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THE STRUCTURE OF ω-LIMIT SETS FOR CONTINUOUS FUNCTIONS

1. INTRODUCTION

Even for well-behaved self-maps of an interval, ω-limit sets (attractors) can exhibit rather interesting (and, for the uninitiated, somewhat bizarre) behaviour. Such a set can be finite, or a countable closed set, or a Cantor set, or an interval, or certain combinations of these. A number of papers, some dating back to the Sixties, indicated some of the possibilities as well as some of the limitations (see, for example $[S_1]$, $[S_2]$, and [HOLE]). Much of the work on the subject involves a good deal of technical machinery that might be difficult for nonspecialists. And there are some troublesome errors in the literature. For example, Šarkovskii seems to say in $[S_1]$ that if an infinite ω-limit set has an isolated point, then it has infinitely many isolated points; see Example 1 below.

Our main purpose is to characterize the closed sets which can be ω-limit sets for continuous functions.

We begin with an elementary example which illustrates how various nowhere dense closed sets can be realized as, ω-limit sets of continuous functions. We do this via one rather simple function mapping [0,1] into itself. This function actually illustrates a rudimentary form of our theorem that every compact, nowhere dense set M is an ω-limit set for some continuous function. More precisely, we show that for every such set M there is a homeomorphic copy M_0 of M contained in $\left[\frac{1}{3},\frac{4}{9}\right]$ such that for some x, $\omega(x,f) \cap \left[\frac{1}{3},\frac{4}{9}\right] = M_0$; that is, a copy of M is that part of the ω-limit set of f which lies in $\left[\frac{1}{3},\frac{4}{9}\right]$.

Our proof that every nowhere dense closed set is an ω-limit set proceeds in stages. Our main tool is Theorem 1, which provides a condition for a sequence $\{z_n\}$ of numbers to be the orbit of some point under a continuous function. We handle separately three types of closed, nowhere dense sets, each requiring its own treatment. First, we deal with the case where the set is uncountable and intersects some interval in a Cantor set. We then address the countable case. Here we find it convenient to deal first with a certain specialized type of countable set and then invoke results (Lemmas 5 and 6) which show that this type of set suffices for the general countable case. Finally, we treat the case in which the set of isolated points is dense in the entire set.

The results of these three parts are labeled "Propositions". Each depends on some technical lemmas and on a concept involving extension of a homeomorphism from a closed nowhere dense set C to a set whose interior has C in its closure.

NOTATION AND TERMINOLOGY

The symbol I will denote the unit interval $[0,1]$. For $f : I \to I$ and $x \in I$ we define $f^0(x) = x$; and $f^{n+1}(x) = f\left(f^n(x)\right)$ for each natural number n. By the <u>orbit</u> of x under f we mean the set $\gamma(x,f) = \left\{f^n(x) : n \in \omega_0\right\}$, where ω_0 is the set of natural numbers. If A is a set, then $\mathrm{cl}\, A$ or $\bar{A}$ denotes the closure of A, and A' denotes the set of limit points of A.

The expression $|A|$ represents the diameter of A; $h|A$ is the restriction of the function h to A; and $d(x,A)$ is the distance from x to A. The notation $\{x_n\}_{n=0}^{\infty}$ denotes the sequence as a function whereas $\{x_n : n \in \omega_0\}$ is the range of the function. The <u>cluster</u> <u>set</u> of $\{x_n\}_{n=0}^{\infty}$ is the set of subsequential limit points of $\{x_n\}_{n=0}^{\infty}$.

Let f be a continuous function. The <u>ω-limit</u> <u>set</u> (some authors use the term "attractor set") $\omega(x,f)$ is defined to be the cluster set of $\{f^n(x)\}_{n=0}^{\infty}$. In the sequel, whenever we write $\omega(x,f)$, it is understood that $f : I \to I$ is a continuous function and $x \in I$.

AN EXAMPLE

We begin with a simple example which illustrates how various, possibly complicated nowhere dense closed sets can arise as ω-limit sets.

It will be convenient to work with base 3 arithmetic. If the ternary expansion of x is $.x_1x_2x_3\ldots$, then $1 - x = .x_1^*x_2^*x_3^*\ldots$, where $x_i^* = 2 - x_i$. If $A = a_1a_2a_3\ldots a_n$ is a finite string or block in which $a_i \in \{0,1,2\}$ for each i, then we define A^* by $A^* = a_1^*a_2^*a_3^*\ldots a_n^*$. We denote the length of A by $\rho(A)$. We will describe ternary representations in terms of blocks. For example, $.1A_11A_2\ldots 1A_n\ldots$ is the number $.1a_{11}a_{12}\ldots a_{1n_1}1a_{21}a_{22}\ldots a_{2n_2}1\ldots$. This will be abbreviated $.\overline{1A_i}$. Additionally, a ternary number of the form $.aa\ldots aa_{n+1}a_{n+2}\ldots$ will be denoted by $.[a]_n a_{n+1}a_{n+2}\ldots$. Note that the effect of multiplying a ternary number by 3^n is to move the ternary point n places to the right if $n > 0$ and n places to the left if $n < 0$.

Let T be the ordinary Cantor set in I. Let $K = \frac{1}{3}T + \frac{1}{3}$ and put $I_0 = \left[\frac{1}{3},\frac{2}{3}\right]$. Note that $K \subseteq I_0$. Define f on I by

$$f(x) = 1 - 3d(x,K).$$

Then in particular, $f(x) = 3x$ on $\left[0,\frac{1}{3}\right]$ and $f(x) = 3(1-x)$ on $\left[\frac{2}{3},1\right]$.

Parenthetically, if we had defined f to be identically 1 on $\left[\frac{1}{3},\frac{2}{3}\right]$ we would have arrived at a function having simpler dynamics. It is easily verified that in this case all points not in T, as well as those points in T which admit a ternary representation containing a 1, would have orbits which eventually arrive at the fixed point 0. On the remainder of T, which we label $T\hat{}$, the dynamical structure resembles that of the "hat function" h which is defined by $h(x) = 2x$ for $0 \leq x \leq \frac{1}{2}$, and $h(x) = 2(1-x)$ for $\frac{1}{2} \leq x \leq 1$ (cf. [B]). Evidently, the orbits of all points in $T\hat{}$ remain in $T\hat{}$. In particular, it is easy to identify in $T\hat{}$ periodic points of arbitrary period and other points whose orbits are dense in T. There is also a countable, dense subset of T whose orbits eventually land on the fixed point $\frac{3}{4}$. The perturbing effect of the spikes appearing on the graph of f in the interval $\left[\frac{1}{3},\frac{2}{3}\right]$ makes it possible to exhibit a greater variety of iterative behaviours.

We now examine the various possibilities.

When $x \in K$, $f^n(x) = 0$ for all $n \geq 2$. However, when $x \in I_0 - K$ we obtain more interesting orbits. For instance, let $x \in I_0 - K$ be of the form

$$x = .1A_1 1A_2 \ldots = .\overline{1A_i}$$

where each A_i is a finite block of 0's and 2's. The nearest point of K to x is either $.1A_1 1\bar{0}$ or $.1A_1 2\bar{0}$ depending on whether A_2 begins with a 0 or a 2, respectively. It is easily checked that

$$f(x) = .[2]_k A_2^* \overline{1A_i^*} \quad \text{or} \quad f(x) = .[2]_k A_2 \overline{1A_i}$$

depending on whether A_2 begins with a 0 or a 2, and where $k = \rho\left(A_1\right) + 1$. Moreover,

$$f^2(x) = .[0]_m A_2 \overline{1A_i} \quad \text{or} \quad f^2(x) = .[0]_m A_2^* \overline{1A_i^*}$$

depending on whether A_2 begins with a 0 or a 2, and where $m = \rho\left(A_1\right)$.

