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The algebra generated by derivatives which are continuous almost everywhere

In 1982 Zbigniew Grande posed several questions concerning algebras generated by different classes of functions. One of them was:

Problem 1. (Problem 9 of [1]). What is the smallest algebra of functions containing all almost everywhere continuous derivatives? Is it the family of all almost everywhere continuous Baire 1 functions?

In this paper we answer both parts of this problem in the positive. Our result is very closely related to David Preiss's theorem concerning the algebra generated by all derivatives. (The only difference is that we have not proved whether our function h can be chosen to be Lebesgue or not.)

Theorem 2. (Theorem of [4]). Whenever u is a function of the first class there are derivatives f, g and h such that $u = fg + h$. Moreover one can find such a representation so that g is bounded and h is Lebesgue and in case u is bounded, such that f and h are also bounded.

First we develope notation and state some known results which we use later. Then we state our main theorem after a few lemmas used in its proof.

The real line $(-\infty, +\infty)$ is denoted by $\mathbf{R}$. The word function means mapping from $\mathbf{R}$ into $\mathbf{R}$. The words measure, almost everywhere (a.e.), integrable etc. refer to Lebesgue measure in $\mathbf{R}$. For each set $A \subset \mathbf{R}$ let $\mathrm{int}A$ be its interior, $\mathrm{cl}A$ its closure, A^c its complement, χ_A its characteristic function and $\mid A \mid$ its outer Lebesgue measure: if $x \in \mathbf{R}$ and $A \subset \mathbf{R}$, then $\rho(x, A) = \inf\{\mid y - x \mid : y \in A\}$ denotes the distance between x and A; symbols like $\int_a^b f$ or $\int_A f$ will always mean the corresponding Lebesgue integral. A function f is in the first class of Baire (B^1) iff it is a pointwise limit of a sequence of continuous functions: it is called a derivative iff there is a function F (called a primitive of f) so that $F'(x) = f(x)$ for each $x \in \mathbf{R}$. A point $x \in \mathbf{R}$ is a density point of $A \subset \mathbf{R}$ iff $\lim_{h \to 0} \mid A^c \cap (x - h, x + h) \mid /(2h) = 0$. A function f is approximately continuous iff for each $x_0 \in \mathbf{R}$ and each $\varepsilon > 0$ x_0 is a density point of $\{x \in \mathbf{R} : | f(x) - f(x_0) | < \varepsilon\}$. We denote by $a \vee b$ ($a \wedge b$) the larger (the smaller) of the real numbers a and b. If f is any function and $x \in \mathbf{R}$, then $\omega(f, x) = \inf\{\sup\{| f(y) - f(z) | : | y - x | < \varepsilon, | z - x | < \varepsilon\} : \varepsilon > 0\}$ is called the oscillation of f at x. We let $\| f \| = \sup\{| f(x) | : x \in \mathbf{R}\}$ and $D(f)$ denotes the set of points of discontinuity of f. We write $\sum_n f_n$, $\bigcup_n A_n$ etc. instead of $\sum_{n \in \mathbf{N}} f_n$, $\bigcup_{n \in \mathbf{N}} A_n$ when there is no possible misunderstanding. Finally by $\mathcal{F}$ we denote the family of all sets $A \subset \mathbf{R}$ such that $| A - \text{int} A | = 0$. (Note that each interval and each set of measure 0 belong to $\mathcal{F}$.)

Symbols like 1., 2. etc. denote the corresponding lemma, theorem or corollary, while (1), (2) etc. refer to conditions marked in the text.

Theorem 3. (Theorem 4.14. of [3]) If $H \subset [0,1]$, $| H | = 0$ and $u \in B^1$, then there is a derivative f so that $f(x) = u(x)$ for $x \in H$.

Theorem 4. (Lemma 4.4. of [3]) Assume that $H \subset [0,1]$ is nowhere dense and closed and f is a derivative. Then there is a derivative g so that g is continuous in $[0,1] - H$ and $f(x) = g(x)$ for $x \in H$.

Corollary 5. Whenever $u \in B^1$, $H \subset \mathbf{R}$ is closed and $| H | = 0$, there is a derivative f so that f is continuous in H^c and $f(x) = u(x)$ whenever $x \in H$.

Theorem 6. (Remark to Theorem 1. of [2]) Let $H \subset [0,1]$ be closed, $| H | = 0$ and let $u \in B^1$ be bounded. Then there is a bounded approximately continuous function φ which is continuous in H^c so that $\varphi(x) = u(x)$ for $x \in H$.

Remark 7. Combining the proofs of Theorem 3.2 of [3] and of Theorem 1. of [2], and using that each bounded approximately continuous function is a derivative we get easily that if the assumptions of 6. are met, then we can find a bounded derivative φ which is continuous in H^c and moreover $\| \varphi \| \leq \| u \|$.

Lemma 8. Assume that A, $B \subset \mathbf{R}$, A is closed and $B \in \mathcal{F}$. Then $B - A \in \mathcal{F}$.

