Real Analysis Exchange Vol 15 (1989-90)

Vasile Ene, Institute of Mathematics, str. Academiei 14, 70109 Bucharest, Romania

Finite Representation of Continuous Functions, Nina Bary's Wrinkled Functions and Foran's Condition M.

In [1](pp. 222;229;237;611), Nina Bary shows the following chain of inclusions: quasi-derivable $\subsetneqq S + S \subsetneqq \mathcal{C} = S + S + S$, for continuous functions on $[0,1]$.

It can be shown that above, Banach's condition S can be replaced by $GE(1) \cap T_1 \subsetneqq S$, where $GE(1)$ is defined using condition $E(1)$ of [6].

In our paper we define conditions $GAC_*D_1^{\#} \subsetneqq GAC_*D_1 \subset GE(1) \cap T_1$ for continuous functions on $[0,1]$, with which we improve the above results. (Following Nina Bary's proof of [1],p.222, conditions $GAC_*D_1^{\#}$ and GAC_*D_1 are very natural.)

To prove that $S + S \neq \mathcal{C}$, Nina Bary introduced the wrinkled functions W (she called them "fonctions ridées", [1],p.236) and showed that $W \neq \emptyset$ (see the example of [1],pp.241-248; see also [5] or [3]) and $W \cap (N +$ quasi-derivable$) = \emptyset$ (see [1],p.237), hence $W \cap (N + N) = \emptyset$, for continuous functions on $[0,1]$.

In our paper we give characterizations of the wrinkled functions which show that between Foran's condition M (introduced in 1979 in [8]) and these functions there is a very close relationship. So we improve Nina Bary's results on wrinkled functions. Finally we construct a wrinkled function which is approximately derivable at no point of $[0,1]$ and for which each level set is perfect.

Let $\mathscr{C} = \{F : F$ is continuous$\}$. Banach's conditions T_1, T_2, S, Lusin's condition N and conditions VB_*, VB, AC_*, AC, VBG_*, VBG, ACG_*, ACG are defined in [13]; E(N) and $\mathscr{E}$ in [6].

<u>Definition 1.</u>([1],p.236). Let Q be a measurable real set and let $f:Q \longrightarrow R$. f is a wrinkled function, $f \in W$, if for every measurable subset $A \subset Q$, $|A| > 0$, f is monotone on some $B \subset A$, where $|B| = 0$, f(B) is measurable and $|f(B)| > 0$. (Without loss of generality A may be supposed to be perfect, since a measurable set is the union of a F_σ-set and a null set.)

<u>Definition 2.</u>([1],p.178). A continuous function $f:[0,1] \longrightarrow R$ is quasi-derivable if on each interval f'(x) exists and is finite at every point x of a set which has positive measure.

<u>Definition 3.</u>([8]). A continuous function fulfils Foran's condition M if it is AC on any set on which it is VB.

<u>Definition 4.</u>([12],p.406). A function f is D_1 (resp. D_1^*) on a set E if for every $\varepsilon > 0$ there exists a sequence $\{I_i\}$ of nonoverlapping closed (resp. of open) intervals which covers E and $\sum_i O(f;E \cap I_i) < \varepsilon$ (resp. $\sum_i O(f;I_i) < \varepsilon$).

A function f is $E_1(1,1)$ on E if $f \in D_1$ on Z, whenever $Z \subset E$, $|Z| = 0$.

<u>Remark 1.</u> a) In [12], Lee calls condition D_1, $D_1(1)$ and he shows that $E_1(1,1)$ and E(1) (see [6]) are equivalent (see [12], Remark 14,p.416). Another condition which is equivalent with E(1) is given by Iseki (see [12],pp.415-416).
b) Clearly $D_1^* \subset D_1$.

<u>Definition 5.</u>([12],p.416). For a function property P (resp. for function properties P_1 and P_2) on sets we say that a function

f is generalized P (resp. generalized P_1P_2) on E, writing $f \in GP$ (resp. $f \in GP_1P_2$) on E, if E can be written as the union of countably many sets on each of which f is P (resp. f is P_1 or f is P_2). Thus we have properties like GD_1^* ; GD_1 ; $GAC_*D_1^*$; GAC_*D_1; GE(1).

Remark 2. a) $GD_1^* = D_1^*$ on a set.

b) If $f \in D_1$ on a set E then $|f(E)| = 0$ and $f \in E(1)$ on E. Hence, if $f \in GD_1$ on E then $|f(E)| = 0$ and $f \in GE(1)$ on E.

c) If f is a Darboux function and $f \in GD_1$ on an interval then f is a constant.

d) Let f be a nonconstant continuous function on $[0,1]$. If A is a countable dense subset of $[0,1]$ then $f \notin D_1^*$ on $[0,1]$ and $f \in D_1^*$ on A.

e) $\mathcal{C} \cap GAC_*D_1 \subset T_1$ on an interval (see [13], Theorem 7.2, p.230, Theorem 6.2, p.278 and Remark 2,b)).

Remark 3. For continuous functions on $[0,1]$, we have:

$$\mathcal{E} \overset{(1)}{\subsetneq} N \overset{(2)}{\subsetneq} M \overset{(3)}{\subsetneq} \text{quasi-derivable} \overset{(4)}{\subsetneq} S{+}S \overset{(5)}{\subsetneq} \text{quasi-derivable} + \text{quasi-derivable} \overset{(6)}{=} \mathcal{C} \overset{(7)}{=} S{+}S{+}S.$$

Proof. For (1) see [6],p.208; for (2) see [8],p.84 ; for (3) see [8],p.87; for (4) see [1],p.222,p.229; for (6) see [1],p.599, hence (5) follows by (6) and [1],p.237; for (7) see [1],p.611.

Proposition 1. For continuous functions on $[0,1]$ we have:

$$GAC_*D_1^* \overset{(1)}{\subsetneq} GAC_*D_1 \overset{(2)}{\subset} GE(1)\cap T_1 \overset{(3)}{\subset} \mathcal{E} \cap T_1 \overset{(4)}{\subsetneq} S.$$

Proof. For (3), see the definitions and for (4) see [6],p. 208. Clearly $GAC_*D_1^* \subset GAC_*D_1 \subset GE(1)\cap T_1$ (see Remark 2,e)). It remains to show that (1) is strict. Let C be the Cantor ternary set and let φ be the Cantor ternary function. Let $\{I_n^k\}$, $n = 1,2,\ldots,2^{k-1}$ be the open intervals excluded at the step k in the

Cantor ternary process. Let c_n^k be the middle point of I_n^k. Let $f: [0,1] \longrightarrow R$, $f(x) = 0$, $x \in C$; $f(c_n^k) = 1/2^k$. Extending f linearly, we have f defined and continuous on $[0,1]$. Clearly $f \in GAC_* D_1$ on $[0,1]$, and $f \in ACG_*$ on $[0,1] - C$. Suppose that $f \in GAC_* D_1^*$ on C. Then there exists a sequence of sets $\{E_n\}$ such that $C = \bigcup E_n$ and either $f \in AC_*$ on E_n or $f \in D_1^*$ on E_n. Let p be a natural number such that f is AC_* on E_p. Since $f \in \mathcal{C}$ it follows that f is AC_* on $\overline{E}_p$. We prove that $f \in D_1^*$ on $\overline{E}_p$. Let $\varepsilon > 0$ and let δ be given by the fact that $f \in AC_*$ on $\overline{E}_p$. Since $f \in \mathcal{C}$ and $|\overline{E}_p| = 0$ we can cover E_p with a sequence of nonoverlapping intervals $\{I_n\}$ such that $\sum |I_n| < \delta$ and $\sum O(f; I_n) < \varepsilon$. Hence $f \in D_1^*$ on $\overline{E}_p$. It follows that $f \in GD_1^*$ on C, hence $f \in D_1^*$ on C. We show that $f \notin D_1^*$ on C. Let $C \subset \bigcup_{i=1} (a_i, b_i)$. For each i let J_i be the greatest excluded open interval (in the Cantor ternary process) contained in $[a_i', b_i']$, where $a_i' = \inf((a_i, b_i) \cap C)$ and $b_i' = \sup((a_i, b_i) \cap C)$. Suppose that J_i is excluded at the step k. Then

$$J_i = \left(\sum_{i=1}^{k-1} c_i/3^i + \sum_{i=k+1}^{\infty} 2/3^i, \; \sum_{i=1}^{k-1} c_i/3^i + 2/3^k\right). \text{ Let}$$

$$J_i' = \left[\sum_{i=1}^{k-1} c_i/3^i, \; \sum_{i=1}^{k-1} c_i/3^i + \sum_{i=k}^{\infty} 2/3^i\right]. \text{ Then } [a_i', b_i'] \subset J_i',$$

hence $C \subset \bigcup J_i'$. We have $O(f; J_i) = O(f; J_i') = |\varphi(J_i')| = 1/2^k$,

$[0,1] = \varphi(C) \subset \bigcup \varphi(J_i')$, hence $\sum_{i=1}^{\infty} O(f; (a_i b_i)) \geqslant \sum_{i=1}^{\infty} O(f; J_i) = \sum_{i=1}^{\infty} |\varphi(J_i')| \geqslant 1$ and $f \notin D_1^*$ on C.

