, it can easily be deduced that $\mathcal{L} \subset M_o$.

At present, we shall show that F^* is a point or an arcwise connected set.

Indeed, let $0^* = M_o \cap \mathrm{Fr}\, K(p,r)$. Define mapping $\psi^*: 0^* \xrightarrow{\text{onto}} F^*$ in the following way: let $\psi^*(x)$ be a point forming a set $H^x \cap \mathrm{Fr}\, F$ where H^x stands for a half-line with the initial point p, passing through x. Since, for any $x \in 0^*$, $H^x \subset M_o$, therefore (by lemma 2) this function is well-defined. Of course, ψ^* is a homeomorphism, thus F^* is an arcwise connected set, which allows us to conclude that:

(4) $\qquad f^*(F^*)$ is a connected set.

We shall now prove that:

(5) if M' is a half-line with the initial point p, contained in M_o, and $k_o \in M' \cap F^*$ and $c_o \in$ $\in M' \cap \mathcal{L}$, then $f(k_o)$ belongs to that component C of the set $f^*(\mathcal{L})$ which contains $f^*(c_o)$.

Indeed, if $c_o \in F^*$, then, by lemma 1, $c_o = k_o$ and this fact is self-evident. So, assume that $c_o \notin F^*$. Let $\alpha_o > 0$ be a number such that $c_o \in A^F_{\alpha_o}$. Consider the following cases:

1. There exists a positive integer n_o such that $\frac{1}{n_o+1} < \alpha_o <$ $< \frac{1}{n_o}$ or $\alpha_o > 1$. In this part of our reasoning, let us adopt the notations used when constructing the function g_{n+1} - our conside-

rations will be restricted only to the case when $\alpha_o < 1$ (when $\alpha_o > 1$, the proof is analogous). Then there exists a positive integer t_o such that $k_o \in F_{t_o}$. Thus $c_o \in P_{t_o}$ or $c_o \in P'_{t_o}$ or $c_o \in P'_{t_o+1}$. Since, in accordance with the assumption adopted,

$$Ł \not\subset \bigcup_{\frac{1}{n_o+1}<\alpha<\frac{1}{n_o}} A^F_\alpha, \quad \text{therefore:}$$

If $c_o \in P'_{t_o}$, then, for each equivalent class $\mathfrak{a} \in \hat{\mathfrak{a}}_{t_o}$, there exists a set $A \in \mathfrak{a}$ such that $Ł \cap A \neq \emptyset$.

If $c_o \in P'_{t_o+1}$, then, for each equivalnet class $\mathfrak{a} \in \hat{\mathfrak{a}}_{t_o+1}$, there exists a set $A \in \mathfrak{a}$ such that $Ł \cap A \neq \emptyset$.

If $c_o \in P_{t_o}$, there may occur two cases:

$-Ł \cap P_{t_o} \subset A^F_{\alpha_o}$. Then $Ł \cap P'_{t_o} \neq \emptyset$ or $Ł \cap P'_{t_o+1} \neq \emptyset$.

Then, however, for each equivalence class $\mathfrak{a} \in \hat{\mathfrak{a}}_{t_o}$, there exists a set $A \in \mathfrak{a}$ such that $Ł \cap A \neq \emptyset$ or, for each equivalence class $\mathfrak{a} \in \hat{\mathfrak{a}}_{t_o+1}$, there exists a set $A \in \mathfrak{a}$ such that $Ł \cap A \neq \emptyset$.

$$-Ł \cap P_{t_o} \not\subset A^F_{\alpha_o}.$$

Then, for each equivalence class $\mathfrak{a} \in \hat{\mathfrak{a}}^{t_o}$, there exists a set $A \in \mathfrak{a}$ such that $Ł \cap A \neq \emptyset$.

In each of the situations described above, relationship (5) really takes place (because the component C of the set $f^*(Ł)$, $f^*(c_o)$, always contains F'_{t_o}, as well).