Proof.

$$\begin{aligned} | B - A - \text{int}(B - A) | &= | B - A - \text{int}(B \cap A^c) | = | B - A - (\text{int} B \cap A^c) | \\ &= | B \cap A^c \cap ((\text{int} B)^c \cup A) | = | B \cap A^c \cap (\text{int} B)^c | \\ &\leq | B - \text{int} B | = 0. \end{aligned}$$

Lemma 9. Whenever $A \in \mathcal{F}$ is an F_σ-set, there are closed sets $A_1, A_2 \dots \in \mathcal{F}$ so that $A = \bigcup_n A_n$.

Proof. Using that each open interval is a countable union of a family of closed intervals we get that there are closed intervals $B_1, B_2, \cdots$ such that $\text{int}A = \bigcup_n B_n$. (Certainly $B_n \in \mathcal{F}$ for $n \in \mathbf{N}$.) Since $A - \text{int}A$ is an F_σ-set, there are closed sets $C_1, C_2, \cdots$ such that $A - \text{int}A = \bigcup_n C_n$. We have for each $n \in \mathbf{N}$ $| C_n | \leq | A - \text{int}A | = 0$. Thus $C_n \in \mathcal{F}$, which together with the previous observation completes the proof.

Lemma 10. Whenever $v \in B^1$ is an almost everywhere continuous function and $\varepsilon > 0$, there is an almost everywhere continuous function $v_1 \in B^1$ so that $D(v_1)$ is closed and $\| v - v_1 \| < \varepsilon$.

Proof. Put $B_k = \{x \in \mathbf{R} : (k-1)\varepsilon < v(x) < (k+1)\varepsilon\}$ for $k \in \mathbf{Z}$. Since $v \in B^1$ and since $| D(v) | = 0$, for each $k \in \mathbf{Z}$ B_k is an F_σ-set and $B_k \in \mathcal{F}$. By 9. let $B_k = \bigcup_l B_{kl}$, where each B_{kl} is closed and $B_{kl} \in \mathcal{F}$. Make a sequence $\{C_n : n \in \mathbf{N}\}$ of all sets B_{kl}, $k \in \mathbf{Z}$, $l \in \mathbf{N}$. Put $\bar{C}_1 = \emptyset$, $\bar{C}_n = C_1 \cup \cdots \cup C_{n-1}$ for $n > 1$ and let $v_1(x) = k\varepsilon$ if for some $r \in \mathbf{N}$ $x \in C_n - \bar{C}_n$ and $C_n \subset B_k$. Note that $\bigcup_n (C_n - \bar{C}_n) = \bigcup_n C_n = \bigcup_k B_k = \mathbf{R}$.) Then

1) $v_1 \in B^1$ because for any $a \in \mathbf{R}$ $\{x \in \mathbf{R} : v_1(x) > a\}$ is the union of $\{C_n - \bar{C}_n : C_n \subset B_k \text{ and } k > a/\varepsilon\}$ so it is an F_σ-set, while $\{x \in \mathbf{R} : v_1(x) < a\}$ is the union of $\{C_n - \bar{C}_n : C_n \subset B_k \text{ and } k < a/\varepsilon\}$ so it is also an F_σ-set.

2) $D(v_1)$ is closed since it is equal to $\{x \in \mathbf{R} : \omega(v_1, x) \geq \varepsilon\}$,

3) $| D(v_1) | \leq | \bigcup_n (C_n - \bar{C}_n - \text{int}(C_n - \bar{C}_n)) | = 0$ since by 8. all $C_n - \bar{C}_n \in \mathcal{F}$.

The statement $\| v - v_1 \| < \varepsilon$ is obvious.

Lemma 11. Assume that $u \in B^1$ is continuous almost everywhere. Then there are almost everywhere continuous functions $u_1, u_2, \cdots \in B^1$ so that

i) $D(u_1)$, $D(u_2), \ldots$ are closed,

ii) $\| u_k \| < 2^{-k}$ if $k \geq 2$,

iii) $u = \sum_k u_k$

Proof. For $k = 1, 2, \ldots$ use 10. with $v = u - u_1 - \cdots - u_{k-1}$ ($v = u$ if $k = 1$) and $\varepsilon = 3^{-k-1}$ writing the result as u_k. Then i) is met.

$$\begin{aligned} \| u_k \| &\leq \| u - u_1 - \cdots - u_{k-1} - u_k \| + \| u - u_1 - \cdots - u_{k-1} \| \\ &\leq 3^{-k-1} + 3^{-k} < 2^{-k} \qquad (k \geq 2) \end{aligned}$$

proves ii) and

$$\begin{aligned}\| u - \sum_k u_k \| &\leq \inf\{\| u - u_1 - \cdots - u_n \| + \| u_{n+1} \| \\ &\qquad + \| u_{n+2} \| + \cdots : n \in \mathbf{N}\} \\ &\leq \inf\{3^{-n-1} + 2^{-n-1} + 2^{-n-2} + \cdots : n \in \mathbf{N}\} = 0\end{aligned}$$

completes the proof.