<u>Theorem 1.</u> <u>Let</u> $F: [0,1] \longrightarrow R$, $F \in \mathcal{C} \cap$ quasi-derivable. <u>Let</u> $a > 0$, $\emptyset \subsetneq P \subset [0,1]$ <u>be a perfect nowhere dense set and let</u> $D = \{x \in [0,1] - P : F$ <u>is derivable at</u> $x\}$. <u>Then there exist a set</u> Q <u>of</u>

F_σ-*type*, $Q \subset D$, $|Q| = |D|$ *and two continuous functions* f_1 *and* f_2 *such that*: a) $F(x) = f_1(x) + f_2(x)$ *on* $[0,1]$; b) $f_1, f_2 \in D_1^*$ *on* $Q^c = [0,1] - Q$; c) $f_1, f_2 \in ACG_*$ *on* Q ; d) $|f_2(x)| < 3a$ *on* $[0,1]$ *and* $f_2(0) = f_2(1) = 0$.

Proof. Let P_1 be a perfect nowhere dense subset of D, $|P_1| > 0$. We shall construct a strictly increasing sequence $\{P_k\}$, $k = 2,3,\ldots$, of nowhere dense perfect subsets of D such that $P_k - P_{k-1}$ is a nowhere dense subset of positive measure in each $\bar{I}_n^{k-1}$ and $|Q| = |D|$, where I_n^k are the intervals contiguous to P_k, $k = 2,3,\ldots$ and $Q = \bigcup_{k=1}^{\infty} P_k$. Let $g_1 : [0,1] \longrightarrow R$ be a continuous function such that: $g_1(x) = F(x)$ on P_1; g_1 is a bounded derived number on each I_n^1; g_1 is constant on each I_n^2 ; $|h_1(x)| < a/2^{n+1}$ on each I_n^1 , where $h_1(x) = F(x) - g_1(x)$. The existence of g_1 follows by [1] ,pp.222-224. Since $h_1 = 0$ on P, by [13] (Theorem 8.5,p.232), it follows that $h_1 \in AC_*$ on P_1. By [13] (Theorem 10.5,p.235), $F \in ACG_*$ on P_2. Clearly $g_1 \in AC_*$ on each I_n^1. Since $g_1 = F - h_1$ it follows that $g_1 \in AC_*$ on P_1, hence $g_1 \in ACG_*$ on $[0,1]$. Since $F \in ACG_*$ on P_2 it follows that $h_1 \in ACG_*$ on P_2. Since $h_1 - F$ is constant on each I_n^2, it follows that h_1 is derivable on $D - P_2$. Replacing F by h_1 we construct a continuous function g_2, analogously to the construction of g_1, such that $g_2 = h_1$ on P_2, g_2 is ACG_* on $[0,1]$, g_2 is constant on each interval I_n^3 and $|h_2(x)| < a/2^{n+2}$ on each I_n^2 , where $h_2(x) = h_1(x) - g_2(x)$. Then $h_2(x) = 0$ on P_2; h_2 is ACG_* on P_3; h_2 is derivable on $D - P_3$; $|h_2(x)| < a/2^2$ on $[0,1]$. Continuing in this way we obtain two sequences of continuous functions $\{g_i\}$, $\{h_i\}$, $i = 2,3,\ldots$ such that:

A. $g_i = h_{i-1}$ on P_i; $g_i \in ACG_*$ on $[0,1]$; g_i is constant on each

I_n^{i+1} ;

B. $h_i = 0$ on P_i; $|h_i(x)| < a/2^{n+i}$ on I_n^i; $h_i \in ACG_*$ on P_{i+1} ;

$h_i = h_{i-1} - g_i$ is derivable on $D - P_{i+1}$.

Clearly

(1) $|h_i(x)| < a/2^i$ on $[0,1]$.

Then we have $F(x) = g_1(x) + \dots + g_m(x) + h_m(x)$, for each natural number m and by (1), $\sum_{i=1}^{\infty} g_i(x)$ converges uniformly to $F(x)$. Let

$F_1(x) = \sum_{i=1}^{\infty} g_{2i-1}(x)$; $F_2(x) = \sum_{i=1}^{\infty} g_{2i}(x)$; $R_m(x) = \sum_{i=m}^{\infty} (h_i(x) - h_{i+1}(x))$. Then $F_1, F_2 \in \mathcal{C}$ on $[0,1]$ and $F(x) = F_1(x) + F_2(x)$.

We have

(2) $F_1(x) = \sum_{i=1}^{k} g_{2i-1}(x) + R_{2k}(x)$;

(3) $F_2(x) = \sum_{i=1}^{k} g_{2i}(x) + R_{2k+1}(x)$;

(4) $R_i(x) = 0$ on P_i ;

(5) $\sum_{n=1}^{\infty} O(R_i; I_n^i) < a/2^{i-2}$.

By (4), (5) and [13] (Theorem 8.5, p.232) it follows that

(6) $R_i(x)$ is AC_* on P_i .

Since $\sum_{i=1}^{k} g_{2i-1}(x)$ is constant on each I_n^{2k}, it follows that

(7) $O(F_1; I_n^{2k}) = O(R_{2k}; I_n^{2k})$.

By A., (2) and (6), it follows that F_1 is ACG_* on P_{2k}. Hence F_1 is ACG_* on $Q = \bigcup_{i=1}^{\infty} P_i$. Analogously F_2 is ACG_* on Q. Moreover,

(8) $|F_2(x)| = |R_1(x)| < \sum_{i=1}^{\infty} |h_i(x)| < a$.

Let $\varepsilon > 0$ and let k be a natural number such that $a/2^{2k} < \varepsilon$.

Then $Q^c \subset \bigcup_{n=1}^{\infty} I_n^{2k+2}$ and by (5) and (7) it follows that

$\sum_{n=1}^{\infty} O(F_1; I_n^{2k+2}) < a/2^{2k} < \varepsilon$, hence F_1 is D_1^* on Q^c. Analogously, F_2 is D_1^* on Q^c. Therefore we obtain: $F = F_1 + F_2$ on $[0,1]$; F_1, F_2 are D_1^* on Q^c; F_1, F_2 are ACG_* on Q; $|F_2(x)| < a$ on $[0,1]$. Let $H(x) = F_2(0) + ((F_2(1) - F_2(0))/|P_1|) \int_0^x K_{P_1}(t)dt$, where K_{P_1} is the characteristic function of P_1. Clearly H is AC on $[0,1]$; $|H(x)| < 2a$; H is constant on each I_n^1. Let $f_2 = F_2 - H$ and $f_1 = F_1 + H$ on $[0,1]$.

__Corollary 1. Let__ $a > 0$ __and let__ P __be a perfect nowhere dense subset of__ $[0,1]$. __Let__ $F: [0,1] \longrightarrow R$, $F \in \mathcal{C}$. __Then there exist two continuous functions__ F_1, F_2 __on__ $[0,1]$ __such that:__ a) $F = F_1 + F_2$ __on__ $[0,1]$; b) $F_2 \in D_1^*$ __on__ P __and__ $F_1 \in D_1$ __on__ P; c) $F_2(0) = F_2(1) = 0$; d) $|F_2(x)| < a$ __on__ $[0,1]$.