2. There exists a positive integer n_o such that $\alpha_o = \frac{1}{n_o}$. Similarly as in case 1, as shall apply the notations used when constructing the function g_{n+1}, understanding that they concern the function g_{n_o+1}. Similar notations with symbol (n_o) will concern the function g_{n_o} (for example, $F_{t_o}^{(n_o)'}$ denotes the set F_{t_o}' construed for the function g_{n_o}). There exists a positive integer t_o such that $k_o \in F_{t_o}$. Then $c_o \in P_{t_o}$ or $c_o \in P_{t_o}'$ or $c_o \in P_{t_o+1}'$. Then:

If $c_o = m_{n_o}^{t_o}$ $(m_{n_o}^{t_o+1})$, then $k_o = k_{t_o}$ (k_{t_o+1}) and relationship (5) is true.

If $c_o \in A_{t_o}''$ (as $k_o \in F_{t_o}$, therefore $c_o \notin A_{t_o}'$), let us consider the following cases:

- there exist $a,b \in \mathcal{L}$ such that $L_{\mathcal{L}}(a,b) \subset A_{t_o}''$; then, for each equivalence class $[x] \in \hat{\mathfrak{a}}_{t_o}''$, $\mathcal{L} \cap [x] \neq \emptyset$;

- there exist $a,b \in \mathcal{L}$ such that $L_{\mathcal{L}}(a,b) \subset A_{t_o+1}'$; then, for each equivalence class $[x] \in \hat{\mathfrak{a}}_{t_o+1}'$, $\mathcal{L} \cap [x] \neq \emptyset$;

- for any neighbourhood V of the point c_o, $\mathcal{L} \cap V \cap$ $\cap \bigcup_{\frac{1}{n_o+1}<\alpha<\frac{1}{n_o}} A_\alpha^F \neq \emptyset$; then, for each equivalence class $\mathfrak{a} \in \hat{\mathfrak{a}}^{t_o}$,

there exists a set $A \in \mathfrak{a}$ such that $\mathcal{L} \cap A \neq \emptyset$ or, for each equivalence class $\mathfrak{a} \in \hat{\mathfrak{a}}_{t_o}$, there exists a set $A \in \mathfrak{a}$ such that $\mathcal{L} \cap A \neq \emptyset$ or, for each equivalence class $\mathfrak{a} \in \hat{\mathfrak{a}}_{t_o+1}$ there exists a set $A \in \mathfrak{a}$ such that $\mathcal{L} \cap A \neq \emptyset$;

- for any neighbourhood V of the point c_o, $\mathcal{L} \cap V \cap$

$\cap \bigcup_{\alpha > \frac{1}{n_o}} A_\alpha^F \neq \emptyset$. Then, for each equivalence class $\mathfrak{a}^{(n_o)} \in \hat{\mathfrak{a}}^{(n_o)}_{t_o'}$ (t_o' is an index such that $H_{t_o'}^{(n_o)} = H_{t_o}$ or H_{t_o} is contained in the angle between $H_{t_o'}^{(n_o)}$ and $H_{t_o'+1}^{(n_o)}$), there exists a set $A^{(n_o)} \in \mathfrak{a}^{(n_o)}$ such that $A^{(n_o)} \cap Ł \neq \emptyset$ (then $F_{t'}^{(n_o)'} \subset f^*(Ł)$).

If $c_o \in A'_{t_o+1}$, the reasoning runs analogously as above.

It is easily observed that, in each of the situations considered, relationship (5) is reall true (of course, $F'_{t_o} \subset F_{t_o}^{(n_o)'}$) since always if $c_o \in A''_{t_o} \cup A'_{t_o+1}$, then $f(k_o)$, $f(c_o) \in F'_{t_o}$ and $F'_{t_o} \subset f^*(Ł)$.