__Lemma 12.__ Assume that $A \subset \mathbf{R}$ is closed and nowhere dense and v is a function so that $D(v) \subset A$. Then there is a closed set $B \subset \mathbf{R}$ so that

i) each $x \in A$ is both a left and right limit point of $B - A$.

ii) $B - A$ is isolated.

iii) $B^c = \bigcup_n G_n$, where $\{G_n : n \in \mathbf{N}\}$ are pairwise disjoint, nonvoid, bounded open intervals,

iv) $c_n = \| v\chi_{G_n} \| < +\infty$,

v) if

1) $f_1, f_2, \ldots$ are summable derivatives,
2) $f_n(x) = 0$ if $x \notin G_n \quad (n = 1, 2, \ldots)$,
3) $\int_{\mathbf{R}} f_n = 0 \quad (n = 1, 2, \ldots)$,
4) there is $N \in \mathbf{R}$ so that for $n = 1, 2, \ldots$ $\| f_n \| \leq N(c_n \vee \sqrt{c_n})$, then $f = \sum_n f_n$ is a derivative and $D(f) \subset A \cup \bigcup_n D(f_n)$.

__Proof.__ Choose within each open interval contiguous to A a sequence of real numbers increasing to its right endpoint and another one decreasing to its left endpoint. Let E equal the union of all those sequences. Let $\{(e_{k1}, e_{k2}) : k \in \mathbf{N}\}$ be components of $(A \cup E)^c$. Due to the choice of sequences it is clear that $\{e_{k1} : k \in \mathbf{N}\} = \{e_{k2} : k \in \mathbf{N}\} = E \subset (D(v))^c$. So $M_k = \| v\chi_{(e_{k1}, e_{k2})} \| < +\infty$. Put $d_k = \rho(e_{k1}, A) \wedge \rho(e_{k2}, A)$ for $k \in \mathbf{N}$ and let

$$B = A \cup E \cup \{e_{k1} + id_k^2/(1 \vee M_k) : i < (e_{k2} - e_{k1})(1 \vee M_k)/d_k^2, \quad i, k \in \mathbf{N}\}.$$

Then conditions i) - iv) are met. To prove v) examine the function

$$F(x) = \begin{cases} \int_{-\infty}^{x} f_n & \text{if } x \in G_n, \ n = 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

We will prove that F is a primitive of f. Choose any $x_0 \in \mathbf{R}$. If $x_0 \in G_n$ for some $n \in \mathbf{N}$, then for each x close enough to x_0

$$\frac{F(x) - F(x_0)}{x - x_0} = \frac{\int_{x_0}^{x} f_n}{x - x_0} \xrightarrow[x \to x_0]{} f_n(x_0) = f(x_0).$$

If $x_0 \in B - A$, then there are n_1, $n_2 \in \mathbf{N}$ so that $x_0 \in \mathrm{cl}G_{n_1} \cap \mathrm{cl}G_{n_2}$. Then for each x close enough to x_0

$$\frac{F(x) - F(x_0)}{x - x_0} = \left\{ \begin{array}{ll} \frac{\int_{x_0}^{x} f_{n_1}}{x - x_0} & (x \in G_{n_1}) \\ & \\ \frac{\int_{x_0}^{x} f_{n_2}}{x - x_0} & (x \in G_{n_2}) \end{array} \right\} \xrightarrow[x \to x_0]{} 0 = f(x_0).$$

Finally if $x_0 \in A$, then for each x

- $x \in B$ implies $\frac{F(x) - F(x_0)}{x - x_0} = 0 = f(x_0)$,
- if there is $n \in \mathbf{N}$ such that $x \in G_n$, then there is $k \in \mathbf{N}$ such that $x \in (e_{k_1}, e_{k_2})$. So

$$\begin{aligned} \left| \frac{F(x) - F(x_0)}{x - x_0} \right| &= \left| \frac{\int_{-\infty}^{x} f_n}{x - x_0} \right| \leq \frac{\int_{G_n} \mid f_n \mid}{\mid x - x_0 \mid} \leq \frac{\| f_n \| \mid G_n \mid}{\mid x - x_0 \mid} \\ &\leq \frac{N(M_k \vee \sqrt{M_k} \frac{d_k^2}{1 \vee M_k}}{\mid x - x_0 \mid} \leq N d_k \\ &\leq N \mid x - x_0 \mid \xrightarrow[x \to x_0]{} 0 = f(x_0). \end{aligned}$$

The rest of the proof is easy.

Lemma 13. Assume that B is closed, $A = \mathrm{cl}(B - A) - (B - A)$, $B - A$ is isolated and v is any function. Then there is a function ψ so that

i) $\psi(x) = v(x)$ if $x \in B - A$,

ii) $\psi(x) = 0$ if $x \in A$,

iii) $D(\psi) \subset A$,

iv) ψ is a derivative,

v) if E is a closed interval whose endpoints belong to B and if $v\chi_E$ is bounded, then $\psi\chi_E$ is also bounded and moreover $\| \psi\chi_E \| \leq \| v\chi_E \|$.