__Proof.__ Let $f: [0,1] \longrightarrow R$ be a continuous function defined as follows: $f(x) = F(x)$, $x \in P \cup \{0,1\}$; f is linear on the closure of each interval contiguous to $P \subset [0,1]$. By Theorem 1, $f = f_1 + f_2$ on $[0,1]$; $f_1, f_2 \in \mathcal{C}$ on $[0,1]$; $f_1, f_2 \in D_1^*$ on P; $f_2(0) = f_2(1) = 0$; $|f_2(x)| < a$ on $[0,1]$. Let $F_2 = f_2$ and $F_1 = F - f_2$. Then $F_1 = f_1$ on P, hence $F_1 \in D_1$ on P.

__Corollary 2. Let__ $a > 0$ __and let__ P __be a perfect nowhere dense subset of__ $[0,1]$. __Let__ $F: [0,1] \longrightarrow R$, $F \in \mathcal{C}$. __Then there exists a perfect nowhere dense set__ $Q \supset P$ __such that__ $Q - P$ __is a perfect nowhere dense set of positive measure in each interval contiguous to__ P __and there exist two continuous functions__ F_1, F_2 __on__ $[0,1]$ __such__ that: a) $F = F_1 + F_2$ on $[0,1]$; b) $F_1, F_2 \in D_1$ on P; F_1 is AC on each interval contiguous to P; F_1 is constant on each interval contiguous to Q; c) $F_2(0) = F_2(1) = 0$; d) $|F_2(x)| < a$ on $[0,1]$.

Proof. By Corollary 1, for $a/2$, there exist two continuous functions f_1, f_2 on $[0,1]$ such that: a') $F = f_1 + f_2$ on $[0,1]$; b') $f_2 \in D_1^{\#}$ on P and $f_1 \in D_1$ on P; c') $f_2(0) = f_2(1) = 0$; d') $|f_2(x)| < a/2$ on $[0,1]$. Let $\{I_n\}$ be the intervals contiguous to P with respect to $[0,1]$. For each natural number n there exists a perfect nowhere dense subset Q_n of $\bar{I}_n$, $|Q_n| > 0$ and a continuous function F_1 on $[0,1]$ such that: $F_1(x) = f_1(x)$ on P; $F_1(0) = F(0)$; $F_1(1) = F(1)$; $F_1 \in AC$ on each I_n; F_1 is constant on each interval contiguous to Q_n with respect to $\bar{I}_n$; $|F_2(x)| < a$ on $[0,1]$, where $F_2(x) = F(x) - F_1(x)$. These follow by [1], p.609-611 (if we put $I_n = \delta_n$; $f_1 = \bar{\varphi}$; $f_2 = \bar{\Psi}$; $F_1 = \varphi$; $F_2 = \Psi$; $a/2 = \varepsilon/2$). By a') and c') it follows that $F_1(x) = f_1(x)$ on $P \cup \{0,1\}$, hence $F_2(x) = f_2(x)$ on $P \cup \{0,1\}$. It follows that $F_2(0) = F_2(1) = 0$ and by b'), $F_1, F_2 \in D_1$ on P.

Remark 4. a) Theorem 1 extends Nina Bary's theorem of [1], p.222 and Corollary 1 extends Nina Bary's theorem of [1], p.603 (instead of condition "$|F_1(P)| = |F_2(P)| = 0$" we put condition b)). b) Corollary 2 is an extension of Nina Bary's Corollary of [1], p.609.

Remark 5. For continuous functions on $[0,1]$ we have: quasi-derivable $\subseteq GAC_*D_1^{\#} + GAC_*D_1^{\#} \subseteq GAC_*D_1 + GAC_*D_1 \subseteq S + S$ (see Theorem 1 and Proposition 1). We don't know if the inclusions are strict.

Proposition 2. For continuous functions on $[0,1]$ we have: quasi-derivable $\subsetneqq GAC_*D_1 + GAC_*D_1$.

Proof. The inclusion follows by Remark 5. To prove that the inclusion is strict we shall construct two continuous functions $f,g:[0,1]\longrightarrow[0,1]$ such that $f,g\in GAC_*D_1$ and $f+g$ is not derivable a.e. on $[0,1]$. Then $f+g$ is not quasi-derivable on $[0,1]$. Let P_1 be a perfect nowhere dense subset of $[0,1]$, symmetrical with respect to 1/2 such that $0,1\in P_1$ and $|P_1| = 1/2$. We shall construct a strictly increasing sequence of subsets of $[0,1]$, P_k, $k\geq 2$. Let $I_n^k = (a_n^k,b_n^k)$ be the intervals contiguous to P_k with respect to $[0,1]$. Suppose that $P_1,\dots,P_k$ have been defined and let's define P_{k+1}. Let P_n^k be a perfect nowhere dense subset of $\bar{I}_n^k$ such that $a_n^k,b_n^k\in P_n^k$, $|P_n^k| = |I_n^k|/2$ and P_n^k is symmetrical with respect to $c_n^k = (a_n^k + b_n^k)/2$. Then $P_{k+1} = P_k\cup(\bigcup_{n=1}^{\infty}P_n^k) = P_k\cup S^{k+1}$. By [1],pp. 229-230 (for $\delta = I_n^k$, $Q = P_n^k$ and $\varepsilon = 1/2^{n+k}$), there exists a continuous function $f_k:[0,1]\longrightarrow[0,\ 1/2^k]$ with the following properties: $f_k(x) = 0$ on P_k; f_k is ACG on each I_n^k; $|f_k(x)| < 1/2^{n+k}$ on I_n^k; $f_k(x) = 0$ on P_n^k; f_k is AC on I_n^{k+1}; f_k is constant on each I_n^{k+2}; f_k is not derivable on P_n^k. Let $A = P_1\cup(\bigcup_{i=1}^{\infty}S^{2i+1})$; $B = \bigcup_{i=1}^{\infty}S^{2i}$; $E = [0,1]-(A\cup B) = [0,1] - (\bigcup_{i=1}^{\infty}P_i)$. Then $|A\cup B| = 1$.

Let $f(x) = \sum_{i=1}^{\infty} f_{2i-1}(x)$ and $g(x) = \sum_{i=1}^{\infty}f_{2i}(x)$. Clearly f, g and $f+g$ are continuous on $[0,1]$. It follows that: a) f is ACG_* on A and f is not derivable a.e. on B; b) g is ACG_* on B and g is not derivable a.e. on A; c) $f\in GD_1$ on B; $f\in D_1^*$ on E; $g\in GD_1$ on A; $g\in D_1^*$ on E. By a) and b), $f+g$ is not derivable a.e. on $[0,1]$. By a) and c), $f\in GAC_*D_1$ on $[0,1]$ and by b) and c), $g\in GAC_*D_1$ on $[0,1]$. We prove only the part with f. Let $R_{2k+1}(x) = \sum_{i=k}^{\infty} f_{2i+1}(x)$. Then

$f(x) = \sum_{i=1}^{k} f_{2i-1}(x) + R_{2k+1}(x)$. We have

(9) $R_{2k+1}(x) = 0$ on P_{2k+1} and

(10) $\sum_{n=1} O(R_{2k+1}; I_n^{2k+1}) < 1/2^{2k-1}$,

hence, by [13] (Theorem 8.5, p.232) ,

(11) R_{2k+1} is AC_* on P_{2k+1}.

Since $\sum_{i=1}^{k} f_{2i-1}(x)$ is constant on each I_n^{2k+1}, it follows that

(12) $O(f; I_n^{2k+1}) = O(R_{2k+1}; I_n^{2k+1})$.

By (9), it follows that

(13) $f(x) = \sum_{i=1}^{k} f_{2i-1}(x)$ on P_{2k+1}.

a) Since $f(x) = R_1(x)$ on $[0,1]$, $f \in AC_*$ on P_1. Since $f_1, f_3, \ldots, f_{2k-1}$ are AC_* on each I_n^{2k}, it follows that $\sum_{i=1}^{k} f_{2i-1}(x)$ is AC_* on each P_n^{2k}. By (11), f is AC_* on each P_n^{2k}. Therefore f is ACG_* on S^{2k+1}. Since $\sum_{i=1}^{k-1} f_{2i-1}(x)$ is constant on each I_n^{2k-1}, f_{2k-1} is not derivable on P_n^{2k-1} and R_{2k+1} is derivable a.e. on P_{2k+1}. It follows that f is not derivable a.e. on S^{2k}.

c) Let $\varepsilon > 0$ and let k be a natural number such that $1/2^{2k-1} < \varepsilon$.