Further calculations yield

$$f^M(x) = .A_2\overline{1A_i} \quad \text{or} \quad f^M(x) = .A_2^*\overline{1A_i^*}$$

depending on whether A_2 begins with a 0 or a 2; here, $M = \rho\left(A_1\right) + 2$. And, when $N = \rho\left(A_1\right) + \rho\left(A_2\right) + 2$, we have

$$f^N(x) = .1A_3\overline{1A_i} \quad \text{or} \quad f^N(x) = .1A_3^*\overline{1A_i^*}.$$

The form of $f^N(x)$ is determined as follows. Let α be the number of pairs of different consecutive integers in A_2. Then $f^N(x)$ has the first form if and only if A_2 begins with a 0 and α is even, or A_2 begins with a 2 and α is odd.

Thus, $f^N(x)$ will be the first iterate after x itself to appear in I_0. Note that the number of iterates necessary to return to I_0 depends on the lengths of both A_1 and A_2, while the nearness of x to K depends primarily on the length of A_1. We shall exploit these facts below. By controlling the lengths of the blocks, we can control the location of the repeated returns of the orbit of x to I_0.

<u>Example 1</u>. Let $Q_0 = \frac{1}{9}\,T + \frac{1}{3}$ and $Q_n = 3^{-n}\,Q_0$ for $n \geq 1$. Then $\{0,1\} \cup \left(\cup_{n=0}^{\infty} Q_n\right)$ is an ω-limit set for f.

<u>Proof</u>. Note that $Q_0 \subseteq \left[\frac{1}{3},\frac{4}{9}\right]$. Any point x in $\left[\frac{1}{3},\frac{4}{9}\right]$ can be written as $x = .10a_3a_4\ldots$. If $x \in Q_0$, then each $a_i \in \{0,2\}$. Let $\mathcal{A}$ consist of all non-empty finite blocks of 0's and 2's which begin with 0. Enumerate $\mathcal{A}$ in such a way that each set A_{2k} consists of a single 0. Write each A_{2k-1} as $0B_k$. Let

$$\begin{aligned} x &= .1A_11A_2\ldots &&= .\overline{1A_i} \\ &= .1A_1101A_3\ldots &&= .1A_1\overline{101A_{2i-1}} \\ &= .10B_11010B_2\ldots &&= .10B_1\overline{1010B_i}. \end{aligned}$$

Then $x \in \left[\frac{1}{3}, \frac{4}{9}\right]$ and

$$f(x) = .[2]_k 212B_2^* \overline{1212B_i^*}$$

where $k = \rho(A_1) + 1$, and

$$f^2(x) = .[0]_k 010B_2 \overline{1010B_i}$$

where $k = \rho(A_1)$. Moreover,

$$f^N(x) = .10B_2 \overline{1010B_i}$$

when $N = \rho(A_1) + 3$ since A_2 consists of a single 0. From this it follows that orbit points are of the following types for some j and m:

$$.[0]_j 10B_m \overline{1010B_i} \quad \text{where} \quad j \leq \rho(B_{m-1}) + 2,$$

or

$$.[2]_j 12B_m^* \overline{1212B_i^*} \quad \text{where} \quad j = \rho(B_{m-1}) + 3.$$

Hence it is not difficult to see that $\omega(x,f) \subseteq \{0,1\} \cup \left(\bigcup_{n=0}^{\infty} Q_n\right)$.

Now let $\mathcal{B} = \left\{B_i : i \in \omega_0\right\}$. It is clear that for each k, $\left\{(3^{-k})(.10B) : B \in \mathcal{B}\right\}$ is dense in Q_k. Since it is clear that $\{0,1\} \subseteq \omega(x,f)$ it will suffice to show that for each $k \in \omega_0$ and $B \in \mathcal{B}$ there exists $z \in \gamma(x,f)$ such that $d\left(z, 3^{-k}(.10B)\right)$ is arbitrarily small. Let $k \in \omega_0$ and $B \in \mathcal{B}$ and suppose $n > 1$. Then we can find m and j such that $k \leq \rho(B_{m-1}) + 2$, $n \leq j$, and $B_m = B[0]_j$. Then $z = 3^{-k}\left(.10B_m \overline{1010B_i}\right) \in \gamma(x,f)$ and $d\left(z, 3^{-k}(.10B)\right) < 3^{-n}$. Since n was arbitrary the proof is complete.

Since $\{0\} \cup \left(\bigcup_{n=0}^{\infty} Q_n\right)$ is a Cantor set, Example 1 shows that a Cantor set plus an isolated point is an ω-limit set for f.

What other sets can be obtained as ω-limit sets via the function f? Our next example shows that we can "replace" the set Q_0 of Example 1 by any nowhere dense compact set.

<u>Example 2</u>. Let M be any closed, nowhere dense set. Then there exists an $M_0 \subseteq I_0$ such that M is homeomorphic to M_0 and $\{0,1\} \cup \left(\bigcup_{n=0}^{\infty} 3^{-n} M_0\right)$ is

an ω-limit set for f.

<u>Proof</u>. Let M_0 be a homeomorphic copy of M inside Q_0. Let $\mathcal{B}$ be the set of all finite blocks of 0's and 2's as in Example 1. Choose $\mathcal{C} \subseteq \mathcal{B}$ such that $\mathrm{cl}\ \{.10B : B \in \mathcal{C}\} = M_0$. Enumerate $\mathcal{C}$ as $\{C_n\}_{n=0}^{\infty}$ and put $x_0 = .10C_1\overline{1010C_i}$. Now, repeating the argument of Example 1 with some slight modifications we obtain the result.

By virtue of Example 2 the ω-limit sets of the function f include all homeomorphs of closed nowhere dense sets in the sense that $\omega(x_0, f) \cap I_0 = M_0$. We have been unable to determine whether there exists a single continuous function g for which each closed nowhere dense set is homeomorphic to some ω-limit set for g.

Now we digress to consider some other useful properties of the function f. The nature of f makes it easy to specify points in $\left(\frac{1}{3}, \frac{2}{3}\right)$ having preassigned period. For example,

$$\frac{1}{2} = .1111\ldots = .\overline{11} \quad \text{has period 2;}$$
$$\frac{6}{13} = .110110\ldots = .\overline{110} \quad \text{has period 3; and}$$
$$\frac{9}{20} = .11001100\ldots = .\overline{1100} \quad \text{has period 4.}$$

In general, let $N > 1$ be a positive integer. If we choose A_1 to be empty and A_2 to be $N - 2$ zeros, and put $A_{2k-1} = A_1$, $A_{2k} = A_2$ for $k = 2,3\ldots$, we have $N = \rho(A_1) + \rho(A_2) + 2$ and $f^N(.\overline{1A_i}) = .\overline{1A_i}$. Since $f^N(.\overline{1A_i})$ is the first iterate to reappear in I_o the point $.\overline{1A_i}$ has period N.

It is also easy to obtain points which are eventually periodic or asymptotically periodic.

A set V is said to be <u>scrambled</u> with respect to a function f if for each x and y in V, $\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0$ and

$\limsup_{n\to\infty} |f^n(x) - f^n(y)| > 0$. In [LY] it is shown that a function yields an uncountable scrambled set whenever it has a point of period 3. From this follows the existence of c pairwise disjoint scrambled sets each bilaterally c-dense-in-itself. However, for our function f such scrambled sets can be described explicitly. For example, suppose A and B are fixed, distinct blocks of 0's and 2's having the same length. Consider the set T of all points of the form $.\overline{10A_i10}$, where $A_i \in \{A,B\}$ such that there are arbitrarily long strings of consecutive A's, and likewise for B's. If $x = .\overline{10A_i10}$ is an element of T and A < B when A and B are viewed as ternary numbers, then $\inf\left\{\omega(x,f) \cap I_0\right\} = .\overline{10A10}$ and $\sup\left\{\omega(x,f) \cap I_0\right\} = .\overline{10B10}$. If x and y are distinct points in T having arbitrarily long strings of consecutive A's in matching positions and x has arbitrarily long strings of consecutive A's which match strings of B's in y, then $\liminf |f^n(x) - f^n(y)| = 0$ and $\limsup |f^n(x) - f^n(y)| = |.\overline{10B10} - .\overline{10A10}| > 0$. So {x,y} is scrambled.