__Proof.__ Let $B - A = \{b_n : n \in \mathbb{N}\}$. For $n \in \mathbb{N}$ put $c_n = d_n \wedge \frac{D_n^2}{(2| v(b_n) | V1)}$, where $d_n = \rho(b_n, B - \{b_n\})/3$ and $D_n = \rho(b_n - d_n, A) \wedge \rho(b_n + d_n, A)$. Put

$$e(x) = \begin{cases} 0 & x \in (-\infty, -1] \cup (1, +\infty) \\ x + 1 & x \in (-1, 0] \\ -8x + 1 & x \in (0, 1/4] \\ -1 & x \in (1/4, 1/2] \\ 2x - 2 & x \in (1/2, 1]. \end{cases}$$

Then e is continuous everywhere, $\| e \| = 1$ and $\int_{\mathbb{R}} e = 0$. Put $\psi_n(x) = v(b_n)e((x - b_n)/c_n)$ $(n = 1, 2, \ldots)$ and $\psi = \sum_n \psi_n$. Then i), ii) and iii) are satisfied. To prove iv) examine the function

$$\Psi(x) = \begin{cases} \int_{-\infty}^{x} \psi_n & \text{if } x \in (b_n - c_n, b_n + c_n),\ n = 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

We will prove that Ψ is a primitive of ψ. Choose any $x_0 \in \mathbb{R}$. If $x_0 \in (b_n - c_n, b_n + c_n)$ for some $n \in \mathbb{N}$, then for any x close enough to x_0

$$\frac{\Psi(x) - \Psi(x_0)}{x - x_0} = \frac{\int_{x_0}^{x} \psi_n}{x - x_0} \underset{x \to x_0}{\longrightarrow} \psi_n(x_0) = \psi(x_0).$$

If $x_0 = b_n - c_n$ for some $n \in \mathbb{N}$, then for each x close enough to x_0

- if $x > x_0$, then

$$\frac{\Psi(x) - \Psi(x_0)}{x - x_0} = \frac{\int_{x_0}^{x} \psi_n}{x - x_0} \underset{x \to x_0}{\longrightarrow} \psi_n(x_0) = 0 = \psi(x_0),$$

- if $x < x_0$, then $\frac{\Psi(x) - \Psi(x_0)}{x - x_0} = 0 = \psi(x_0)$.

Similarly if $x_0 = b_n + c_n$ for some $n \in \mathbb{N}$, then

$$\lim_{x \to x_0} \frac{\Psi(x) - \Psi(x_0)}{x - x_0} = \psi(x_0).$$

If $x_0 \in A$, then for each $x \in \mathbb{R}$

- if $x \in (b_n - c_n, b_n + c_n)$ for some $n \in \mathbf{N}$, then

$$\begin{aligned}
\left|\frac{\Psi(x)-\Psi(x_0)}{x-x_0}\right| &= \left|\frac{\int_{-\infty}^{x}\psi_n}{x-x_0}\right| = \left|\frac{\int_{b_n-c_n}^{x}\psi_n}{x-x_0}\right| \\
&\le \frac{\int_{b_n-c_n}^{x} |\psi_n|}{|x-x_0|} \le \frac{2c_n \| \psi_n \|}{|x-x_0|} \\
&= \frac{2c_n |v(b_n)|}{|x-x_0|} \le \frac{D_n^2}{2|v(b_n)| \vee 1}\frac{2|v(b_n)|}{|x-x_0|} \\
&\le D_n \le |x-x_0| \xrightarrow[x\to x_0]{} 0 = \psi(x_0),
\end{aligned}$$

- $x_0 \notin \bigcup_n (b_n - c_n, b_n + c_n)$ implies $\frac{\Psi(x)-\Psi(x_0)}{x-x_0} = 0 = \psi(x_0)$.

Finally $x_0 \in \operatorname{int}(\bigcup_n [b_n - c_n, b_n + c_n])^c$ implies that for each x close enough to x_0 $\frac{\Psi(x)-\Psi(x_0)}{x-x_0} = 0 = \psi(x_0)$.

Now take a closed interval E with both endpoints belonging to B such that $v\chi_E$ is bounded. Then

$$\begin{aligned}
\| \psi\chi_E \| &= \sup\{|\psi(b_n)| : b_n \in E, n \in \mathbf{N}\} \\
&= \sup\{|v(b_n)| : b_n \in E, n \in \mathbf{N}\} \le \| v\chi_E \|,
\end{aligned}$$

which completes the proof.

Lemma 14. Assume that $G = (a_1, a_2)$ is an open bounded nonvoid interval, functions $f_0, \bar{f}, g_0$ and $\bar{g}$ are summable over G and w is so that $w\chi_G$ is a bounded summable derivative, $\| w\chi_G \| = C$. Then there are functions g and h continuous everywhere and a summable derivative f so that

i) $w\chi_G = fg + h$,

ii) $f(x) = g(x) = h(x) = 0$ whenver $x \notin G$,

iii) $\int_{\mathbf{R}} f = \int_{\mathbf{R}} g = \int_{\mathbf{R}} h = \int_{\mathbf{R}}(fg_0) = \int_{\mathbf{R}}(f\bar{g}) = \int_{\mathbf{R}}(gf_0) = \int_{\mathbf{R}}(g\bar{f}) = 0$,

iv) $\| f \| \le 50(C \vee \sqrt{C})$, $\| g \| \le 1 \wedge \sqrt{C}$, $\| h \| \le 73C$.