Since $E \subset \bigcup_{n=1}^{\infty} I_n^{2k+1}$, by (10) and (12), it follows that

$\sum_{n=1}^{\infty} O(f; I_n^{2k+1}) < \varepsilon$, hence $f \in D_1^*$ on E. Since $\sum_{i=1}^{k-1} f_{2i-1}(x)$ is constant on each I_n^{2k+1} and $f_{2k-1}(x) = 0$ on each P_n^{2k-1}, it follows that f is constant on each P_n^{2k-1}, hence $f \in D_1$ on S^{2k}. Thus $f \in GD_1$ on B.

Using Corollary 2 instead of Nina Bary's Corollary [1],p. 609, the following theorem can be proved:

Theorem 2. (An extension of the theorem of [1],p.611). $\mathcal{C} = GAC_*D_1 + GAC_*D_1 + GAC_*D_1$ for continuous functions on $[0,1]$.

Remark 6. We don't know if Theorem 2 remains true if GAC_*D_1 is replaced by $GAC_*D_1^*$.

Theorem 3. (An extension of the theorem of [1],p.237). Let P be a perfect set, $P \subset [0,1]$, $|P| > 0$ and let $F: P \longrightarrow [0,1]$. If $F \in W \cap \mathcal{C}$ then F cannot be written as the sum of two continuous functions F_1 and F_2 such that $F_1 \in M$ and F_2 is approximately differentiable on a set of positive measure. Hence $W \cap (M + M) = \emptyset$ for continuous functions on $[0,1]$ (see Remark 3).

Proof. Suppose that there exists a set $E \subset P$ of positive measure such that F_2 is approximately differentiable on E. By [13](Theorem 10.14,p.239) F_2 is ACG on E. Since $F \in W$ there exists $E_1 \subset E$, $|E_1| = 0$ such that $F(E_1)$ is measurable, $|F(E_1)| > 0$ and F is monotone on E_1. Then $F_1 = F - F_2$ is VBG on E_1. Since $F_1 \in M$, $F_1 \in$ ACG on E_1. Hence $F_1 + F_2$ is ACG on E_1 and $|F(E_1)| = 0$, a contradiction.

Remark 7. If P is a perfect nowhere dense set of positive measure, $P \subseteq [0,1]$ then there exists a continuous function in D_1^* on P which is approximately differentiable on no set of positive measure (see Theorem 1 and Theorem 3).

Corollary 3. Let $F: P \subseteq [0,1] \longrightarrow [0,1]$, where $F \in W \cap \mathcal{C}$ and P is a measurable set of positive measure. Then F is approximately differentiable on no set of positive measure.

Theorem 4. a) Let $F:[0,1]\longrightarrow R$, $F\in\mathcal{C}$ and let $P\subseteq[0,1]$ be a measurable set, $|P|>0$. Then $F\in W$ on P if and only if $F\in M$ on no closed subset Q of P, $|Q|>0$.

b) Let $f:S\longrightarrow R$, $f\in\mathcal{C}$, where $S\subset[0,1]$ is a measurable set. Then $f\in W$ if and only if for every subset $A\subset S$ of positive measure, f is strictly monotone on some perfect subset $B\subset A$ such that $|f(B)|>0$ and $|B|=0$.

Proof. a) "$\Longrightarrow$" Let Q be a closed subset of positive measure of P. Since $F\in W$ on P it follows that there exists $Q_1\subset Q$, $|Q_1|=0$ such that F is monotone on Q_1, $F(Q_1)$ is measurable and $|F(Q_1)|>0$, hence $F\notin M$ on Q.

"$\Longleftarrow$" Let A be a perfect subset of P of positive measure. Since $F\notin M$ on A, by Theorem 1 of [8](p.83), it follows that there exists $B\subset A$ such that F is monotone on B and $F\notin AC$ on B. Since $F\in\mathcal{C}$, F is monotone on $\overline{B}$. We prove that $|\overline{B}|=0$, hence $F(\overline{B})$ is measurable and $|F(\overline{B})|>0$. Suppose that $|F(\overline{B})|=0$ then $F\in\mathcal{C}\cap VB\cap N$ on $\overline{B}$ and by Theorem 6.7 of [13](p.227), $F\in AC$ on $\overline{B}$, hence $F\in AC$ on B, a contradiction. Suppose that $|\overline{B}|>0$ then F is approximately differentiable on a measurable set $E\subset B$, $|E|=|\overline{B}|$, hence $F\in ACG$ on E. It follows that there exists a closed set Q, $Q\subset E$, $|Q|>0$ such that $F\in ACG\subset M$ on Q, a contradiction.

b) "$\Longrightarrow$" is evident.

"$\Longleftarrow$" Let A be a perfect subset of S, $|A|>0$. Since $f\in W$, there exists $B_1\subset A$, $|B_1|=0$, $f_{|B_1}$ is monotone, $f(B_1)$ is measurable and $|f(B_1)|>0$. We prove that $|\overline{B}_1|=0$. Suppose that $|\overline{B}_1|>0$. Since $f\in\mathcal{C}$, $f_{|\overline{B}_1}$ is monotone. Hence f is approximately derivable a.e. on $\overline{B}_1$, a contradiction (see Corollary 3). Let $C=\{y\in f(\overline{B}_1):$ $f^{-1}(y)\cap\overline{B}_1$ contains more than one point$\}$. Then C is countable.

Suppose $C = \{y_1, y_2, \dots\}$. Let $\varepsilon < (|f(B_1)|)/4$, $a_n = \inf(\overline{B}_1 \cap f^{-1}(y_n))$, $b_n = \sup(\overline{B}_1 \cap f^{-1}(y_n))$. Since $f \in \mathcal{C}$ it follows that there exist $\delta_n > 0$ such that $f(\overline{B}_1 \cap (a_n - \delta_n, b_n + \delta_n)) \subset (y_n - \varepsilon/2^{n+1}, y_n + \varepsilon/2^{n+1})$. Let $G = \bigcup_n (a_n - \delta_n, b_n + \delta_n)$. Hence $|f(\overline{B}_1 \cap G)| < \varepsilon$. Let B be the set of points of accumulation of the closed set $\overline{B}_1 - G$. Then B is a perfect subset of A, $|B| = 0$, $f_{|B}$ is strictly monotone, $f(B)$ is a compact set (since $f \in \mathcal{C}$) and $|f(B)| > (3/4)|f(B_1)| > 0$.

Lemma 1. Let A be a perfect subset of $[0,1]$, $|A| > 0$ and let $f: A \to R$, $f \in \mathcal{C}$. Let $E = \{x \in A : f$ is approximately differentiable at x and $f'_{ap}(x) > 0\}$. If E has positive measure then there exists a perfect subset B of E, $|B| = 0$, such that $f_{|B}$ is strictly increasing.

Proof. That E is measurable follows by [13], p.299. Let $E_n = \{x \in E : 0 < h < 1/n$ implies $|\{t : 1/n \le (f(t)-f(x))/(t-x),\ 0 < |t-x| < h\}| > (3/4)\cdot 2h\}$. Let $E_{in} = E_n \cap [i/n, (i+1)/n]$ for each natural number i. Then $E = \bigcup_i \bigcup_n E_{in}$. Let p, j such that $|E_{pj}| > 0$. If $x < y$, $x, y \in E_{pj}$ then $f(y)-f(x) \geq (1/p)(y-x)$. Since $f \in \mathcal{C}$ it follows that $f(y)-f(x) \geq (1/p)(y-x)$, for $x < y$, $x, y \in \overline{E}_{pj}$. Let B be a perfect subset of positive measure of $\overline{E}_{pj}$. Then f is strictly increasing on B.

Theorem 5. Let P be a perfect subset of $[0,1]$, $|P| > 0$. Let $F: P \to [0,1]$, $F \in W \cap \mathcal{C}$; $g: P \to Q$, $g(P) = Q$; $g \in \mathcal{C}$; $f: Q \to [0,1]$ and $D_{ap} = \{x \in Q : f$ is approximately differentiable at $x\}$. If $F = f \circ g$, $f \in \mathcal{C}$ and $|f(Q - D_{ap})| = 0$ then $g \in W$.