Using pairs of such blocks we can, employing arguments identical to those in [BH], find an uncountable, bilaterally c-dense-in-itself scrambled set S(A,B). Moreover, S(A,B) and S(C,D) are disjoint when $(A,B) \neq (C,D)$.

A similar argument shows that if $\mathfrak{B}$ is any collection (of at least two elements) of blocks of 0's and 2's and $T(\mathfrak{B})$ consists of all points $.\overline{10A_i10}$ where $A_i \in \mathfrak{B}$, and such that there are arbitrarily long strings of consecutive A's for each $A \in \mathfrak{B}$, then we can construct a bilaterally c-dense-in-itself scrambled subset $S(\mathfrak{B})$ of $T(\mathfrak{B})$. Moreover, $S(\mathfrak{B}_1)$ and $S(\mathfrak{B}_2)$ are disjoint when $\mathfrak{B}_1 \neq \mathfrak{B}_2$. Since there are c possible collections $\mathfrak{B}$ we see that f has c pairwise disjoint scrambled sets, each bilaterally c-dense in itself.

GENERAL RESULTS

In this section we obtain a number of general results concerning the structure of ω-limit sets of continuous functions. In Proposition 1, below, we further exploit the function f of Example 1 by showing that the set Q_0 can be augmented by a countable closed set. We begin by obtaining a simple but powerful tool for constructing functions whose ω-limit sets are prescribed. It will allow us to pick a sequence which does what we want and in addition is the orbit of some continuous function. In essence, the requirement is that for an isolated sequence $\{z_n\}_{n=0}^{\infty}$ the mapping $z_n \to z_{n+1}$ has removable discontinuities on the closure of the set $\{z_n : n \in \omega_0\}$. We state this theorem in the form in which it is used.

<u>Theorem 1</u>. Let M be a closed nowhere dense set. Suppose $\{z_n\}$ is a sequence of distinct points not in M but whose set of subsequential limit points is M. Then there exists a continuous function f and $z_0 \in \text{dom } f$ such that $\omega(z_0,f) = M$ provided the following condition is fulfilled:

for all numbers α and β, and $\lambda \in M$ and subsequences $\{n_k\}_{k=0}^{\infty}$ and $\{m_k\}_{k=0}^{\infty}$, $\alpha = \beta$ whenever $\lim_{k\to\infty} \left(z_{n_k}, z_{n_k+1}\right) = (\lambda,\alpha)$ and $\lim_{k\to\infty} \left(z_{m_k}, z_{m_k+1}\right) = (\lambda,\beta)$.

<u>Proof</u>. Define $\Gamma = \text{cl}\left\{\left(z_n,z_{n+1}\right) : n \in \omega_0\right\}$. Then $\text{dom } \Gamma = M \cup \{z_n : n \in \omega_0\}$ and by hypothesis each z_n is an isolated point of $\text{dom } \Gamma$. The condition implies that Γ is a function. Since Γ is closed it is continuous. Define f to be Γ on $\text{dom } \Gamma$; when (a,b) is a component of $I - \text{dom } \Gamma$ define $f(\theta a+(1-\theta)b) = \theta\Gamma(a) + (1-\theta)\Gamma(b)$ for $\theta \in (0,1)$; if $[0,d)$ (or similarly $(d,1]$ is a component of $I - \text{dom } \Gamma$ define f to have the constant value $\Gamma(d)$ on that component. Then f has the desired properties.

Lemma 1. Let M be a closed, nowhere dense ω-limit set, $M = \omega(y_0,f)$, and let W be an open set such that $M \subseteq \bar{W}$. Then there exists $x \in I$ and a continuous function $g : I \to I$ such that $\omega(x,g) = M$ and $\gamma(x,g) \subseteq W$. Moreover, the set of fixed points of f and g coincide.

Proof. Let $\gamma(y_0,f) = \{y_n\}_{n=0}^{\infty}$. Pick ϵ_0 so that $y_0 + \epsilon_0$ belongs to W. Having picked $\epsilon_0,\epsilon_1,\ldots,\epsilon_m$, choose ϵ_{m+1} as follows: the set $U = W - \{y_j + \epsilon_j : j \leq m\}$ is dense in the open set W. So we may select $\nu_{m+1} \in U$ such that $|y_{m+1} - \nu_{m+1}| < d(y_{m+1},M) + (m+1)^{-1}$. Now put $\epsilon_{m+1} = \nu_{m+1} - y_{m+1}$.

Define $z_n = y_n + \epsilon_n$ and apply Theorem 1 to get the desired function g.

If h is a homeomorphism between two topological spaces X and Y and $f : X \to Y$ then $g = h\circ f\circ h^{-1}$ is said to be *conjugate* to f, and g exhibits topological dynamics identical to those of f. If on the other hand X and Y are homeomorphic subsets of the same space it is not immediately clear that if X is an ω-limit set for some f, then Y is an ω-limit set for some g. The difficulty is that the homeomorphism between subsets of X and Y of $[0,1]$ might not be extendable to a homeomorphism of $[0,1]$ onto itself, and the orbits that cluster on, say, X need not be contained in X, although they are "near" X. This suggests that something less than a homeomorphism between X and Y which can be extended to all of I is needed. We shall see that the type of extension which suffices is available at least in the case where X is nowhere dense and closed.

A homeomorphism h between two closed, nowhere dense sets E and F is called a *W-homeomorphism* if there exists an open set W such that $E \subseteq \bar{W}$

and h can be extended so that h is a homeomorphism on $\bar{W}$. Clearly a W-homeomorphism can be extended to a continuous function on I. For example, if K_1 and K_2 are Cantor sets and h is any increasing continuous function from K_1 onto K_2, then h is a W-homeomorphism.

We shall make frequent use of the following lemma.

Lemma 2. Suppose h is a W-homeomorphism between closed, nowhere dense sets E and F. If E is an ω-limit set, $\omega(x,f)$, then F is also an ω-limit set $\omega(y,g)$. Moreover, if $f(\lambda) = \lambda$ for all $\lambda \in E'$, then $g(\lambda) = \lambda$ whenever $\lambda \in F'$ and $h(\lambda) = \lambda$.

Proof. Extend h so that it is a continuous function on I. Let W be open with $E \subseteq \bar{W}$, and $h|\bar{W}$ a homeomorphism. By Lemma 1 we may assume that $\gamma(x,f) \subseteq W$. Since $\bar{W}$ is compact, $h(\bar{W})$ is closed. Define $g = hfh^{-1}$ on $h(\bar{W})$ and extend g to the components of $I - h(\bar{W})$ by linearity. Evidently g is continuous.

We have $h(x) \in h(\bar{W})$ so that $gh(x) = hfh^{-1}h(x) = hf(x) \in h(\bar{W})$. Then $g^2h(x) = hfh^{-1}gh(x) = hfh^{-1}hf(x) = hf^2(x)$, and $hf^2(x) \in h(\bar{W})$. By induction it follows that $g^nh(x) = hf^n(x)$ and $g^nh(x) \in h(\bar{W})$ for all n.

Next we show that $F = \omega(h(x),g)$. Suppose $\lambda \in F$. Then $h^{-1}(\lambda) \in E$ and there exists a sequence $\{n_k\}_{k=0}^{\infty}$ such that $f^{n_k}(x) \to h^{-1}(\lambda)$. Hence $hf^{n_k}(x) \to \lambda$ by continuity of h. It follows that $g^{n_k}(h(x)) \to \lambda$, so that $\lambda \in \omega(h(x),g)$. Therefore, $F \subseteq \omega(h(x),g)$.