Proof. For $i = 1, \ldots, 5$ put $e_i(x) = \sin(ix)\chi_{[0,2\pi]}$ and put

$$e(x) = \sum_{i=1}^{5} a_i e_i\left(2\pi\frac{x-a_1}{a_2-a_1}\right)$$

where $\alpha_1, \alpha_2, \dots, \alpha_5$ are some real numbers. There are $\alpha_1, \alpha_2, \dots, \alpha_5$ so that $\int_{\mathbf{R}} e^2 = \mid \int_G w \mid$ and $\int_{\mathbf{R}} e = \int_{\mathbf{R}}(eg_0) = \int_{\mathbf{R}}(e\bar{g}) = \int_{\mathbf{R}}(ef_0) = \int_{\mathbf{R}}(e\bar{f}) = 0$. Indeed, $\int_{\mathbf{R}} e = 0$ for any $\alpha_1, \alpha_2, \dots, \alpha_5$ and the following system of equations

$$\begin{cases} x_1 \int_{\mathbf{R}} e_1 \left(2\pi \frac{x-a_1}{a_2-a_1}\right) g_0(x)dx + \cdots + x_5 \int e_5 \left(2\pi \frac{x-a_1}{a_2-a_1}\right) g_0(x)dx = 0 \\ x_1 \int_{\mathbf{R}} e_1 \left(2\pi \frac{x-a_1}{a_2-a_1}\right) \bar{g}(x)dx + \cdots + x_5 \int e_5 \left(2\pi \frac{x-a_1}{a_2-a_1}\right) \bar{g}(x)dx = 0 \\ x_1 \int_{\mathbf{R}} e_1 \left(2\pi \frac{x-a_1}{a_2-a_1}\right) f_0(x)dx + \cdots + x_5 \int e_5 \left(2\pi \frac{x-a_1}{a_2-a_1}\right) f_0(x)dx = 0 \\ x_1 \int_{\mathbf{R}} e_1 \left(2\pi \frac{x-a_1}{a_2-a_1}\right) \bar{f}(x)dx + \cdots + x_5 \int e_5 \left(2\pi \frac{x-a_1}{a_2-a_1}\right) \bar{f}(x)dx = 0 \end{cases}$$

is linear, homogeneous and the number of unknowns exceeds the number of equations so it has a non-zero solution, say $\beta_1, \beta_2, \dots, \beta_5$. Since

$$\begin{aligned} &\int_{\mathbf{R}} \left(\beta_1 e_1 \left(2\pi \frac{x-a_1}{a_2-a_1}\right) + \cdots + \beta_5 e_5 \left(2\pi \frac{x-a_1}{a_2-a_1}\right)\right)^2 dx \\ = \;& \beta_1^2 \int_{\mathbf{R}} e_1^2 \left(2\pi \frac{x-a_1}{a_2-a_1}\right) dx + \cdots \beta_5^2 \int_{\mathbf{R}} e_5^2 \left(2\pi \frac{x-a_1}{a_2-a_1}\right) dx \\ = \;& \beta_1^2(a_2-a_1)/2 + \cdots + \beta_5^2(a_2-a_1)/2 \\ = \;& (\beta_1^2 + \cdots + \beta_5^2)(a_2-a_1)/2, \end{aligned}$$

$\alpha_1 = \gamma\beta_1, \dots, \alpha_5 = \gamma\beta_5$, where $\gamma = \left(\frac{2|\int_g w|}{(a_2-a_1)(\beta_1^2+\cdots+\beta_5^2)}\right)^{1/2}$ satisfy our requirements. For $i = 1, 2, \dots, 5$ we have

$$\begin{aligned} \alpha_i^2(a_2-a_1)/2 \;&\le\; (\alpha_1^2 + \cdots + \alpha_5^2)(a_2-a_1)/2 = \int_{\mathbf{R}} e^2 \\ &= \; \mid \int_G w \mid \le (a_2-a_1) \parallel w\chi_G \parallel = (a_2-a_1)C. \end{aligned}$$

So $\mid \alpha_i \mid \le \sqrt{2C}$. Hence $\parallel e \parallel = \parallel \alpha_1 e_1 + \cdots + \alpha_5 e_5 \parallel \le 5\sqrt{2C}$. Put $f(x) = 5\sqrt{2}(\sqrt{C} \vee 1)e(x)$, $g(x) = \frac{e(x)\operatorname{sgn}(\int_G w)}{5\sqrt{2}(\sqrt{C}\vee 1)}$ and $h = w\chi_G - fg$. Then i) - iii) are obviously satisfied and the following proves iv) completing the proof:

$$\begin{aligned} \parallel g \parallel = \parallel e \parallel /(5\sqrt{2}(\sqrt{C} \vee 1)) \le 5\sqrt{2C}/(5\sqrt{2}(\sqrt{C} \vee 1)) \le 1 \wedge \sqrt{C}, \\ \parallel f \parallel \le 5\sqrt{2}(\sqrt{C} \vee 1) \parallel e \parallel \le 5\sqrt{2C} \times 5\sqrt{2}(\sqrt{C} \vee 1) \le 50(C \vee \sqrt{C}), \\ \parallel h \parallel = \parallel w\chi_G - fg \parallel \le C + \parallel f \parallel \parallel g \parallel \le C + 50C \le 51C. \end{aligned}$$

Theorem 15. Whenever $u \in B^1$ is continuous almost everywhere there are derivatives f, g and h which are continuous almost everywhere so that g is bounded and $u = fg + h$ and in case u is bounded so that f and h are also bounded.