By (5), in view of (4), we may infer that, also in this case, relationship (3) is true and, moreover, that

$$(6) \qquad f(F^*) \subset f^*(Ł) \quad \text{for any arc } Ł \subset \overline{R^2 \setminus F}.$$

In order to show the varacity of statement (2) definitely assume that $Ł$ is an arc such that $Ł \cap \operatorname{Int} F \neq \emptyset \neq Ł \setminus F$. Let h be a homeomorphism mapping $[0,1]$ onto $Ł$. Let further

$$A_Ł = \{x \in [0,1] : h(x) \in F\} \quad \text{and} \quad B_Ł = [0,1] \setminus A_Ł.$$

Then $B_Ł$ is an open set in $[0,1]$, thus it possesses open components. Suppose that Γ is such a component. Then $\Gamma = (a,b)$ $(0 \leq \leq a < b \leq 1)$, $\Gamma = [0,b)$ or $\Gamma = (a,1]$. Assume first that $\Gamma = = (a,b)$. Then the arc $L_{\mathfrak{L}}(h(a),h(b))$ possesses the property that $L_{\mathfrak{L}}(h(a),h(b)) \cap F = \{h(a),h(b)\}$. Let M_a (M_b) be a half-line with the initial point p, passing through the point $h(a)$ $(h(b))$. Denote by the symbol M_Γ a closed angle between M_a and M_b with the property that, for any $x \in M_\Gamma \cap \mathrm{Fr}\, F$ and any closed half-line H_x with the initial point x, such that $H_x \cap F = \{x\}$, we have $H_x \cap L_{\mathfrak{L}}(h(a),h(b)) \neq \emptyset$ (such an angle exists in virtue of lemma 3). Since $h(a) \neq h(b)$, therefore $\mathrm{Fr}\, F \cap M_\Gamma$ is an arc.

Let $h_\Gamma : [a,b] \to \mathrm{Fr}\, F \cap M_\Gamma$ be a homeomorphism such that $h_\Gamma(a) = h(a)$ and $h_\Gamma(b) = h(b)$.

Now, consider the situation when $\Gamma = [0,b)$. Let M_a (M_b) be a half-line with the initial point p, passing through $h(0)$ $(h(b))$. In the case when $M_a \neq M_b$, let M_Γ denote a closed angle between M_a and M_b with the property that, for any $x \in M_\Gamma \cap \mathrm{Fr}\, F$ and for any half-line H_x with the initial point p, passing through x, $H_x \cap L_{\mathfrak{L}}(h(0),h(b)) \neq \emptyset$ (the proof of the existence of such arc is analogous to that of lemma 3). It is easy to check that $\mathrm{Fr}\, F \cap M_\Gamma$ is an arc. So, let $h_\Gamma : [0,b] \to \mathrm{Fr}\, F \cap M_\Gamma$ be a homeomorphism such that $h_\Gamma(b) = h(b)$. In the case when $M_a = M_b$, adopt $M_\Gamma = \{h(b)\}$. Let then

$$h_\Gamma = \mathrm{const}_{h(b)}^{[0,b],R^2} \quad .$$

In a similar way one constructs the function h_Γ when $\Gamma = (\alpha,1]$.

Let $g = h_{|A_{\mathcal{L}}} \nabla \underset{\Gamma}{\nabla} h_{\Gamma} : [0,1] \xrightarrow{\text{onto}} (F \cap \mathcal{L}) \cup \bigcup_{\Gamma} (\text{Fr } F \cap M_{\Gamma})$ where the combination of functions and the union of sets are considered over all components of the set $B_{\mathcal{L}}$.

We shall demonstrate that g is a continuous function. Let $x_o \in [0,1]$ and let $\varepsilon' > 0$. If x_o is an element of any component Γ of the set $B_{\mathcal{L}}$, then, of course, $x_o \in C_g$ and, similarly, if $x_o \in \text{Int } A_{\mathcal{L}}$. Consequently, assume that $x_o \in A_{\mathcal{L}} \cap B_{\mathcal{L}}^d$.