Now suppose $g^{n_k}(h(x)) \to \lambda$ for some subsequence $\{n_k\}_{k=0}^{\infty}$. Then $hf^{n_k}(x) \to \lambda$ and $\lambda \in \overline{h(W)} \subseteq h(\bar{W})$. Hence, since $\gamma(x,f) \subseteq W$, $h^{-1}hf^{n_k}(x) = f^{n_k}(x) \to h^{-1}(\lambda)$; and $h^{-1}(\lambda) \in E$. Therefore, $\lambda \in h(E) = F$, and $\omega(h(x),g) \subseteq F$.

Finally, if $\lambda \in F'$ and $h^{-1}(\lambda) = \lambda$, then $h^{-1}(\lambda) \in E'$. So $fh^{-1}(\lambda) = \lambda$ and $g(\lambda) = hfh^{-1}(\lambda) = \lambda$.

Note. As a consequence of Theorem 3 it will turn out that Lemma 2 can be strengthened so that if any two sets are homeomorphic and one of them is an ω-limit set, then so is the other.

Referring back to Example 1, choose a sequence $\{W_n\}_{n=-1}^{\infty}$ of pairwise disjoint relatively open intervals in I such that $Q_n \subseteq W_n$ for $n > 0$ and $1 \in W_{-1}$ and choose x_0 so that $\gamma(x_0,f) \subseteq \cup_{n=-1}^{\infty} W_n$. Note that if $f^m(x_o) \in W_n$ then $f^{m+1}(x_0) \in W_{n-1}$ and if $f^m(x_0) \in W_{-1}$ then $f^{m+1}(x_0) \in W_k$ for some k. The sequence $\{W_n\}_{n=-1}^{\infty}$ will be used in Lemma 3.

Lemma 3 Suppose M_0 is a nonempty closed subset of Q_0 such that each isolated point of M_0 is a one-sided limit point of Q_0. Suppose further that if G is a component of $W_0 - M_0$ either $G \cap Q_0 = \phi$ or $(G \cap Q_0)'$ contains exactly one endpoint of G. Then $\{0,1\} \cup \left(\cup_{n=1}^{\infty} Q_n\right) \cup M_0$ is an ω-limit set.

Proof From Example 1 there is an x such that $\omega(x,f) = \{0,1\} \cup \left(\cup_{n=0}^{\infty} Q_n\right)$. Enumerate the family $\mathcal{G}$ of components of $W_0 - M_0$ as $\{G_n\}_{n=1}^{\infty}$. (Without loss of generality we may assume that $\mathcal{G}$ is infinite.) If $G_n \cap Q_0 \neq \phi$ let a_n be the endpoint of G_n such that $a_n \in (G_n \cap Q_0)'$. For each such n let $\{\lambda_{nk}\}_{k=0}^{\infty}$ be a sequence in $G_n - \gamma(x,f) - Q_0$ converging to a_n. Then $G_n \cap \gamma(x,f) = \{x_{h_n(k)} : k \in \omega_0\}$ for some subsequence $\{h_n(k)\}_{k=0}^{\infty}$. Here $x_m = f^m(x)$.

Define g on $\gamma(x,f) \cap W_0$ by

$$g(y) = \begin{cases} \lambda_{nk} & \text{if } y = x_{h_n(k)} \\ y & \text{otherwise.} \end{cases}$$

Define $\left\{z_\alpha\right\}_{\alpha=0}^{\infty}$ as follows:

if $x_\alpha \in W_m$, where $m \neq 0$, put $z_\alpha = x_\alpha$;

if $x_\alpha \in W_0$, put $z_\alpha = g\left(x_\alpha\right)$.

Since g is one-to-one, $z_\alpha \neq z_\beta$ for $\alpha \neq \beta$. The set of subsequential limit points of $\left\{z_\alpha\right\}_{\alpha=0}^{\infty}$ is readily seen to be $\{0,1\} \cup \left(\cup_{n=1}^{\infty} Q_n\right) \cup M_0$.

To complete the proof it will suffice to show that the condition of Theorem 1 is satisfied. Note $z_\alpha \notin \{0,1\} \cup \left(\cup_{n=1}^{\infty} Q_u\right) \cup M_0$ since $z_\alpha \in \gamma(\alpha,F)$ or $z_\alpha = \lambda_{nk} \notin Q_0$.

So suppose $\left(z_{n_k}, z_{n_k+1}\right) \to (\lambda,\beta)$. If $\left\{z_{n_k}\right\}_{k=0}^{\infty}$ is not eventually in some W_m then $\lambda = 0$; hence $\beta = 0$.

So assume $\left\{z_{n_k}\right\}_{k=0}^{\infty}$ lies in W_m. If $m \geq 2$ then $\beta = f(\lambda)$. If $m = 1$ then $z_{n_k} = x_{n_k} \to \lambda$ and $x_{n_k+1} \to f(\lambda)$. Now consider the sequence $\left\{x_{n_k+1}\right\}_{k=0}^{\infty}$ in W_0. If $f(\lambda) \in G_n$ for some n, then $z_{n_k+1} = g\left(x_{n_k+1}\right) \to a_n$ so $\beta = a_n$. If $f(\lambda) \in M_0 - \cup \mathcal{G}$, then each $x_{n_k+1} \in G_{m_k}$ for some m_k and $m_k \to \infty$. Since $z_{n_k+1} \in G_{m_k}$ and $|G_{m_k}| \to 0$ and $G_{m_k} \to f(\lambda)$, it follows that $z_{n_k+1} \to f(\lambda)$ and $\beta = f(\lambda)$.

Finally, in case $z_{n_k} \in W_0$ for each k, then $n_k \to \infty$ and $z_{n_k+1} = x_{n_k+1} \to 1$. So $\beta = 1$.

In each case, β is uniquely determined, as required.

<u>Proposition 1</u> Let D be a countable set such that $D \subseteq \left[\frac{1}{3},\frac{4}{9}\right]$ and $Q_0 \cup D$ is closed. Then $\{0,1\} \cup D \cup \left(\cup_{n=0}^{\infty} Q_n\right)$ is an ω-limit set.

<u>Proof</u>. It suffices by Lemma 2 to show that the set $\{0,1\} \cup D \cup \left(\cup_{n=0}^{\infty} Q_n\right)$ is W-homeomorphic to some set $\{0,1\} \cup M_0 \cup \left(\cup_{n=1}^{\infty} Q_n\right)$, where M_0 satisfies the hypothesis of Lemma 3.

Let $\{d_n\}_{n=0}^{\infty}$ be the isolated points of D. Choose intervals $\left\{\left[d_n, e_n\right]\right\}_{n=0}^{\infty}$ which are disjoint and miss $Q_0 \cup D$. Let H_n be a Cantor set such that $\inf H_n = d_n$ and $\sup H_n = \frac{1}{2}\left(d_n + e_n\right)$. The set $B = Q_0 \cup D \cup \left(\cup_{n=0}^{\infty} H_n\right)$ is a Cantor set, so there exists an increasing continuous function g from this set onto Q_0. This function is a W-homeomorphism from B onto Q_0. Moreover, $Q_0 \cup D$ is W-homeomorphic to $M_0 = g\left(Q_0 \cup D\right)$ and M_0 satisfies the hypothesis of Lemma 3.

If we define $h(x) = x$ on $\{0,1\} \cup \left(\cup_{n=1}^{\infty} Q_n\right)$ and $h(x) = g(x)$ on $Q_0 \cup D$, then h will be a W-homeomorphism from $\{0,1\} \cup D \cup \left(\cup_{n=0}^{\infty} Q_n\right)$ onto $\{0,1\} \cup M_0 \cup \left(\cup_{n=1}^{\infty} Q_n\right)$.