Proof. By 11. there are functions $u_1, u_2, \ldots$ with the properties described there. Now we will define inductively (on k) almost everywhere continuous derivatives $\varphi_k, \psi_k, f_k, g_k, \bar{f}_k, \bar{g}_k$ and h_k, sets A_k, G_{kn} and positive numbers c_{kn} and C_{kn} $(k = 1, 2, \ldots; n = 1, 2, \ldots)$ so that:

(1) $u_k = f_k g_k + h_k + \varphi_k + \psi_k$,

(2) $\| g_1 \| \le 1$, $\| f_k \| \le 2^{8-k/2}$, $\| g_k \| \le 2^{2-k/2}$, $\| h_k \| \le 2^{7-k}$ $\quad (k \ge 2)$,

(3) $D(f_k) \subset A_k$, $D(g_k) \subset A_k$, $D(h_k) \subset A_k$,

(4) $\bar{f}_1 = \bar{g}_1 = \bar{f}_2 = \bar{g}_2 = 0$,

(5) $\bar{f}_k = \bar{f}_{k-1} + f_{k-1}$, $\bar{g}_k = \bar{g}_{k-1} + g_{k-1}$ $\quad (k \ge 3)$,

(6) sets B_k, $\{G_{kn} : n \in \mathbf{N}\}$ and numbers $\{c_{kn} : n \in \mathbf{N}\}$ are picked to the set A_k and function $u_k - \varphi_k$ according to 12.,

(7) $C_{kn} = \| (u_k - \varphi_k - \psi_k)\chi_{G_{kn}} \|$ $\quad (n = 1, 2, \ldots)$,

(8) $f_k\bar{g}_k$, $g_k\bar{f}_k$, $f_1 g_k$ and $g_1 f_k$ are almost everywhere continuous derivatives.

First step. Put $A_1 = D(u_1)$. Since $| A_1 | = 0$ and since A_1 is closed, it is nowhere dense. According to 5. there is a derivative φ_1 so that

(9) $\{x \in \mathbf{R} : \varphi_1(x) = u_1(x)\} \supset A_1$ and $D(\varphi_1) \subset A_1$.

Hence $D(u_1 - \varphi_1) \subset A_1$. So we can use 12. with $A = A_1$ and $v = u_1 - \varphi_1$ getting a closed set B_1, a family of open bounded intervals $\{G_{1n} : n \in \mathbf{N}\}$ and a sequence of real numbers $\{c_{1n} : n \in \mathbf{N}\}$ satisfying 12. i) - v). By 12. i) $A_1 \subset \mathrm{cl}(B_1 - A_1) - (B_1 - A_1)$ and since B_1 is closed, $\mathrm{cl}(B_1 - A_1) - (B_1 - A_1) \subset B_1 - (B_1 - A_1) = A_1 \cap B_1 = A_1$. Hence we can take $A = A_1$, $B = B_1$ and $v = u_1 - \varphi_1$ in 13. and find an almost everywhere continuous derivative ψ_1 satisfying 13. i) - v). By (9) and 13. iii) $D(u_1 - \varphi_1 - \psi_1) \subset A_1$, so by 13. i) $(u_1 - \varphi_1 - \psi_1)\chi_{G_{1n}}$ is continuous everywhere. Put $C_{1n} = \| (u_1 - \varphi_1 - \psi_1)\chi_{G_{1n}} \|$ $(n = 1, 2, \ldots)$ and note that by the above and 13. v)

(10) $C_{1n} \le \| (u_1 - \varphi_1)\chi_{G_{1n}} \| + \| \psi_1\chi_{G_{1n}} \| \le 2c_{1n}$ $\quad (n = 1, 2, \ldots)$.

For $n = 1, 2, \ldots$ use 14. with $G = \chi_{G_{1n}}$, $f_0 = \bar{f} = \bar{f}_1 = g_0 = \bar{g} = \bar{g}_1 = 0$ and $w = u_1 - \varphi_1 - \psi_1$ getting functions f_{1n}, g_{1n} and h_{1n} satisfying 14. i) - iv). (Note that $C = C_{1n}$.) We will use 12. v) for each of the sequences $\{f_{1n} : n \in \mathbb{N}\}$, $\{g_{1n} : n \in \mathbb{N}\}$ and $\{h_{1n} : n \in \mathbb{N}\}$. We check the assumptions:

1) f_{1n}, g_{1n} and h_{1n} are continuous everywhere ($n = 1, 2, \ldots$),

2)-3) are included in 14. ii) - iii),

4) by (10) and 14. iv) we get

$$\| f_{1n} \| \leq 50(C_{1n} \vee \sqrt{C_{1n}}) \leq 100(c_{1n} \vee \sqrt{c_{1n}}),$$
$$\| g_{1n} \| \leq 1 \wedge \sqrt{C_{1n}} \leq \sqrt{2}(c_{1n} \vee \sqrt{c_{1n}}),$$
$$\| h_{1n} \| \leq 51C_{1n} \leq 102(c_{1n} \vee \sqrt{c_{1n}}).$$

Hence $f_1 = \sum_n f_{1n}$, $g_1 = \sum_n g_{1n}$ and $h_1 = \sum_n h_{1n}$ are derivatives and $D(f_1) \subset A_1$, $D(g_1) \subset A_1$, $D(h_1) \subset A_1$. Certainly the other requirements are also met.