Let us first assume that, for some component Γ of the set $B_{\mathcal{L}}$, the equality $\Gamma = (x_o, b)$ (where $x_o < b$) takes place. Then there exists $\delta > 0$ such that $x_o + \delta < b$ and $g[x_o, x_o+\delta) = h_{\Gamma}[x_o, x_o+\delta) \subset$ $\subset K(g(x_o), \varepsilon')$. A similar argumentation can, of course, be carried out when $\Gamma = (a, x_o)$ where $a < x_o$. So, assume that x_o is not an (without loss of generality - left-hand) endpoint of any component of the set $B_{\mathcal{L}}$, and that $x_o \neq 1$. Then there exists a sequence $\{\Gamma_n\}$ of components of the set $B_{\mathcal{L}}$, such that $\lim_{n\to\infty} \Gamma_n = \{x_o\}$ and $\Gamma > x_o$ for $n=1,2,\ldots$. We then have

(7) $\quad h([x_o-\delta, x_o+\delta] \cap [0,1]) \subset K(h(x_o), \varepsilon')$ for some $\delta > 0$.

It can be assumed here that if the component Γ of the set $B_{\mathcal{L}}$ is not non-disjoint from $[x_o-\delta, x_o+\delta]$ and $\Gamma > x$, then $\Gamma \subset [x_o, x_o+\delta]$ and $\delta < \rho(x_o, 1)$. It is easily noticed that $g([x_o, x_o+\delta] \cap A_{\mathcal{L}}) \subset$ $\subset K(g(x_o), \varepsilon')$. We shall now show that

(8) $\quad g([x_o, x_o+\delta] \cap B_{\mathcal{L}}) \subset K(g(x_o), \varepsilon')$.

Let $\Gamma = (a,b)$ be any component of the set $B_{\mathcal{L}}$, such that $\Gamma \subset$ $\subset [x_o, x_o+\delta]$ and let $c \in \Gamma$. Then $g(c) = h_\Gamma(c) \in \mathrm{Fr}\, F \cap M_\Gamma$. Let N stand for a half-line with the initial point $g(x_o)$, passing through $g(c)$. There may occur two cases:

1^* $[g(x_o), g(c)] \cap \mathrm{Int}\, F \neq \emptyset$. Then the closed half-line $N^* \subset N$ with the initial point $g(c)$ possesses, by lemma 2, the property that $N^* \cap \mathrm{Fr}\, F = \{g(c)\}$.

2^* $[g(x_o), g(c)] \cap \mathrm{Int}\, F = \emptyset$. Then $[g(x_o), g(c)] \subset \mathrm{Fr}\, F$. So, let N^* denote a closed half-line with the initial point $g(c)$, perpendicular to N and such that $N^* \cap F = \{g(c)\}$.

Continuing our reasoning identically for both the cases, we note that, in virtue of lemma 3, $N^* \cap L_{\mathcal{L}}(h(a), h(b)) \neq \emptyset$. Let $\xi \in$ $\in N^* \cap L_{\mathcal{L}}(h(a), h(b))$. By (7), $\xi \in K(g(x_o), \varepsilon')$, which means that $g(c) \in K(g(x_o), \varepsilon')$.

A similar argumentation can be carried out when x_o is not a right endpoint of any component of the set $B_{\mathcal{L}}$ and $x_o \neq 0$.

The above considerations enable us to deduce that g is really a continuous function. This allows us to find that $F_* = (F \cap \mathcal{L}) \cup$ $\cup \bigcup_\Gamma (\mathrm{Fr}\, F \cap M_\Gamma)$ is an arcwise connected set, thus $f(F_*)$ is a connected set.

Note that $f^*(\mathcal{L}) = f^*(h(A_{\mathcal{L}})) \cup \bigcup_\Gamma f^*(\overline{h(\Gamma)})$. Note also that, for any component Γ, $\overline{h(\Gamma)} = L_{\mathcal{L}}(h(a), h(b))$ because $\bar{\Gamma} = [a,b]$, that is, $\overline{h(\Gamma)}$ is an arc such that $\overline{h(\Gamma)} \not\subset \bigcup_{\frac{1}{n+1} < \alpha \leq \frac{1}{n}} A_\alpha^F$ for any n; therefore, in view of (6),

(9) $$f^*(\mathcal{L}) = f(F_*) \cup \bigcup_{\Gamma} f^*(\overline{h(\Gamma)}).$$

On the ground of (3), we may infer that, for any component Γ of the set $B_{\mathcal{L}}$, $f^*(\overline{h(\Gamma)})$ is a connected set, whereas, in virtue of (6), we may additionally observe that $f^*(\overline{h(\Gamma)}) \cap f(F_*) \neq \emptyset$, which, in view of the connectedness of the set $f(F_*)$ shown before, proves (2) by (9).