For a Cantor set K let $\mathcal{G}(K)$, or $\mathcal{G}$ where no confusion arises, denote the set of components of $[\inf K, \sup K] - K$. We will say that a subfamily $\mathcal{H}$ of $\mathcal{G}$ is <u>dense</u> in K iff each nonempty open set which hits K contains a member of $\mathcal{H}$. This is of course equivalent to saying that the set of endpoints of members of $\mathcal{H}$ is dense in K.

The next result is a more general version of Proposition 1.

<u>Proposition 1'</u> Let K be any Cantor set. Let M be any closed nowhere dense set such that for each $G \in \mathcal{G}$ ($\mathcal{G}$ as above) the set $M \cap G$ is countable. If $\{G \in \mathcal{G} : M \cap G \neq \phi\}$ is not dense, then $K \cup M$ is an

ω-limit set.

Proof. Case 1: $M - K \neq \phi$. If $\mathcal{H} = \{G \in \mathcal{G} : M \cap G \neq \phi\}$ is not dense there exist members (a,b), (c,d) of $\mathcal{G}$ such that $S = [b,c] \cap K$ misses M.

Let $\Gamma = \{0,1\} \cup \left(\cup_{n=0}^{\infty} Q_n\right)$ be the set of Example 1. Let z be an isolated point of M and suppose $z \in (d, \sup K)$. Let (α,β) be a component of $I - \Gamma$ where $(\alpha,\beta) \subseteq \left(\frac{1}{3},\frac{4}{9}\right)$.

We may pick homeomorphisms h_1, h_2, and h_3 such that

$h_1\left([\inf K, a]\right) = \left[\frac{1}{3},\alpha\right]$, $h_1\left([\inf K, a] \cap K\right) = \left[\frac{1}{3},\alpha\right] \cap \Gamma$

$h_2\left([b,c]\right) = \left[0,\frac{4}{27}\right]$, $h_2\left([b,c] \cap K\right) = \left[0,\frac{4}{27}\right] \cap \Gamma$

$h_3\left([d, \sup K]\right) = \left[\beta,\frac{4}{9}\right]$, $h_3\left([d, \sup K] \cap K\right) = \left[\beta,\frac{4}{9}\right] \cap \Gamma$.

Define h on $K \cup M$ as follows:

$$h(x) = \begin{cases} h_1(x) & \text{if } x \in [\inf K, a] \cap (K \cup M) \\ h_2(x) & \text{if } x \in [b,c] \cap K \\ h_3(x) & \text{if } x \in [d, \sup K] \cap (K \cup M - \{z\}) \\ 1 & \text{if } x = z \end{cases}$$

Put $D = h(M-K) - \{1\}$. Then D is closed and h is a W-homeomorphism from $K \cup D$ onto $\Gamma \cup D$. By Proposition 1 $\Gamma \cup D$ is an ω-limit set; by Lemma 2, $K \cup M$ is an ω-limit set.

Case 2: $M - K = \phi$. Then $K \cup M = K$ is a Cantor set. It is well known that every Cantor set is an ω-limit set.

The Countable Case

This completes the first of the three stages in our program; that is, we have shown that an uncountable nowhere dense closed set whose intersection with some interval is a Cantor set, is an ω-limit set. We turn now to the second stage in which we establish that every non-empty countable closed set is an ω-limit set. The crucial argument is contained in Lemma 6, which deals with a special type of closed set. Lemmas 4 and 5 reveal that this special case suffices.

Lemma 4. A homeomorphism between closed nowhere dense sets E and F is a W-homeomorphism if $E - E'$ is dense in E.

Proof. If E is finite, the result is obvious so we may assume $E - E'$ is infinite. If h is such a homeomorphism then under the stated hypothesis $h(E') = F'$ and $h(E-E') = F - F'$. Moreover, $E - E'$ and $F - F'$ are countable. Let $\{c_n\}_{n=0}^{\infty}$ be an enumeration of $E - E'$. Then $F - F' = \{h(c_n)\}_{n=0}^{\infty}$.

We may choose two sequences $\{S_n\}_{n=0}^{\infty}$ and $\{T_n\}_{n=0}^{\infty}$ of open intervals with the following properties:

c_n and $h(c_n)$ are the midpoints of S_n and T_n, respectively;

$\bar{S}_m \cap \bar{S}_n = \phi$ whenever $m \neq n$;

$\bar{T}_m \cap \bar{T}_n = \phi$ whenever $m \neq n$;

$|S_n| \to 0$ and $|T_n| \to 0$ as $n \to \infty$; and

$S_n \cap E' = \phi$ and $T_n \cap F' = \phi$ for all n.

Let L_n be a linear mapping of $\bar{S}_n$ onto $\bar{T}_n$.

Define

$$h_1 = \cup \{L_n : n \in \omega_0\} \cup f|E'.$$

Let $W = \cup \{S_n : n \in \omega_0\}$; then $\bar{W} = \cup \{\bar{S}_n : n \in \omega_0\} \cup E'$ and $\operatorname{dom} h_1 = \bar{W}$. Clearly h_1 is one-to-one and continuous on $\bar{W}$. Now extend h_1 linearly to a function continuous on all of I. Since $h_1|\bar{W}$ is a homeomorphism and $h_1(E) = F$, the proof is complete.

Let F be a set. We say that $x \in F$ is <u>isolated from the left</u> if $x \notin ([0,x) \cap F)'$.

<u>Lemma 5</u>. Any countable closed set not containing 0 is homeomorphic to a countable closed set having no limit points isolated from the left.

<u>Proof</u>. Let D be a countable closed set and let L be the set of left-isolated limit points of D; assume without loss of generality that L is infinite. Enumerate L as $\{r_k\}_{k=0}^{\infty}$. Let $J = D - D'$. By induction we may pick for each k a sequence $\{r(k,n)\}_{n=0}^{\infty}$ in J such that $|r(k,n) - r_k| < 2^{-n}$ for each n and $r(k,n) \neq r(j,m)$ whenever $(k,n) \neq (j,m)$. Let $E = \{r(k,2n) : k \in \omega_0, n \in \omega_0\}$.

Also by induction we may pick for each k a sequence $\{p_{kn}\}_{n=0}^{\infty}$ in $I - D$ such that $0 < r_k - p_{kn} < 2^{-n}$ for each n and $p_{kn} \neq p_{jm}$ whenever $(k,n) \neq (j,m)$.

Put $P = \{p_{kn} : k \in \omega_0, n \in \omega_0\}$ and let $C = D \cup P - E$. It is clear that C is closed and countably infinite. Moreover, $D' = C'$ and C has no left-isolated limit points.

Define a function $h : D \to C$ as follows:

$$h(x) = \begin{cases} p_{kn} & \text{if } x = r(k,2n) \text{ for some } k,n \\ x & \text{if } x \in D - E. \end{cases}$$

Then h is a homeomorphism from D onto C.

<u>Lemma 6</u>. If F is a nonempty, closed nowhere dense set having no limit points isolated from the left, then F is an ω-limit set, $\omega(x,f)$, and

$f(\lambda) = \lambda$ whenever $\lambda \in F'$.

<u>Proof</u>. Let $a = \inf F$ and $b = \sup F$. Our hypotheses imply that a is isolated. Let $\mathcal{G}$ be the family of components of $[a,b] - F$. For each $n \geq 1$ let $\mathcal{G}_n = \{G \in \mathcal{G} : 2^{-n} \leq |G|\}$. In view of Lemma 2 we may assume without loss of generality that $\mathcal{G}_1$ consists of exactly one interval having a as its left endpoint.