Proof. The proof is similar with that of [1] (Theorem of p. 238), using Lemma 1 instead of the lemma of [1], p.239.

Let A be a perfect subset of P, $|A| > 0$. Since $F \in W$, by Theorem 4,b), there exists a perfect subset B of A, $|B| = 0$, such that $F|_B$ is strictly monotone and $|F(B)| > 0$. Let $B' = g(B)$. By [13] (Theorem 10.14,p.239), it follows that $f \in ACG$ on D_{ap}. Since $|f(Q - D_{ap})| = 0$ it follows that $f \in N$ on Q, hence $|B'| > 0$. Indeed, if $|B'| = 0$ then $|F(B)| = |f(B')| = 0$, a contradiction. Let $D_o = \{x \in Q : f'_{ap}(x) = 0\}$. By [13](Lemma 9.2,p.290), it follows that $|f(D_o)| = 0$, hence $B' - D_o$ is measurable and $|B' - D_o| > 0$ (if $|B' - D_o| = 0$ then $|F(B)| = 0$, a contradiction). It follows that B' contains a subset $E \subset D_{ap}$ of positive measure where f'_{ap} does not change the sign. Suppose that $f'_{ap}(x) > 0$, for each $x \in E$. By Lemma 1 there exists a perfect subset C' of E, $|C'| > 0$, such that $f|_{C'}$ is strictly increasing. Let $C = g^{-1}(C')$. Since F is strictly monotone on C, it follows that g is strictly monotone on C, $|C| = 0$ and $|g(C)| = |C'| > 0$, hence $g \in W$.

<u>Remark 8.</u> Theorem 5 is an extension of the theorem of [1], p.238 (there, $f \in AC$).

<u>Theorem 6.</u> <u>Let</u> $P \subseteq [0,1]$ <u>be a perfect set</u>, $|P| > 0$. <u>Let</u> $F: P \longrightarrow [0,1]$; $g: P \longrightarrow Q \subset [0,1]$, $Q = g(P)$; $|Q| > 0$; $f: Q \longrightarrow [0,1]$ <u>and let</u> $D_{ap} = \{x \in P : g$ <u>is approximately differentiable at</u> $x\}$. <u>If</u> $F = f \circ g$, $F, g, f \in \mathcal{C}$, $F \in W$ <u>and</u> $|g(P - D_{ap})| = 0$ <u>then</u> $f \in W$.

<u>Proof.</u> Let A be a perfect subset of Q, $|A| > 0$. Let $A_1 = g^{-1}(A)$, then A_1 is a closed subset of P. But g is ACG on P, hence g satisfies Lusin's condition N on P. It follows that $|A_1| > 0$. Since $|A| > 0$, $D_o = \{x \in P : g'_{ap}(x) = 0\}$, $|g(D_o)| = 0$ and $|g(P - D_{ap})| = 0$, it follows that $A_1 \cap D_{ap}$ contains a subset E of positive measure where f'_{ap} doesn't change the sign. Suppose that $f'_{ap}(x) > 0$ for all $x \in E$. By Lemma 1 it follows that there exists a perfect

subset C of E, $|C| > 0$ such that $g_{|C}$ is strictly increasing. $F \in W$ implies that there exists a perfect subset B of C such that $|B| = 0$, $F_{|B}$ is strictly monotone and $|F(B)| > 0$. Let $B_1 = g(B) \subset A$. Since g is ACG on D_{ap} it follows that $|B_1| = 0$. Since $g_{|B}$ is strictly increasing and $F_{|B}$ is strictly monotone it follows that $f_{|B_1}$ is strictly monotone and $|f(B_1)| = |f(g(B))| = |F(B)| > 0$, hence $f \in W$ on Q.

Definition 6. Let $F:[0,1] \longrightarrow R$, $F \in \mathcal{C}$. F is said to be W^* if for every subinterval I of $[0,1]$, there exists a perfect subset P of I, $|P| = 0$, $F_{|P}$ - monotone, such that $|F(P)| > 0$. Clearly $W \subset W^*$.

Remark 9. By Corollary 2 of [2](p.213), a typical continuous function $f:[0,1] \longrightarrow R$ does not have finite or infinite derivative at any point. By [8](Theorem 3,p.87), if f is a continuous function on $I \subset [0,1]$ and if $\{x \in I : f'(x) \text{ exists}\}$ has measure 0 then there is a perfect set P, $|P| = 0$ such that f is increasing on P and $|f(P)| > 0$. Hence a typical continuous function is W^*. Is W typical for continuous functions on $[0,1]$?

Remark 10. There exists a function $g \in W^* - W$. By [10] (Example 2,p.41), there exists a continuous function g defined on $[0,1]$ whose graph has σ - finite Hausdorff length and such that g is nowhere differentiable but has approximate derivative 0 almost everywhere (g will satisfy condition T_1). Since the set $E = \{x : g'(x) = +\infty\}$ has measure 0 (see [13], Theorem 4.4,p.270), by [8] (Theorem 3,p.87), it follows that $g \in W^*$. By Corollary 3, $g \notin W$.

Lemma 2. There exist a continuous function $F:[0,1] \longrightarrow [0,1]$ and a symmetric perfect nowhere dense subset Q of $[0,1]$, $0,1 \in Q$, $|Q| = 1/2$, such that: a) $F \in W$ on P; b) For each $y \in [0,1]$, $Q \cap F^{-1}(y)$ is a nonempty perfect subset of Q; c) $F_{|Q}$ has finite or infinite derivative at no point $x \in Q$; d) $F|_Q$ has finite approximate derivative at no point $x \in Q$; e) F is linear and strictly decreasing on each interval contiguous to Q.

Proof. We shall define the set Q. Let $a_{2i} = (1 - \sum_{k=2}^{i+1} 1/2^k)/4^i = 1/(2\cdot 4^i) + 1/(2\cdot 8^i)$, $i \geq 0$; $a_{2i-1} = 2a_{2i}$, $i \geq 1$, $c_i = a_{i-1} - a_i$, $i \geq 1$. Then $a_{2i} = c_{2i} = \sum_{k=2i+1}^{\infty} c_k$, $c_{2i-1} = 1/4^i + 3/8^i$, $i \geq 1$, $a_{2i-1} = \sum_{k=2i}^{\infty} c_k$. Let $Q = \{x :$ There exists $e_i(x)$ taking on 0 or 1 and $x = \sum e_i(x)c_i\}$. The open intervals deleted in the s-step of the construction of Q are $O_{e_1 \ldots e_{s-1}} = (\sum_{i=1}^{s-1} e_i c_i + a_s, \sum_{i=1}^{s-1} e_i c_i + c_s)$, $(e_1, \ldots e_{s-1}) \in \{0,1\}^{s-1}$. $O_{e_1 \ldots e_{s-1}} \neq \emptyset$ iff $s = 2p-1$, $p \geq 1$ and in this case $|O_{e_1 \ldots e_{2p-2}}| = 2/8^p$. The remaining intervals of the s-step are $R_{e_1 \ldots e_s} = [\sum_{i=1}^{s} e_i c_i , \sum_{i=1}^{s} e_i c_i + a_s]$, where $(e_1, \ldots, e_s) \in \{0,1\}^s$. Then $Q = \lim_{s \to \infty} 2^s a_s = 1/2$. Let $F(x) = \sum_{i=1}^{\infty} e_{2i}(x)/2^i$, $x \in Q$. Extending F linearly on each interval contiguous to Q we have F defined and continuous on $[0,1]$. We have:

$$(14) \quad F(R_{e_1 \ldots e_{2s}}) = F(Q \cap R_{e_1 \ldots e_{2s}}) = [\sum_{i=1}^{s} e_{2i}/2^i, \sum_{i=1}^{s} e_{2i}/2^i + 1/2^s] .$$

(See fig.1 for the representation of the first two steps in the construction of the graph of F.)