At present, we shall prove that

(10) $$C_{f^*} = C_f.$$

We shall adopt the notations used when constructing the function g_{n+1}, with that the encircled index denotes an index of the function a given object has relation to (for instance, $H_t^{(n_1)}$ stands for the half-line H_t corresponding to the function g_{n_1}).

It can easily be noticed that $D_f \subset D_{f^*}$, $\operatorname{Int} F \cap C_f \subset C_{f^*}$ and $R^2 \setminus F \subset D_{f^*}$, and so, in order to prove equality (10), it is sufficient to show that

(11) $$\operatorname{Fr} F \cap C_f \subset C_{f^*}.$$

Let $x_o \in \operatorname{Fr} F \cap C_f$ and let $\varepsilon^* > 0$. Then there exists $\delta > 0$ such that $f^*(K(x_o,\delta) \cap F) \subset K(f^*(x_o), \frac{\varepsilon^*}{2})$. Let T stand for a half-line with the initial point p, passing through x_o. It is not difficult to verify that x_o is the endpoint of the arcs $\mathcal{L}_1$

and $Ł_2$ contained in Fr F and lying on different sides of the line containing T. Consequently, there exist arcs $\hat{Ł}_1 \subset Ł_1$ and $\hat{Ł}_2 \subset Ł_2$ having their endpoints at x_o and such that $\hat{Ł}_1, \hat{Ł}_2 \subset K(x_o,\delta)$ So, let n_1 be a positive integer such that, among the hal- -lines $H_t^{(n_1)}$ occuring in the construction of the function g_{n_1}, there are four - denote them by H_o, H_o^*, H_*^o, H^o - such that $H_o \cap \hat{Ł}_1 \neq \emptyset \neq H_o^* \cap \hat{Ł}_1$, $H^o \cap \hat{Ł}_2 \neq \emptyset \neq H_*^o \cap \hat{Ł}_2$ and the open convex angle Γ_o formed by the half-lines H_o and H^o contains the open convex angle Γ^o formed by H_o^* and H_*^o, and besides, $T \setminus \{p\} \subset \Gamma^o$.

Let $n_o \geq n_1$ be a positive integer such that $\frac{\varepsilon}{n_o} < \frac{\varepsilon^*}{2}$.

Adopt $V_o = K(x_o,\delta) \cap \Gamma \cap (\bigcup_{0 \leq \alpha < \frac{1}{n_o}} A_\alpha^F \cup F)$. Evidently, V_o is an open set containg x_o. We shall demonstrate that

(12) $$f^*(V_o) \subset K(f^*(x_o),\varepsilon^*).$$

Indeed, let $x \in V_o$. If $x \in F$, then $f^*(x) \in K(f^*(x_o), \frac{\varepsilon^*}{2})$. So, assume that $x \notin F$. Since $n_o \geq n_1$, therefore the half-lines H_o, H_o^*, H_*^o and H^o are included in all decompositions made when constructing $g_{n_o}, g_{n_o+1}, \ldots$. Denote $\hat{F} = \overline{\Gamma_o \cap \text{Fr } F}$. Then $\hat{F} \subset K(x_o,\delta)$.

Let $x \in A_{\alpha_x}^F$. Then $0 < \alpha_x < \frac{1}{n_o}$, so, let $\frac{1}{n^*+1} < \alpha_x \leq \frac{1}{n^*}$. Then

$f^*(x) \in \bigcup_{y \in F \cap K(x_o,\delta)} K(f(y), \frac{\varepsilon}{n^*+1}) \subset K(f(x_o),\varepsilon^*)$. By (12), in view of the free choice of x_o, we may infer that inclusion (11) is true, which ends the proof of the theorem in the case when $\text{Int } F \neq \emptyset$

and F contains no half-plane.