Each $\mathcal{G}_n$ consists of a finite number of intervals which we enumerate as $G_{n1}, G_{n2}, \ldots, G_{n\alpha(n)}$ with G_{ni} to the left of G_{nj} if $i > j$. Observe that each $G \in \mathcal{G}$ belongs to each $\mathcal{G}_k$ for sufficiently large k. We will obtain a sequence $\{z_j\}_{i=1}^{\infty}$ which intuitively "travels" throughout each $\mathcal{G}_n$ "from right to left" and then moves into the beginning of $\mathcal{G}_{n+1}$, reminiscent of a typewriter with finitely many words on each line and infinitely many lines. Our assumption that $G_{n\alpha(n)}$ is the leftmost member of $\mathcal{G}_n$ will guarantee that the "typewriter" does not go to the line $\mathcal{G}_{n+1}$ before coming to the last word of the line $\mathcal{G}_n$.

For each n and k let a_{nk} be the left endpoint of G_{nk}. Since $\{\alpha(n)\}_{n=1}^{\infty}$ is an increasing sequence, each positive integer $j \geq 1$ can be uniquely expressed as $j = (k-1) + \sum_{i=1}^{n} \alpha(i)$ for some $n \geq 1$ and some k such that $1 \leq k \leq \alpha(n+1)$. To each $j \geq 1$ we associate the pair (n,k) and put

$$z_j = a_{nk} + |G_{nk}|\, 2^{-j}.$$

Note that $z_i \neq z_j$ when $i \neq j$.

We will show that the sequence $\{z_n\}_{n=1}^{\infty}$ satisfies the requirements of Theorem 1 with respect to F, from which the desired result will follow.

First of all, $z_n \notin F$ for all n and it is clear that the cluster set of $\{z_n\}_{n=1}^{\infty}$ is F.

Now suppose for some subsequence $\{j_k\}_{k=0}^{\infty}$ we have $\left(z_{j_k}, z_{j_k+1}\right) \to (\lambda,\beta)$, where $\lambda \in F$. We will show that β is uniquely determined. There will be two cases, depending upon whether λ is isolated or not.

Case 1: λ is isolated in F. If $\lambda \neq a$, then there exist μ and ξ such that $(\lambda,\xi) \in \mathcal{G}$ and $(\mu,\lambda) \in \mathcal{G}$. For sufficiently large k, $z_{j_k} \in (\lambda,\xi)$ and $z_{j_k+1} \in (\mu,\lambda)$. Thus $z_{j_k+1} \to \mu$. In case $\lambda = a$, one sees similarly that $z_{j_k+1} \to b$.

Case 2: λ is a limit point of F. First, assuming $\lambda \neq b$, we show that for each $\epsilon > 0$ there exists a k_0 such that $\lambda - \epsilon < z_{j_k+1} < z_{j_k} < \lambda + \epsilon$ whenever $k > k_0$. To see this, observe that λ is by hypothesis a left limit point of F so there is a $G \in \mathcal{G}$, say $G = (c,d)$, such that $\lambda - \epsilon < c < d < \lambda$. Since $z_{j_k} \to \lambda$ we may choose k_0 such that for $k > k_0$, $z_{j_k} \in (d,\lambda+\epsilon)$ and the member of $\mathcal{G}$ which contains z_{j_k} has length less than $d - c$. From the definition of z_{j_k+1} we must have $c < z_{j_k+1} < z_{j_k}$. It follows that $\lambda - \epsilon < z_{j_k+1} < z_{j_k} < \lambda + \epsilon$ whenever $k > k_0$. Since ϵ was arbitrary we see that $z_{j_k+1} \to \lambda$ and $\beta = \lambda$. In case $\lambda = b$ a similar analysis shows that $z_{j_k+1} \to b$.

Now apply Theorem 1 to get $F = \omega(x,f)$ and note from case 2 that $f(\lambda) = \lambda$ whenever $\lambda \in F'$.

Remark. Our hypothesis that F has no limit points isolated from the left was used in only one place, namely in case 2. Without that assumption the construction of the sequence $\{z_n\}_{n=1}^{\infty}$ might not have met the hypothesis of Theorem 1 in the case where λ is a right endpoint of an interval contiguous to F.

Proposition 2. Any nonempty countable closed set C is an ω-limit set $\omega(x,f)$ with $f(\lambda) = \lambda$ for all $\lambda \in C'$.

Proof. If $0 \notin C$ then we see from Lemma 5 that C is homeomorphic to a countable closed set with no limit points isolated from the left. By Lemma 4 the homeomorphism is a W-homeomorphism since the set of isolated points of C is dense in C. The result now follows from Lemma 2 and Lemma 6.

If on the other hand 0 is a element of C we first map C onto $\left[\frac{1}{2},1\right]$ by a nonconstant linear function L with $L(0) = \frac{1}{2}$. Since $0 \notin L(C)$ there exists a continuous function f such that $L(C) = \omega(x,f)$ for some x. By Lemma 1 we may assume that $\gamma(x,f) \subseteq \left(\frac{1}{2},1\right)$. Thus $C = \omega\left(L^{-1}(x),L^{-1}fL\right)$.

This completes stage 2 of our program.

The Dense Case

The final stage is devoted to showing that an uncountable nowhere dense closed set whose isolated points form a dense subset, is an ω-limit set.

Lemma 7. Let K be any Cantor set. Suppose D is a countable, nowhere-dense subset of $\cup \mathcal{G}$. If $K \cup D$ is closed and no limit point of $K \cup D$ except $\inf K$ is isolated from the left, then $K \cup D$ is an ω-limit set $\omega(z,g)$ such that $g(\lambda) = \lambda$ whenever $\lambda \in K \cup D'$.

Proof. Let us first consider the case where $0 \notin K$. Let $a = \inf K$. Choose $\left\{d_n\right\}_{n=0}^{\infty}$ to be a sequence in D converging to a and choose $\left\{e_m\right\}_{m=0}^{\infty}$ to be a sequence in $[0,a)$ converging to a. Put $D_1 = D - \left\{d_n : n \in \omega_0\right\}$ and $E = \left\{e_m : m \in \omega_0\right\} \cup D_1$. Then $K \cup E$ is W-homeomorphic to $K \cup D$ via the following function f:

$$f\left(e_m\right) = d_m,$$
$$f(\lambda) = \lambda \quad \text{if } \lambda \in K \cup D_1.$$

By Lemma 6, $K \cup E$ is an ω-limit set. By Lemma 2, $K \cup D$ is an ω-limit set $\omega(y,g)$ and $g(\lambda) = \lambda$ whenever $\lambda \in K = K \cup D'$.

To finish the proof consider the case where $0 \in K$. Then there is a linear mapping $L : I \to I$ sending K into $\left[\frac{1}{2},1\right]$. Then $L(K \cup D)$ is an ω-limit set $\omega(z,g)$ by the above argument. By Lemma 1 we can assume that $\gamma(z,g) \subseteq \left(\frac{1}{2},1\right)$. Then $K \cup D = \omega\left(L^{-1}(z),L^{-1}gL\right)$. Clearly $L^{-1}gL(\lambda) = \lambda$ if $\lambda \in D$.

Lemma 8. Let K be any Cantor set. Suppose C is a countable subset of $\cup \mathcal{G}$ such that $C \cap G$ is finite for each $G \in \mathcal{G}$. If $\mathcal{H} = \{G \in \mathcal{G} : C \cap G \neq \phi\}$ is dense in K, then $K \cup C$ is an ω-limit set $\omega(z,h)$ where $h(\lambda) = \lambda$ for each $\lambda \in K \cup C'$ $(= K = C')$.

Proof. Enumerate $\mathcal{G}$ as $\left\{G_n\right\}_{n=0}^{\infty}$ and put $G_n = \left(b_n,e_n\right)$. Let g be a one-to-one function from ω_0 onto $\omega_0 \times \omega_0$. We will define a sequence $\left\{H_n\right\}_{n=0}^{\infty}$ in $\mathcal{H}$ as follows. If $g(0) = (m,k)$, select H_0 to be a member of $\mathcal{H}$ in $\left(e_m,e_m + 2^{-k-m}\right)$. Having chosen H_j for each $j < n$, if $g(n) = (m,k)$ pick H_n to be any member of $\mathcal{H} - \left\{H_j : j < n\right\}$ included in the interval $\left(e_m,e_m + 2^{-k-m}\right)$.