a) Let $a_i' = 1/2^{i+1} + 1/4^{i+1}$, $i \geq 0$, $c_i' = a_{i-1}' - a_i'$, $i \geq 1$, hence $c_i' = 1/2^{i+1} + 3/4^{i+1}$. Let $P = \{x :$ There exists $e_i(x)$ taking on 0

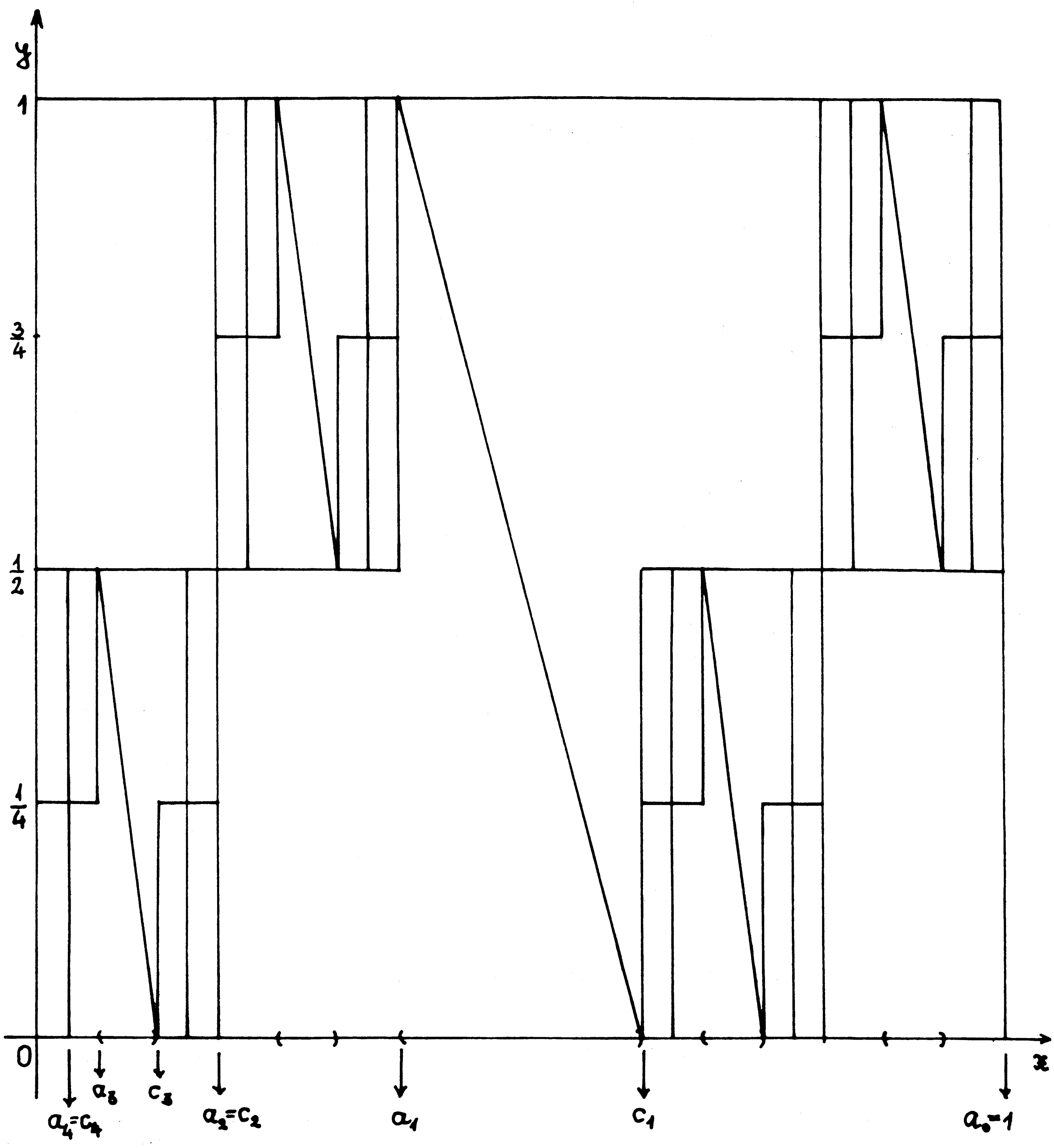

fig 1

or 1 and $x = \sum e_i(x)c_i'\}$. Clearly P is a symmetric perfect nowhere dense subset of $[0, 3/4]$. The open intervals deleted in the s-step of the construction of P are $O'_{e_1\ldots e_{s-1}} = (\sum_{i=1}^{s-1} e_i c_i' + a_s'$, $\sum_{i=1}^{s-1} e_i c_i' + c_s')$, $(e_1,\ldots,e_{s-1}) \in \{0,1\}^{s-1}$ and the remaining intervals of the s-step are $R'_{e_1\ldots e_s} = [\sum_{i=1}^{s} e_i c_i' , \sum_{i=1}^{s} e_i c_i' + a_s']$, where $(e_1,\ldots,e_s) \in \{0,1\}^s$, $|P| = \lim_{s\to\infty} 2^s a_s' = 1/2$. Let $F_1(x) = \sum_{i=1}^{\infty} e_{2i}(x)/2^i$ if $x \in P$. Extending F_1 linearly on each interval contiguous to P, we have F_1 defined and continuous on $[0, 3/4]$ (see [5]). If s is odd (resp. even) then F_1 is linear and strictly decreasing (resp. constant) on each $O'_{e_1\ldots e_s}$. Let $h:P \to Q$, $h(x) = h(\sum_{i=1}^{\infty} e_i(x)c_i') = \sum_{i=1}^{\infty} e_i(x)c_i$. Extending h linearly on each interval contiguous to P we have h defined, continuous and increasing on $[0, 3/4]$, $h(0) = 0$, $h(3/4) = 1$; h = constant on each $O_{e_1\ldots e_{s-1}}$ if s is even; $h(P \cap R'_{e_1\ldots e_s}) = Q \cap R_{e_1\ldots e_s}$; $h(R'_{e_1\ldots e_s}) = R_{e_1\ldots e_s}$. We prove that $h \in AC$ on $[0, 3/4]$. Since h is increasing it suffices to show that $\int_0^{3/4} h'(x)dx = 1$, hence $\int_P h'(x)dx = |Q|$. The function h is derivable a.e. on P. Let $x_0 \in P$ be a point at which h is derivable. Then $h'(x_0) = \lim_{s\to\infty} |R_{e_1\ldots e_s}| / |R'_{e_1\ldots e_s}| = 1$, hence $\int_P h'(x)dx = |Q|$. Since $F_1(x) = F(h(x))$ and $F_1 \in W$ on P (see Lemma 3 of [5]), by Theorem 6, $F \in W$ on Q.

b) Let $y \in [0,1]$. If y is uniquely represented in base 2, y =

$\sum y_i/2^i$, then $A_y = \{x \in Q : F(x) = y\} = \{x \in Q : e_{2i}(x) = y_i\}$ is a nowhere perfect subset of Q. If y has two representations in base 2, $y = \sum y_i/2^i = \sum y_i'/2^i$ then $A_y = \{x \in Q : F(x) = y\} = \{x \in Q : e_{2i}(x) = y_i\} \cup \{x \in Q : e_{2i}(x) = y_i'\}$ is a nonempty perfect subset of Q.

c) By b) it follows that 0 is a derived number for $F|_Q$ at $x \in Q$. Let $x_0 \in Q$. Then for each $s \geq 1$ there exist $e_1, \ldots, e_{2s}$ such that $x_c \in R_{e_1 \ldots e_{2s}}$. Since $O(F; Q \cap R_{e_1 \ldots e_{2s}}) = 1/2^s$ and $(1/2^s)/a_{2s} \to \infty$, $s \to \infty$ it follows that $F|_Q$ has finite or infinite derivative at no point $x_0 \in Q$.