Consequently, let us now assume that $\operatorname{Int} F \neq \emptyset$ and F contains some half-plane. Then there exists a closed half-plane $\pi_1 \subset \subset \operatorname{Int} F$ with edge S, such that $\pi_2 = R^2 \setminus \operatorname{Int} \pi_1 \not\subset F$. Then $F^* = F \cap \pi_2$ is a closed convex set with non-empty interior and F^* contains no half-plane. In virtue of the fact proved above, there exists a Darboux function $f_* : R^2 \to R^2$ being an ε-extension of the function $f_{|F^*}$, such that $C_{f_*} = C_{f|_{F^*}}$. Let us then adopt

$$f^*(x) = \begin{cases} f_*(x) & \text{when} \quad x \in \pi_2, \\ f(x) & \text{when} \quad x \in \pi_1. \end{cases}$$

It is easily noticed that f^* satisfies all conditions of the assertion of our theorem.

So, consider the situation when F is a closed and convex boundary set. Then F is a point, a closed segment, a closed half-line or a line. Let S stand for a line containing F. Let $f_* : S \to R^2$ be the $\frac{\varepsilon}{2}$-extension of the function f, such that $C_{f^*} = C_f$. The line S cuts R^2 into two half-planes U and V. Define a function $\hat{f} : \bar{U} \to R^2$ in the following manner: $\hat{f}(x) = f_* \ (\operatorname{proj}_S(x))$. Then f is a Darboux function and one can easily observe that

$$x \in C_{\hat{f}} \Longleftrightarrow \operatorname{proj}_S(x) \in C_{f_*} \Longleftrightarrow \operatorname{proj}_S(x) \in C_f.$$

In accordance with the part of the theorem we have already proved, there exists a Darboux $\frac{\varepsilon}{2}$-extension $\tilde{f} : R^2 \to R^2$ of the function $\hat{f}$, such that $C_{\tilde{f}} = C_{\hat{f}}$. Let $S_S : \bar{U} \to \bar{V}$ be a symmetry with respect to the line S. Then S_S is a continuous function. Define $f^* : R^2 \to R^2$ as follows:

$$f^*(x) = \begin{cases} \tilde{f}(x) & \text{when} \quad x \in \bar{V}, \\ \tilde{f}(S_S(x)) & \text{when} \quad x \in \bar{U}. \end{cases}$$

It is easy to see that f^* is a Darboux function being an ε-extension of the function f. Besides, note that $C_{f^*} = C_f$, which completes the proof of the theorem.

Now, the following question may arise: is in the above theorem the assumption of the convexity of the set F essential, or is it sufficient to assume about F that it is a closed and connected set or a continuum (for simplicity, we shall assume that all the sets under consideration are contained in the cube $[0,1] \times [0,1]$)? In 1922 B. Knaster in paper [5] showed an example of a <u>non-one-element continuum whose no subcontinuum is decomposable</u> (i.e. <u>it cannot be represented in the form of the union of two subcontinua different from</u> K). It is easy to define on this continuum a Darboux function which does not possess the 1-extension to another Darboux function. It turns out that such a situation is not incidental since on the basis of the Mazurkiewicz theorem proved in [7], it can be observed that:

PROPOSITION 5. In the space composed of continua contained in the square $[0,1] \times [0,1]$ with the Hausdorff metric of the exponential space, the set F of continua for which there exists a Darboux function $f : F \to R^2$ not possessing the 1-extension to another Darboux function is a dense set.

Consequently, there may be raised a further question: for what sets $F \subset R^2$ do the Darboux function f defined on F possess (ordinary) extensions to the Darboux function f^* in such a way that $C_f = C_{f^*}$? A partial answer to the above question is give by the succesive two propositions being, in substance, corollaries from theorem 4.