Inductive step. Assume that we have already defined functions $\varphi_i, \psi_i, f_i, \bar{f}_i, g_i, \bar{g}_i, h_i$, sets A_i, B_i, G_{in} and numbers c_{in} and C_{in} for $i = 1, 2, \ldots, k-1; n = 1, 2, \ldots$, where $k \geq 2$. Put $A_k = B_{k-1} \cup D(u_k)$. A_k is closed and $| A_k | = 0$, so it is nowhere dense. According to 7. there is a derivative φ_k so that

(11) $\{x \in \mathbb{R} : \varphi_k(x) = u_k(x)\} \supset A_k$, $\| \varphi_k \| \leq \| u_k \|$ and $D(\varphi_k) \subset A_k$.

Hence $D(u_k - \varphi_k) \subset A_k$. So we can use 12. with $A = A_k$ and $v = u_k - \varphi_k$ and find a closed set B_k, a family of open bounded intervals $\{G_{kn} : n \in \mathbb{N}\}$ and a sequence of real numbers $\{c_{kn} : n \in \mathbb{N}\}$ satisfying 12. i) - v). By 12. i) we get $A_k \subset \mathrm{cl}(B_k - A_k) - (B_k - A_k)$ and since B_k is closed, $\mathrm{cl}(B_k - A_k) - (B_k - A_k) \subset B_k - (B_k - A_k) = A_k \cap B_k = A_k$. Hence we can take $A = A_k$, $B = B_k$ and $v = u_k - \varphi_k$ in 13. and find an almost everywhere continuous derivative ψ_k satisfying 13. i) - v). By (11) and 13. iii) we have $D(u_k - \varphi_k - \psi_k) \subset A_k$, so by 13. i) $(u_k - \varphi_k - \psi_k)\chi_{G_{kn}}$ is continuous everywhere. Put $C_{kn} = \| (u_k - \varphi_k - \psi_k)\chi_{G_{kn}} \|$ $(n = 1, 2, \ldots)$ and note that by 13. v)

(12) $C_{kn} \leq \| (u_k - \varphi_k)\chi_{G_{kn}} \| + \| \psi_k \chi_{G_{kn}} \| \leq 2c_{kn}$ $(n = 1, 2, \ldots)$.

For $n = 1, 2, \ldots$ use 14. with $G = G_{kn}, f_0 = f_1, g_0 = g_1, \bar{f} = \bar{f}_k = f_2 + \cdots + f_{k-1}$ and $\bar{g} = \bar{g}_k = g_2 + \cdots + g_{k-1}$ (If $k = 2$, then $\bar{f} = \bar{f}_2 = \bar{g} = \bar{g}_2 = 0$.) getting functions f_{kn}, g_{kn} and h_{kn} which satisfy 14. i) - iv). (Note that $C = C_{kn}$.) We will use 12. v) for each of the sequences $\{f_{kn} : n \in \mathbb{N}\}$, $\{g_{kn} : n \in \mathbb{N}\}$, $\{h_{kn} : n \in \mathbb{N}\}$, $\{f_{kn}g_1 : n \in \mathbb{N}\}$, $\{g_{kn}f_1 : n \in \mathbb{N}\}$, $\{f_{kn}\bar{g}_k : n \in \mathbb{N}\}$ and $\{g_{kn}\bar{f}_k : n \in \mathbb{N}\}$. We check the assumptions:

1) for $n = 1, 2, \ldots$ functions $f_{kn}, g_{kn}, h_{kn}, f_{kn}g_1, g_{kn}f_1, f_{kn}\bar{g}_k$ and $g_{kn}\bar{f}_k$ are continuous everywhere.

2)-3) are implied by 14. ii) - iii)

4) from (12) and 14.iv) we get

$$\| f_{kn} \| \leq 50(C_{kn} \vee \sqrt{C_{kn}}) \leq 100(c_{kn} \vee \sqrt{c_{kn}}),$$
$$\| g_{kn} \| \leq 1 \wedge \sqrt{C_{kn}} \leq \sqrt{2}(c_{kn} \vee \sqrt{c_{kn}}),$$
$$\| f_{kn}g_1 \| \leq \| f_{kn} \| \| g_1 \| \leq 150(c_{kn} \vee \sqrt{c_{kn}}),$$
$$\| g_{kn}f_1 \| \leq \| g_{kn} \| \| f_1\chi_{G_{kn}} \| \leq \| f_1\chi_{G_{kn}} \| \sqrt{2}(c_{kn} \vee \sqrt{c_{kn}}),$$

(Note that $D(f_1) \subset A_1 \subset A_k$, so $\| f_1\chi_{G_{kn}} \| < +\infty$.), and by (2) and (5)

$$\begin{aligned} \| f_{kn}\bar{g}_k \| &\leq (k-2) \| f_{kn} \| \leq 100(k-2)(c_{kn} \vee \sqrt{c_{kn}}), \\ \| g_{kn}\bar{f}_k \| &\leq (k-2) \| g_{kn} \| \leq (k-2)\sqrt{2}(c_{kn} \vee \sqrt{c_{kn}}). \end{aligned}$$