For each m let $\mathcal{H}_m = \left\{H_n : g(n) = (m,k) \text{ for some } k\right\}$. Then $\mathcal{H}_m \cap \mathcal{H}_j = \phi$ whenever $m \neq j$. Moreover, each $\mathcal{H}_m$ consists of a sequence converging downward to e_m. Let $\mathcal{H}'_m = \mathcal{H}_m$ if $G_m \in \mathcal{H}_m$ and $\mathcal{H}'_m = \mathcal{H}_m \cup \left\{G_m\right\}$ if $G_m \notin \mathcal{H}_m$. Enumerate $\mathcal{H}'_m$ as $\left\{G_{mk}\right\}_{k=0}^{\infty}$. Then we have the following:

$G_{mk} \cap G_{nj} = \phi$ whenever $(m,k) \neq (n,j)$,

$\mathcal{H} = \{G_{mk},\ m \in \omega_0,\ k \in \omega_0\}$,

$|x - e_m| < 2^{-m} + |G_m|$ if $x \in G_{mk}$.

Now choose D to be any countable subset of $\cup \mathcal{G}$ such that for each $G \in \mathcal{G}$, $D \cap G$ consists of a sequence converging to the right-hand endpoint of G. By Lemma 7 $K \cup D$ is an ω-limit set $\omega(z,g)$ where $g(\lambda) = \lambda$ for all $\lambda \in K \cup D'$. We will complete the proof by showing that $K \cup C$ is W-homeomorphic to $K \cup D$, with C mapping into D. Then we invoke Lemmas 2 and 4.

For each m let $\{d_{mk}\}_{k=0}^{\infty}$ be an enumeration of $D \cap G_m$; evidently $d_{mk} \to e_m$. For each m and k let $C \cap G_{mk}$ be written as $\{c_{mk1}, c_{mk2}, \ldots, c_{mk\alpha(m,k)}\}$. For each m let f map the set $\{c_{m11}, \ldots, c_{m1\alpha(m,1)}, c_{m21}, \ldots, c_{m2\alpha(m,2)}, \ldots, c_{mk1}, \ldots, c_{mk\alpha(m,k)}, \ldots\}$ onto the set $\{d_{m1}, d_{m2}, \ldots, d_{mj}, \ldots\}$ in the order indicated. For $\lambda \in K$ put $f(\lambda) = \lambda$. Then $\operatorname{dom} f = K \cup C$ and $f(C) = D$. Moreover, f is one-to-one. We need only show that f is continuous; then it will be clear that f is a W-homeomorphism from C onto D.

Let $\{x_\alpha\}_{\alpha=0}^{\infty}$ be a sequence with $x_\alpha \to x$. Then $x \in K$, and we consider two cases.

Case 1: $x_\alpha \in K$ for all α. Then $f(x_\alpha) = x_\alpha$ and $f(x_\alpha) \to x$. But $f(x) = x$ when $x \in K$. Hence, $f(x_\alpha) \to f(x)$.

Case 2: $x_\alpha \notin K$ for all α. Then for each α there exist m_α and k_α for which $x_\alpha \in G_{m_\alpha k_\alpha}$. Since $C \cap G_{m_\alpha k_\alpha}$ is finite, if there exists an m such that $\{\alpha : m_\alpha = m\}$ is infinite, then $x = e_m$. Hence, there exists at most one m for which $\{\alpha : m_\alpha = m\}$ is infinite. There are then two subcases.

Subcase 1: $m_\alpha \to \infty$. Since $|x_\alpha - e_{m_\alpha}| < 2^{-m_\alpha} + |G_{m_\alpha}|$ we have $x_\alpha - e_{m_\alpha} \to 0$. By definition $f(x_\alpha) \in G_{m_\alpha}$. Since $|G_{m_\alpha}| \to 0$ we must have $f(x_\alpha) - e_{m_\alpha} \to 0$. Hence $f(x_\alpha) - x_\alpha \to 0$, so that $f(x_\alpha) \to x = f(x)$.

Subcase 2: there exists m such that $A = \{\alpha : m_\alpha = m\}$ is infinite. In this event, $x = e_m$. If $\omega_0 - A$ is finite then eventually $x_\alpha \in G_{mk_\alpha}$ and $k_\alpha \to \infty$; thus it is clear from the definition of f that $f(x_\alpha) \to e_m$. Hence $f(x_\alpha) \to f(x) = x = e_m$. So let us suppose that $\omega_0 - A$ is infinite. Then if $\{n_\beta\}_{n=0}^{\infty}$ enumerates $\omega_0 - A$ we must have $m_{n_\beta} \to \infty$. Then by the argument in Case 1, $f(x_{n_\beta}) \to f(x) = x = e_m$. Now if $\{t_\xi\}_{\xi=0}^{\infty}$ is an enumeration of A, we have $x_{t_\xi} \in G_{mk_{t_\xi}}$ with $k_{t_\xi} \to \infty$. It follows that $f(x_{t_\xi}) \to e_m$. Since $x = e_m$ we have $f(x_{t_\xi}) \to f(x)$. Therefore, $f(x_\alpha) \to f(x)$.

Proposition 3. Let K be any Cantor set. Suppose C is a countable, nowhere-dense subset of $\cup \mathcal{G}$ such that $K \cup C$ is closed. If $\mathcal{H} = \{G \in \mathcal{G} : C \cap G \neq \phi\}$ is dense, then $K \cup C$ is an ω-limit set $\omega(z,h)$ where $h(\lambda) = \lambda$ for all $\lambda \in K \cup C'$.

Proof. Choose C_1 to consist of exactly one isolated point from each member of $\mathcal{H}$. Now carry out the proof of Lemma 8 relative to C_1 instead of C with the only change being that D is chosen to miss $\dot{C}$. Let f be the W-homeomorphism from $K \cup C_1$ onto $K \cup D$. Extend f to the identity on $C - C_1$. Then f is a W-homeomorphism from $K \cup C$ onto $K \cup D_1$ where

$D_1 = D \cup (C - C_1)$. Then for each $G \in \mathcal{G}$, $(D_1 \cap G)' \neq \phi$.

Now, using the argument in Lemma 5, for each $G \in \mathcal{G}$ we can find a set $E_G \subseteq G$ such that (1) E_G is W-homeomorphic to $D_1 \cap G$, (2) E_G has no limit points isolated from the left, and (3) E_G is closed in G. Then, taking the union of these W-homeomorphisms and the identity on K we obtain a W-homeomorphism g from $K \cup D_1$ onto $K \cup E$, where $E = \cup \{E_G : G \in \mathcal{G}\}$. However, $K \cup E$ has no limit points isolated from the left except for $\inf K$. Hence, by Lemma 7, $K \cup E$ is an ω-limit set. Then $g \circ f$ is a W-homeomorphism from $K \cup C$ onto $K \cup E$. Therefore, by Lemma 2, $K \cup C$ is an ω-limit set $\omega(z,h)$.

By Lemmas 5 and 8, $f \circ g(\lambda) = \lambda$ when $\lambda \in K \cup C'$. By Lemma 2, $h(\lambda) = \lambda$ whenever $\lambda \in K \cup C'$.

<u>Theorem 2</u>. Any nonempty, closed, nowhere-dense set is an ω-limit set.

<u>Proof</u>. Let F be any closed, nonempty, nowhere dense subset of I. Then F can be decomposed as $K \cup C$, where C is a countable set and K is either empty or it is a Cantor set. If $K = \phi$, then F is an ω-limit set by Proposition 2.