PROPOSITION 6. Let $F \subset R^2$ be a set such that there exists a homeomorphism $h : R^2 \to R^2$ such that $h(F)$ is a closed convex set. Then, if $f : F \to R^2$ is a Darboux function, then there exists a Darboux function $f^* : R^2 \to R^2$ being an extension of the function f, such that $C_f = C_{f^*}$.

P r o o f. Adopt $g = f \circ h^{-1}_{|h(F)} : h(F) \to R^2$. It is easy to notice that g is a Darboux function. Consequently, by theorem 4, there exists a Darboux function $g^* : R^2 \to R^2$ being an extension of the function g, such that $C_{g^*} = C_g$.

Put $f^* = g^* \circ h : R^2 \to R^2$. It is easily noted that f^* is a Darboux function. Moreover, we have $f^*(x) = g^*(h(x)) = f(x)$ for each $x \in F$. And note that $C_g = h(C_f)$; so, indeed $C_{f^*} = C_f$.

PROPOSITION 7. Let $F = F_1 \cup F_2 \subset I^2$ where F_1, F_2 are closed convex sets such that $F_1 \cap F_2 = \emptyset$. Then, if $f : F \to R^2$ is a Darboux function, then there exists a Darboux function $f^* : R^2 \to R^2$ being an extension of the function f, such that $C_{f^*} = C_f$.

P r o o f. Adopt $U_i = \{x : \rho_{F_i}(x) \in [0, \frac{1}{3}\rho(F_1, F_2))\}$ for $i = 1,2$, Then U_1 and U_2 are open sets such that $F_1 \subset U_1$, $F_2 \subset U_2$ and $\bar{U}_1 \cap \bar{U}_2 = \emptyset$. Besides, adopt $V = R^2 \setminus (\bar{U}_1 \cup \bar{U}_2)$. Then V is also a non-empty open set. Define a function $\hat{f}_o : V \to R^2$ in the following way: Let $\mathfrak{a} = \{A_r = \{x : \rho(x, \bar{U}_1 \cup \bar{U}_2) = r\} : r > 0\}$. Then one can define the equivalence relation $*$ in the family $\mathfrak{a}$:

$$A_{r_1} * A_{r_2} \Longleftrightarrow r_1 - r_2 \in Q.$$

Let $\hat{\mathfrak{a}}$ be the collection of all equivalence classes of this relation. It can easily be verified that, for any class $\mathfrak{a}' \in \hat{\mathfrak{a}}$, there exists a real number r such that $A_r \in \mathfrak{a}'$ and $A_r \neq \emptyset$. Let then $h : \hat{\mathfrak{a}} \xrightarrow{\text{onto}} R^2$ be a bijection. Put $\hat{f}_o(x) = h([A_{r_x}])$ where A_{r_x} denotes the set $\{x' : \rho(x', \bar{U}_1 \cup \bar{U}_2) = \rho(x, \bar{U}_1 \cup \bar{U}_2)\}$, while $[A_{r_x}]$ stands for the equivalence class determined by this set.

Note that $f_{|F_i}$ $(i=1,2)$ is a Darboux function. Consequently, in virtue of theorem 4, there exist Darboux functions $f_i^* : R^2 \to R^2$ which are extensions of the function $f_{|F_i}$ and such that $C_{f_i^*} = C_{f|F_i}$ $(i=1,2)$. Then the functions $\hat{f}_i = f^*_{i|\bar{U}_i}$ $(i=1,2)$ are Darboux functions, too. Adopt $f^* = \mathop{\nabla}\limits_{i=1}^{2} \hat{f}_i : R^2 \to R^2$. It is easy to notice that f^* is a Darboux function and $\bar{V} = D_{f^*}$, and thus, indeed, $C_{f^*} = C_f$.

In the succesive theorem we shall present the possibility of the 0-extension of a Darboux function with the preservation of a suitable class of Baire. To simplify the notation, the fact that f belongs to the Baire class α will be written down: $f \in B_\alpha$ $(0 \leq \alpha < \Omega)$.