d) Let $x_0 \in Q$, and for each $s \geq 1$, let $e_1, \ldots, e_{2s}$ such that $x_c \in R_{e_1 \ldots e_{2s}}$. Then either (i) $F(x_0) \in [\sum_{i=1}^{s} e_{2i}(x)/2^i, \sum_{i=1}^{s} e_{2i}(x)/2^i + 1/2^{s+1}]$ or (ii) $F(x_0)$ $[\sum_{i=1}^{s} e_{2i}(x)/2^i + 1/2^{s+1}, \sum_{i=1}^{s} e_{2i}(x)/2^i + 1/2^s]$. Suppose for example (i). Let $E_{e_1 \ldots e_{2s}} = Q \cap (R_{e_1 \ldots e_{2s}0101} \cup R_{e_1 \ldots e_{2s}0111} \cup R_{e_1 \ldots e_{2s}1101} \cup R_{e_1 \ldots e_{2s}1111})$. Then $|E_{e_1 \ldots e_{2s}}| / |R_{e_1 \ldots e_{2s}}| \to 1/4$ and if $y_s \in E_{e_1 \ldots e_{2s}}$ then $|F(y_s) - F(x_0)|/a_{2s} \geq (1/2^{s+2})/a_{2s} \to \infty$, hence F has a finite approximate derivative at no point $x_0 \in Q$.

e) Let $s = 2p-1$, $p \geq 1$, then

$$(15) \qquad F\left(\sum_{i=1}^{s-1} e_i c_i + a_s\right) - F\left(\sum_{i=1}^{s-1} e_i c_i + c_s\right) = F(a_{2p-1}) - F(c_{2p-1})$$

$$= F\left(\sum_{k=2p}^{\infty} c_k\right) = 1/2^{p-1}.$$

<u>Theorem 7. There exists a continuous function</u> $f:[0,1] \to [0,1]$ <u>with the following properties</u>: a) $f \in W$; b) <u>For each</u> $y \in [0,1]$,

$f^{-1}(y)$ is a nonempty perfect set; c) For each $x \in [0,1]$, $f'(x)$ does not exist (finite or infinite); d) f is approximately derivable at no point $x \in [0,1]$.

Proof. In what follows we use the notations introduced in the proof of Lemma 2. Let $I = [a,b] \subset [0,1]$ and let $h_I : [0,1] \to [a,b]$, $h_I(x) = (b-a)x + a$. Let $Q_I = h_I(Q) = a + (b-a)\cdot Q$. It follows that $|Q_I| = (1/2)\cdot(b-a)$, $a,b \in Q_I$ and Q_I is a symmetric perfect nowhere dense subset of $[a,b]$, which can be obtained on $[a,b]$ exactly as Q was obtained on $[0,1]$. The open intervals deleted in the s-step of the construction of Q_I are $(O_I)_{e_1\ldots e_{s-1}} = a + (b-a)O_{e_1\ldots e_{s-1}}$, which are nonempty if and only if $s = 2p-1$, $p \geqslant 1$. In this case

(16) $$(O_I)_{e_1\ldots e_{s-1}} = (b-a)\cdot(2/8^p).$$

The remaining intervals of the s-step are

(17) $$(R_I)_{e_1\ldots e_s} = a + (b-a)R_{e_1\ldots e_s}.$$

Let $g_I = F \circ h_I^{-1}$. By Theorem 5, $g_I \in W$ on Q_I. (The graph of g_I is similar to the graph of F, see fig.1) We have:

(18) $$g_I(a) = 0;\ g_I(b) = 1;\ g_I(I) = [0,1] \text{ and}$$

(19) $$g_I((R_I)_{e_1\ldots e_{2s}}) = g_I(Q_I \cap (R_I)_{e_1\ldots e_{2s}}) = \Big[\sum_{i=1}^{s} e_{2i}/2^i,\ \sum_{i=1}^{s} e_{2i}/2^i + 1/2^s\Big].$$

By (15), for $s = 2p-1$, we have

(20) $$O(g_I; (O_I)_{e_1\ldots e_{s-1}}) = 1/2^{p-1}.$$

Let $Q_1 = Q$. We shall construct a strictly increasing sequence Q_k, $k \geqslant 2$, of nowhere dense perfect subsets of $[0,1]$ and denote by $I_n^k = (a_n^k, b_n^k)$, $k \geqslant 1$, $n \geqslant 1$, the intervals contiguous to Q_k with respect to $[0,1]$. Let $A_n^k = [a_n^k, c_n^k]$, $B_n^k = [c_n^k, b_n^k]$, where c_n^k is the middle point of I_n^k. Then $Q_k = Q_{k-1} \cup (\bigcup_{n=1}^{\infty} (Q_{A_n^k} \cup Q_{B_n^k}))$. Let $f_1 = F$ on $[0,1]$. Suppose that $f_{k-1} : [0,1] \longrightarrow [0,1]$, $k \geqslant 2$ has already been defined and let's define $f_k : [0,1] \longrightarrow [0,1]$ as follows: $f_k(x) = f_{k-1}(x)$, $x \in Q_{k-1}$; $f_k(x) = f_{k-1}(a_n^{k-1}) + (f_{k-1}(c_n^{k-1}) - f_{k-1}(a_n^{k-1})) \cdot g_{A_n^{k-1}}(x)$, $x \in A_n^{k-1}$; $f_k(x) = f_{k-1}(c_n^{k-1}) + (f_{k-1}(b_n^{k-1}) - f_{k-1}(c_n^{k-1})) \cdot g_{B_n^{k-1}}(x)$, $x \in B_n^{k-1}$. We prove that $\{f_K\}$ is an uniformly convergent sequence of continuous functions on $[0,1]$. Clearly $f_1 \in \mathcal{C}$ on $[0,1]$. Suppose that $f_{k-1} \in \mathcal{C}$, $k \geqslant 2$ on $[0,1]$. We prove that $f_k \in \mathcal{C}$ on $[0,1]$. Since $f_k = f_{k-1}$ on Q_{k-1} it follows that $f_k \in \mathcal{C}$ on Q_{k-1}. We have

(21) $\qquad f_k(a_n^{k-1}) = f_{k-1}(a_n^{k-1})$; $f_k(c_n^{k-1}) = f_{k-1}(c_n^{k-1})$; $f_k(b_n^{k-1}) = f_{k-1}(b_n^{k-1})$; f_{k-1} is linear on $[a_n^{k-1}, b_n^{k-1}]$.

(See (18) and the definition of f_k on A_n^{k-1} and B_n^{k-1}.) Also,

(22) $\qquad f_k(A_n^{k-1}) = f_k(A_n^{k-1}) = [f_{k-1}(a_n^{k-1}), f_{k-1}(c_n^{k-1})]^*$ and

$$f_k(B_n^{k-1}) = f_{k-1}(B_n^{k-1}) = [f_{k-1}(c_n^{k-1}), f_{k-1}(b_n^{k-1})]^* ,$$

where $[x,y]^*$ is either $[x,y]$ or $[y,x]$ (see (18)). By (21) and (22) $O(f_k; [a_n^{k-1}, b_n^{k-1}]) = O(f_{k-1}; [a_n^{k-1}, b_n^{k-1}])$, hence $f_k \in \mathcal{C}$ on $[0,1]$.

Suppose that $|f_{k-1}(a_n^{k-1}) - f_{k-1}(b_n^{k-1})| \leq 1/2^{k-2}$, $k \geqslant 2$ and let's prove that

(23) $\qquad |f_k(a_n^k) - f_k(b_n^k)| \leq 1/2^{k-1}$.

