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ORDERED FAMILIES OF BAIRE-2-FUNCTIONS

We call a (partially) ordered set $(X,<)$ Baire-α-representable if it is possible to associate a Baire-α function f^x to each element $x \in X$ such that $x < y$ iff $f^x < f^y$. Here, for two real functions f and g we write $f < g$ to denote that $f(x) \leq g(x)$ for every x and $f(x) < g(x)$ for some x. A classical result of Kuratowski says that $(\omega_1,<)$ is not Baire-1-representable [5,6]. Obviously, $\mathbb{R}$ is Baire-0-representable. An ordered set quite different from $\mathbb{R}$, from well-ordered or inverse well-ordered is a Souslin-line, i.e. an ordered set which is not separable but there does not exist an uncountable family of pairwise disjoint intervals. The existence of a Souslin-line is independent from our usual axiom system. See [2,3,4]. G. Petruska and J. Gerlits asked if a Souslin-line may be Baire-1-representable.

Theorem 1. No Souslin-line is Baire-1-represented.

Proof. Assume that V is a model of ZFC set theory and $(X,<)$ is a Baire-1-represented Souslin-line in it. It is well-known that there is a Souslin-tree $(T,<)$ consisting of some intervals of $(X,<)$ with the reverse ordering. For these arguments as well as the forcing notions, see [3] or [4]. We generically extend V by $(T,<)$. In the resulting model, there is a cofinal branch in T which gives rise to an ω_1-type subset of X, represented by Baire-1 functions. As $(T,<)$ is *ccc*, the old ω_1 survives. As we forced by a Souslin-tree, new reals will not be introduced, and so the representing functions will still be Baire-1. These statements, however, contradict Kuratowski's theorem.

The class of order types represented by Baire-2 functions is much richer. D. Fremlin pointed out the following reduction by which some results originally proved by the author may be deduced from some known set theoretical theorems. For $f,g:\mathbb{N}\to\mathbb{N}$. ($\mathbb{N}$ is the set of natural numbers) we put $f \leq g$ iff $f(n) \leq g(n)$

holds for all but finitely many $n \in \mathbf{N}$. For $f : \mathbf{N} \to \mathbf{N}$ we put $A(f) = \{g : \mathbf{N} \to \mathbf{N},\ g \leq f\}$ an F_σ set, when the $\mathbf{N} \to \mathbf{N}$ functions are identified by reals.

Lemma 2. For $f, g : \mathbf{N} \to \mathbf{N}$, $f \leq g$ iff $A(f) \subseteq A(g)$.

Proof. Straightforward.

This lemma enables us to quote results about (partially) ordered sets represented in $(\mathbf{N} \to \mathbf{N}, <)$. An easy induction shows that every set of size $\aleph_1$ is so represented. By Parovičenko's theorem (see [2]) this holds even for partially ordered sets. If $MA(\kappa)$ holds, then every partially ordered set of size κ as well as $(\kappa^+, <)$ is so represented. It is consistent that $c = \aleph_2$ and still every ordered set of size $\leq c$ is so represented. See $26Kf$ and $21Nb$ in [2] and [7].

We prove some negative independence results.

Theorem 3. If V is a model of $ZFC + CH$, $(X, <) \in V$ is an ordered set of size $\aleph_2$, and if we generically add $\kappa \geq \omega_2$ Cohen reals, then $(X, <)$ will not be Baire-α-represented in the enlarged model, for any $\alpha < \omega_1$.

Proof. Let $\{r_\xi : \xi < \kappa\}$ be the Cohen reals. For $A \subseteq \kappa$, let $P(A)$ denote the partially ordered set adding the Cohen reals r_ξ for $\xi \in A$. Assume that some condition p forces that the Baire-α functions f^x are ordered similarly to $(X, <)$. Every Baire-α function can be coded by a real. (See the similar coding for Borel sets in [3, Section 42].) By *ccc*, for every $x \in X$ there exists a countable set $A(x) \subseteq \kappa$ such that the real coding f^x is in $V^{P(A(x))}$. By the Erdoós-Rado theorem (See Theorem 2.1.6 in [4].), there is a set $Z \subseteq X$ of size $\aleph_2$ and a set A such that $A(x) \cap A(x') = A$ for $x, x' \in Z$. As $\aleph_2 > c$, there are $x < y$, $x, y \in Z$, and a bijection $\pi : A(x) \to A(y)$ such that if a statement about f^x is forced by some element in $P(A(x))$, then the π-isomorphic copy of it forces the same statement about f^y in $P(A(y))$. If $G \subseteq P(\kappa)$ is the generic filter, let $H \subseteq P(\kappa)$ be gotten by interchanging $G \mid P(A(x) - A(y))$ and $G \mid P(A(y) - A(x))$. H is generic by the product lemma (Lemma 20.1 in [3]), and in $V[H]$ the actual f^x is what f^y is in $V[G]$ and vice versa. But then $x < y$ and $f^y < f^x$, a contradiction.

Corollary 4. It is consistent with $ZFC + c = \aleph_2$ and

(a) $(\omega_2, <)$ is not Baire- α-represented for $\alpha < \omega_1$;

(b) there is an ordered set of cardinality $\aleph_2$, not containing subsets of type ω_2, ω_2^* which is not Baire-α-represented for any $\alpha < \omega_1$.

Proof. (a) Apply Theorem 3 with $(X, <) = (\omega_2, <)$. (b) Apply Theorem 3 with the set of $\omega_1 \to \{0,1\}$ functions as X, ordered lexicografically. It suffices to show that $(X, <)$ does not contain subsets of type ω_2 or ω_2^* after adding Cohen reals. It is enough to show this about the set of all $\omega_1 \to \{0,1\}$ functions in the new model. This is proved in [1, Lemma 2], see also [3, Lemma 29.4] but is probably a result of Sierpiński.

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