THEOREM 8. Let F be a closed and convex subset of the plane and let $f : F \to R^2$ be a Darboux function such that $f \in B_\alpha$ for some $0 \leq \alpha < \Omega$. Then there exists a Darboux function $f^* : R^2 \to R^2$ being the 0-extension of the function f, such that $f^* \in B_\alpha$.

P r o o f. In the case when F is a singleton, it suffices to adopt $f = \mathrm{const}_{f(x_o)}^{R^2, R^2}$. Consequently, assume that F is not a one-element set. Adopt then the following notations:

If $\mathrm{Int}\, F \neq \emptyset$, let p denote a fixed point belonging to $\mathrm{Int}\, F$. Then, for any $p_o \in \mathrm{Fr}\, F$, let H_{p_o} denote a closed half-line with the initial point p_o, such that the line L, containing H_{p_o}, contains the point p, and $H_{p_o} \cap F = \{p_o\}$.

Whereas if $\mathrm{Int}\, F = \emptyset$, let S stand for a line containing F. Then, if $p_o \in \mathrm{Int}_S\, F$, let H_{p_o} denote a line perpendicular to S passing through p_o, and if $p_o \in F \setminus \mathrm{Int}_S\, F$, let H_{p_o} denote a closed half-plane with the edge being a perpendicular to S such that $H_{p_o} \cap F = \{p_o\}$. It is not hard to notice that:

for each $x \in R^2 \setminus \mathrm{Int}\, F$, there exists exactly one point $p_x \in \mathrm{Fr}\, F$ such that $x \in H_{p_x}$.

Define a function $f_* : R^2 \to F$ in the following manner: $f_*(x) = x$ when $x \in F$; if $x \notin F$, let $f_*(x) = p_x$ where p_x is an element of the set $\mathrm{Fr}\, F$ such that $x \in H_{p_x}$.

We shall prove that:

(1) f_* is a continuous function.

This fact is evident when $\mathrm{Inf}\, F = \emptyset$ (f_* can be represented as a superposition of two continuous functions).

So, assume that $\mathrm{Int}\, F \neq \emptyset$. Of course, $\mathrm{Int}\, F \subset C_{f_*}$. Consequently, let $x_o \notin \mathrm{Int}\, F$ and let $\varepsilon > 0$. Adopt $f_*(x_o) = p_{x_o} \in \mathrm{Fr}\, F$. Let K be a closed rectangle with vertices a,b,c,d such that:

1^o $p_{x_o} \in \mathrm{Int}\, K \subset K \subset K(p_{x_o}, \varepsilon)$,

2^o $[a,b] \perp H_{p_{x_o}} \perp [d,c]$ and $[a,d] \parallel H_{p_{x_o}} \parallel [b,c]$,

3^o $[a,b] \subset F$ and $[c,d] \subset R^2 \setminus F$.

Let H^c, H^d denote half-lines with the initial point p, passing through the points c and d, respectively. Let Γ stand for an open convex angle formed by these half-lines. It is easy to observe that then $\Gamma \setminus (F \setminus K(p_{x_o}, \varepsilon))$ is a neighbourhood of the point x_o and, moreover, $f_*(\Gamma \setminus (F \setminus K(p_{x_o},\varepsilon))) \subset K(p_{x_o},\varepsilon) \cap F$, which proves (1) definitively.

Adopt $f^* = f \circ f_* : R^2 \to R^2$. Then, as can easily to seen f^* is a Darboux function being the 0-extension of the function f and, of course, $f^* \in B_\alpha$.

COROLLARY 9. Let $F \subset R^2$ be a set such that there exists a homeomorphism $h : R^2 \to R^2$ such that $h(F)$ is a closed and convex set. Then, if $f : F \to R^2$ is a Darboux function such that $f \in B_\alpha$ $(0 \le \alpha < \Omega)$, there exists a Darboux function $f^*: R^2 \to R^2$ being an extension of the function f, such that $f^* \in B_\alpha$.

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Received April 12, 1989