So let us suppose that $K \neq \phi$ and $[a,b] = [\inf K, \sup K]$. If $C \cap [(0,a) \cup (b,1)] = \phi$, we are finished by Propositions 1' and 3. Suppose now that $C \cap ([0,a) \cup (b,1]) \neq \phi$. Assume that C meets only $(b,1]$; the proof of the more general case will require an obvious extension of the argument. Let $C_1 = C \cap (b,1]$. We will have two cases.

In case $d(C_1,K)$, the distance between C_1 and K, is positive, let D be some homeomorphic copy of C_1 in some member of $\mathcal{G}$, the family of components of $[a,b] - K$. Then it is clear that $K \cup C$ is W-homeomorphic to

$K \cup (C-C_1) \cup D$. By Propositions 1′ and 3 the latter set is an ω-limit set. By Lemma 2, $K \cup C$ is also an ω-limit set.

In case $d(C_1,K) = 0$ choose a sequence $\{W_n\}_{n=0}^{\infty}$ of open intervals in [b,1] converging to b such that $C_1 \subseteq \cup \{W_n : n \in \omega_o\}$ and $\overline{W_n} \cap \overline{W_m} = \phi$ whenever $n \neq m$. Next choose a sequence $\{G_n\}_{n=0}^{\infty}$ in $\mathcal{G}$ converging to b. For each n choose an open interval V_n such that $V_n \subseteq G_n - C$. Let D_n be some homeomorphic copy of $C_1 \cap W_n$ inside V_n. Then it is easily seen that $K \cup (C-C_1) \cup [\cup \{D_n : n \in \omega_0\}]$ is W-homeomorphic to $K \cup (C-C_1) \cup C_1 = K \cup C$. By Propositions 1′ and 3 the former set is an ω-limit set and Lemma 2 is clearly applicable so that $K \cup C$ is also an ω-limit set.

There is an interesting and important contrast between Propositions 1′ and 3. In the "dense" case (Proposition 3) the continuous function may be chosen so that all the limit points of the ω-limit set are fixed points, whereas in the non-dense case (Proposition 1′) there is no guarantee that there are any fixed points. If we translate this into the context of Theorem 2 we can say that a given nonempty, closed, nowhere dense set F can be realized as an ω-limit set $\omega(x,f)$ so that all the limit points of F are fixed points for f whenever F has the property that the family $\{G \in \mathcal{G}(P) : G \cap F \neq \phi\}$ is dense. Here, P denotes the perfect part of F.

In [B] the problem of characterizing the countable ω-limit sets was raised. Our Theorem 2 solves that problem since each closed countable set is nowhere dense. It is now an easy matter to characterize all ω-limit sets of continuous functions. Lemma 9 below and a remark of Šarkovskii [S_1] accomplishes this. Although we imagine Lemma 9 is known, we have not found a proof so we provide one.

Lemma 9. If F is a union of finitely many nondegenerate closed intervals, then F is an ω-limit set.

Proof. We will carry out the proof when $F = \left[0,\frac{1}{3}\right] \cup \left[\frac{2}{3},1\right]$. From this it will be clear how to deal with the general case.

Let h be a function defined on $\left[0,\frac{1}{4}\right]$ by $h(x) = 2x$ if $0 \leq x \leq \frac{1}{8}$, $h(x) = 2\left(\frac{1}{4} - x\right)$ if $\frac{1}{8} \leq x \leq \frac{1}{4}$. Let $\mathcal{B}$ consist of all finite blocks of 0's and 1's; we shall work in binary arithmetic. Using this form of representation, $\{.00B : B \in \mathcal{B}\}$ is dense in $\left[0,\frac{1}{4}\right]$.

Suppose $\left\{B_i\right\}_{i=1}^{\infty}$ is any enumeration of $\mathcal{B}$ and $\left\{k_i\right\}_{i=1}^{\infty}$ is any increasing sequence of positive integers. Consider

$$x = .00B_1[0]_{k_1}B_2[0]_{k_2}\overline{B_i[0]_{k_i}} = .00\overline{B_i[0]_{k_i}}.$$

Let t_n be the length of B_n plus twice the number of 1's in B_n. Then

$$h^{N(2)}(x) = .00B_2[0]_{k_2}\overline{B_i[0]_k}$$

where $N(2) = t_1 + k_1$. In general we have

$$h^{N(m)}(x) = .00B_m[0]_{k_m}\overline{B_i[0]_{k_i}}$$

where $N(m) = \sum_{i=1}^{m-1}\left(t_i + k_i\right)$.

Now select a particular enumeration of $\mathcal{B}$ such that each B of $\mathcal{B}$ is repeated infinitely often and $B_{2k} = B_{2k-1}$ for each $k \geq 1$. Then we may pick a particular increasing sequence $\left\{k_n\right\}_{n=1}^{\infty}$ such that $N(m)$ is odd if and only if m is odd.

Then, clearly, for each $B \in \mathcal{B}$ there exists a sequence $\left\{s_n\right\}_{n=1}^{\infty}$ of odd positive integers and an increasing sequence $\left\{\alpha_n\right\}_{n=1}^{\infty}$ such that $.00B[0]_{\alpha_n}$ is an initial block of $h^{s_n}(x)$. Since there also exists a similar sequence of odd integers, we see that both $\left\{h^n(x) : n \text{ even}\right\}$ and $\left\{h^n(x) : n \text{ odd}\right\}$ are dense in $\left[0,\frac{1}{4}\right]$.

Define f as follows:

$$f(x) = \begin{cases} h(x) & , 0 \leqslant x \leqslant \frac{1}{4} \\ \frac{3}{2}x - \frac{3}{8} & , \frac{1}{4} \leqslant x \leqslant \frac{3}{4} \\ h\left(x - \frac{3}{4}\right) + \frac{3}{4} & , \frac{3}{4} \leqslant x \leqslant 1. \end{cases}$$

Define g as follows:

$$g(x) = \begin{cases} x + \frac{3}{4} & , 0 \leqslant x \leqslant \frac{1}{4} \\ -2x + \frac{3}{2} & , \frac{1}{4} \leqslant x \leqslant \frac{3}{4} \\ x - \frac{3}{4} & , \frac{3}{4} \leqslant x \leqslant 1. \end{cases}$$

The three functions f, g, and $g \circ f$ are all continuous from I onto I. It is easily verified that $(g \circ f)^n(x_0) = h^n(x_0)$ if n is even, and $(g \circ f)^n(x_0) = h^n(x_0) + \frac{3}{4}$ if n is odd. However, $\{h^n(x_0) : n \text{ even}\}$ is dense in $\left[0, \frac{1}{4}\right]$, and $\{h^n(x_0) + \frac{3}{4} : n \text{ odd}\}$ is dense in $\left[\frac{3}{4}, 1\right]$. Therefore, $\omega(x_0, g \circ f) = \left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right]$.

Theorem 3. Let F be a nonempty closed set. Then F is an ω-limit set if and only if either F is nowhere dense, or it is a union of finitely many nondegenerate closed intervals.

Proof. Šarkovskii [S_1] showed that if an ω-limit set has a nonvoid interior then it must consist of finitely many nondegenerate closed intervals. Now apply Lemma 9 and Theorem 2 to finish the proof.

It would be interesting to know to what extent our results carry over to more general compact spaces; for example, to continuous functions mapping the closed unit disk into itself. In this connection we note that since each countable planar set is homeomorphic via some projection onto a line to some countable linear set, it follows that each closed countable set in the unit disk is an ω-limit set for some continuous function mapping the disk into itself. But, in general, for a two-dimensional space there are considerable difficulties in applying our techniques to an arbitrary uncountable nowhere dense compact set, since our proofs depend heavily on linearity.

In a subsequent paper we will investigate the structure of ω-limit sets for more general kinds of functions from I to I; for instance, Darboux functions in the first class of Baire.

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