Hence $f_k = \sum_n f_{kn}$, $g_k = \sum_n g_{kn}$, $h_k = \sum_n h_{kn}$, $f_k\bar{g}_k = \sum_n(f_{kn}\bar{g}_k)$, $g_k\bar{f}_k = \sum_n(g_{kn}\bar{f}_k)$, $f_kg_1 = \sum_n(f_{kn}g_1)$ and $g_kf_1 = \sum_n(g_{kn}f_1)$ are almost everywhere continuous derivatives and

$$\begin{aligned} \| f_k \| &\leq \sup\{\| f_{kn} \|: n \in \mathbb{N}\} \leq 100(c_{kn} \vee \sqrt{c_{kn}}) \\ &= 100(\| u_k - \varphi_k \| \vee \sqrt{\| u_k - \varphi_k \|}) \\ &\leq 200(\| u_k \| \vee \sqrt{\| u_k \|}) \leq 2^{8-k/2}, \\ \| g_k \| &\leq \sup\{\| g_{kn} \|: n \in \mathbb{N}\} \leq \sqrt{2}(c_{kn} \vee \sqrt{c_{kn}}) \\ &= \sqrt{2}(\| u_k - \varphi_k \| \vee \sqrt{\| u_k - \varphi_k \|}) \\ &\leq 3(\| u_k \| \vee \sqrt{\| u_k \|}) \leq 2^{2-k/2}, \\ \| h_k \| &\leq \sup\{\| h_{kn} \|: n \in \mathbb{N}\} \leq 51C_{kn} \leq 102c_{kn} \\ &= 102 \| u_k - \varphi_k \| \leq 2^{7-k}. \end{aligned}$$

Now using the uniform convergence of all the rows below we get

$$\begin{aligned} u &= \sum_k u_k = \sum_k (f_k g_k + h_k + \varphi_k + \psi_k) \\ &= (\sum_k f_k)(\sum_k g_k) - \sum_k (f_k \bar{g}_k) - \sum_k (g_k \bar{f}_k) - f_1 \sum_{k=2}^{\infty} g_k - \sum_{k=2}^{\infty} (f_k g_1) \\ &+ \sum_k h_k + \sum_k \varphi_k + \sum_k \psi_k. \end{aligned}$$

Put $h = \sum_k h_k + \sum_k \varphi_k + \sum_k \psi_k - \sum_k (f_k \bar{g}_k) - \sum_k (g_k \bar{f}_k) - f_1 \sum_{k=2} g_k - g_1 \sum_{k=2} f_k$, $f = \sum_k f_k$ and $g = \sum_k g_k$. The functions $\sum_k f_k$, $\sum_k g_k$, $\sum_k (f_k \bar{g}_k)$, $\sum_k (g_k \bar{f}_k)$, $\sum_{k=2}^{\infty} (f_k g_1)$, $\sum_k h_k$, $\sum_k \varphi_k$ and $\sum_k \psi_k$ are almost everywhere continuous derivatives (They are limits of uniformly convergent rows of such functions.) so we need only show that $f_1 \sum_{k=2}^{\infty} g_k$ is an almost everywhere continuous derivative to complete the proof. We will use 12. v) with $A = A_1$, $B = B_1$, $c_n = c_{1n}$ for the sequence $\{f_{1n} \sum_{k=2}^{\infty} g_k : n \in \mathbb{N}\}$. We check the assumptions:

1) f_{1n} is continuous everywhere and $\sum_{k=2}^{\infty} g_k$ is a bounded derivative. So $f_{1n} \sum_{k=2}^{\infty} g_k$ is a derivative and $| D(f_{1n} \sum_{k=2}^{\infty} g_k) | \leq | \bigcup_{k=2}^{\infty} A_k | = 0$ $(n = 1, 2, \ldots)$,

2) if $x \notin G_{1n}$, then $(f_{1n} \sum_{k=2}^{\infty} g_k)(x) = f_{1n}(x) = 0$,

3) for each $n \in \mathbb{N}$ $\int_{\mathbb{R}} (f_{1n} \sum_{k=2}^{\infty} g_k) = \sum_{k=2}^{\infty} \int_{\mathbb{R}} (f_{1n} g_k) = \sum_{k=2}^{\infty} \sum_l \int_{G_{1n}} (f_{1n} g_{kl}) = 0$, because for each $k \geq 2$ and each $l \in \mathbb{N}$ we have either $G_{kl} \subset G_{1n}$ or $G_{kl} \subset G_{1n}^c$. So $\int_{\mathbb{R}} (f_{1n} g_{kl}) = 0$ either by 14. iii) (the way we have chosen g_{kl}) or by the previous condition,

4) $\| f_{1n} \sum_{k=2}^{\infty} g_k \| \leq \| f_{1n} \| \sum_{k=2}^{\infty} \| g_k \| \leq \| f_{1n} \| \leq 700(c_{1n} \vee \sqrt{c_{1n}})$.

All the assumptions of 12. v) are met. So $f_1 \sum_{k=2}^{\infty} g_k = \sum_n (f_{1n} \sum_{k=2}^{\infty} g_k)$ is a derivative and $|D(f_1 \sum_{k=2}^{\infty} g_k)| \leq |\bigcup_{k=1}^{\infty} A_k| = 0$. Hence f, g and h are almost everywhere continuous derivatives and $u = fg + h$.

In case u is bounded we can also take functions $u_k, \varphi_k, \psi_k, f_k, g_k$ and h_k to be bounded $(k = 1, 2, \ldots)$. Proceeding in the above way we easily prove that the functions f, g and h are also bounded, just as we claim.

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