Let (a_n^k, b_n^k) be an open interval of Q_k. Then (a_n^k, b_n^k) is an open interval either of (i) $Q_{A_p^{k-1}}$ or of (ii) $Q_{B_p^{k-1}}$, for some natural number p. Suppose (i), then by (21) and (22) it follows that $|f_k(a_n^k) - f_k(b_n^k)| \leq |f_{k-1}(A_p^{k-1})| \leq |f_{k-1}(a_p^{k-1}) - f_{k-1}(b_p^{k-1})|/2 \leq 1/2^{k-1}$. Since $f_k(x) - f_{k-1}(x) = 0$ on $Q_{k-1} \cup (\bigcup_n^{\infty}\{e_n^{k-1}\})$ (see (20) and the definition of f_k), by (21), (22) and (23), it follows that

(24) $|f_k(x) - f_{k-1}(x)| \leq 1/2^{k-1}$ on $[0,1]$. Let $f(x) = \lim_{k\to\infty}(f_k(x))$ Then by (24), $f_k \to f$ [unif] on $[0,1]$, hence $f \in \mathcal{C}$ on $[0,1]$.

a) Since $f_k(x) = f(x)$ on Q_k, by Lemma 2,a) it follows that $f \in W$ on Q_k. Since $|\cup Q_k| = 1$, $f \in W$ on $[0,1]$.

b) Suppose that there exists $y_0 \in [0,1]$ such that $E_{y_0} = \{x \in [0,1] : f(x) = y_0\}$ has an isolated point x_0. Since $f(x) = f_k(x)$ on Q_k, by Lemma 2,b) it follows that $x_0 \in [0,1] - \bigcup_{k=1}^{\infty} Q_k$. Since x_0 is isolated, there exists $\delta > 0$ such that $(x_0-\delta, x_0+\delta) \cap E_{y_0} = \{x_0\}$. Let k be a natural number such that $I_{n_k}^k \subset (x_0-\delta, x_0+\delta)$. We may suppose without loss of generality that $x_0 \in A_{n_k}^k$. Let $z_0 = g_{A_{n_k}^k}(x_0) \in [0,1]$. By Lemma 2,b) $E_{z_0} = \{x \in Q_{n_k}^k : g_{A_{n_k}^k}(x) = z_0\}$ is a perfect nonempty set. But $E_{z_0} \subset A_{n_k}^k \subset (x_0-\delta, x_0+\delta)$ and $f(E_{z_0}) = \{y_0\}$, a contradiction.

c) If $x_0 \in \cup Q_k$, since $f = f_k$ on Q_k, by Lemma 2,c) it follows that $f'(x_0)$ does not exist finite or infinite. Let $x_0 \in [0,1] - (\bigcup_{k=1}^{\infty} Q_k)$. Then there exists a sequence of natural numbers $\{n_k\}$, $k \geq 1$, such that $x_0 = \bigcap_{k=1}^{\infty} I_{n_k}^k$ and $I_{n_1}^1 \supset I_{n_2}^2 \supset \ldots$. For $\{n_k\}$, $k \geq 1$ there exists a sequence of natural numbers $\{p_k\}$, $k \geq 1$ such that

(25) $|I^k_{n_k}| = 2/8^{p_1+\dots+p_k}$, $|A^k_{n_k}| = |B^k_{n_k}| = 1/8^{p_1+\dots+p_k}$ and

$O(f;I^k_{n_k}) = 2/2^{p_1+\dots+p_k}$, hence $O(f;A^k_{n_k}) = O(f;B^k_{n_k}) = 1/2^{p_1+\dots+p_k}$.

Indeed, for n_1 there exists $p_1 \geq 1$ such that $I^1_{n_1}$ is an open interval from the step $2p_1-1$ of the construction of Q_1, hence $|I^1_{n_1}| = 2/8^{p_1}$. By (15), $O(f;I^1_{n_1}) = |f_1(b^1_{n_1}) - f_1(a^1_{n_1})| = 1/2^{p_1-1}$. Continuing, for n_k, $k \geq 2$ there exists p_k 1 such that $I^k_{n_k}$ is an open interval from the step $2p_k-1$ of the construction of $Q_{A^{k-1}_{n_{k-1}}}$ (resp. $Q_{B^{k-1}_{n_{k-1}}}$) for $x_0 \in A^{k-1}_{n_{k-1}}$ (resp. $x_0 \in B^{k-1}_{n_{k-1}}$). Hence $|I^k_{n_k}| = |A^{k-1}_{n_{k-1}}| \cdot (2/8^{p_k}) = 2/8^{p_1+\dots+p_{k-1}+p_k}$ and by (20) $O(f;I^k_{n_k}) = O(f;A^{k-1}_{n_{k-1}}) \cdot (2/2^{p_k-1}) = 2/2^{p_1+\dots+p_k}$. By b) it follows that 0 is a derived number for f at x_0. By (25), $O(f;I^k_{n_k})/|I^k_{n_k}| \to \infty$, $k \to \infty$, hence $f'(x_0)$ does not exist, finite or infinite.

d) If $x_0 \in \cup Q_k$, since $f = f_k$ on Q_k, by Lemma 2,d) it follows that $f'_{ap}(x_0)$ does not exist finite. Let $x_0 \in [0,1] - (\cup Q_k)$. Suppose that $f'_{ap}(x_0) = t_0$. It follows that there exists a measurable set E_{x_0} such that $d(E_{x_0}, x_0) = 1$ and $\lim_{\substack{x \to x_0 \\ x \in E_{x_0}}} (f(x)-f(x_0))/(x-x_0) = t_0$.

(Here $d(E;x)$ denotes the density of the set E at x.) Let k be a natural number such that $|t_0| < 4^{k-1}$; $|E_{x_0} \cap J|/|J| > 5/32$ for each interval J with $x_0 \in J$, $|J| \leq 1/8^{p_1+\dots+p_k}$ and $|f(x)-f(x_0)| < 4^{k-1} \cdot |x-x_0|$, for $x \in E_{x_0} \cap J$. We may suppose without loss of generality that $x_0 \in A^k_{n_k}$. By (19), either (i) $g_{A^k_{n_k}}(x_0) \in [0, 1/2]$

or (ii) $g_{A^k_{n_k}}(x_0) \in [1/2\ ,1]$. Suppose for example (i). Let $H_k =$ $(R_{A^k_{n_k}})_{0101} \cup (R_{A^k_{n_k}})_{0111} \cup (R_{A^k_{n_k}})_{1101} \cup (R_{A^k_{n_k}})_{1111}$. Then $H_k \subset A^k_{n_k}$, $g_{A^k_{n_k}}(H_k) \in [3/4,\ 1]$, $|H_k|/|A^k_{n_k}| = 4 \cdot a_4 = 5/128$, hence $E_{x_0} \cap H_k \neq \emptyset$. By (18) and (25), $|f(c^k_{n_k}) - f(a^k_{n_k})| = 1/2^{p_1+\dots+p_k}$. It follows that there exists $x \in E_{x_0} \cap H_k$ such that $|f(x)-f(x_0)| \geq (1/4)(1/2^{p_1+\dots+p_k})$. Hence $|f(x)-f(x_0)|/|A^k_{n_k}| \geq (1/4)(1/2^{p_1+\dots+p_k}) \cdot 8^{p_1+\dots+p_k} \geq 4^{k-1}$ and so $|f(x)-f(x_0)| \geq 4^{k-1} \cdot |x-x_0|$. For $J = A^k_{n_k}$ we have a contradiction.

We are indebted to Professor Solomon Marcus for his help in preparing this article.

References

[1] Bary, N.: Mémoire sur la représentation finie des fonctions continues. Math. Ann., 103 (1930), 185-248 and 598-653.

[2] Bruckner, A.M.: Differentiation of Real Functions. Lecture Notes in Mathematics, 659, Springer-Verlag, New York (1978).

[3] Cater, F.S.: On nondifferentiable wrinkled functions. Real Analysis Exchange, 14 (1988-89), 175-188.

[4] Ene, G. and Ene, V.: Nonabsolutely Convergent Integrals. Real Analysis Exchange, 11 (1985-86), 121-135.

[5] Ene, V.: Construction of a wrinkled function. Real Analysis Exchange, 14 (1988-89), 224-227.

[6] Ene,V.: A study of Foran's conditions A(N) and B(N) and his class $\mathfrak{F}$. Real Analysis Exchange,10(1985),194-211.

[7] Foran,J.: An extension of the Denjoy integral. Proc.Amer. Math.Soc.,49 (1975), 359-365.

[8] Foran,J.: A Generalization of Absolute Continuity. Real Analysis Exchange, 5 (1979-80), 82-91.

[9] Foran,J.: Continuous functions. Real Analysis Exchange, 2 (1977), 85-103.

[10] Foran,J.: On continuous functions with graphs of σ-finite linear measure.Proc.Cambridge Philos.Soc.,76(1974),33-43.

[11] Foran,J.: Differentiation and Lusin's Condition (N). Real Analysis Exchange, 3 (1977-78), 34-37.

[12] Lee,C.M.: Some Hausdorff variants of absolute continuity, Banach's condition (S) and Lusin's condition (N), Real Analysis Exchange, 13 (1988), 404-419.

[13] Saks,S.: Theory of the integral. 2nd rev.ed. Monografie Matematyczne, 7 (1937), PWN, Warsaw.

Received 16 January, 1989