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On the Maximal Multiplicative Family for the Class of Quasicontinuous Functions

Let (X,T) be a topological space with the topology T. We say that a real function $f : X \to \mathsf{R}$ is quasicontinuous at a point $x \in X$ if for every $\varepsilon > 0$ and for every $U \in T$ such that $x \in U$ there exists a nonempty set $V \in T$ such that $V \subset U$ and $|f(t) - f(x)| < \varepsilon$ for each $t \in V$. If Y is a nonempty set and $\mathcal{A}$ is a family of real functions on Y, then $N(\mathcal{A}) = \{g : Y \to \mathsf{R}; gf \in \mathcal{A}$ for every $f \in \mathcal{A}\}$ is called the maximal multiplicative class for $\mathcal{A}$ ([1]). In [2] the following is proved:

If X is a complete metric space, Q is the family of all quasicontinuous real functions on X, then

$$N(Q) = \{f \in Q;\ \text{if } x \notin C(f),\ \text{then } f(x) = 0 \text{ and } x \in Cl\,(C(f) \cap f^{-1}(0))\},$$

where $C(f)$ denotes the set of all continuity points of f and $Cl\,A$ is the closure of A.

In this article this theorem is generalized to real functions defined on topological spaces. The proof of this generalized theorem will follow from Remarks 3 and 4 and Theorem 1. Now let Q be the family of all quasicontinuous real functions on X.

Remark 1. $N(Q) \subset Q$.

Proof. If $g \in N(Q)$, then $g = g \cdot 1 \in Q$.

Remark 2. If f is continuous at $x \in X$, then fg is quasicontinuous at x for every function $g : X \to \mathsf{R}$ quasicontinuous at x.

The proof of this remark is easy.

Remark 3. Let $f \in Q$. Suppose that $f(x) = 0$, $x \notin C(f)$ and for every open neighborhood U of x there is a point $u \in C(f) \cap U$ such that $f(u) = 0$. Then fg is quasicontinuous at x for every $g \in Q$.

The proof of this remark is also easy.

Remark 4. If $f \in Q$, $x \notin C(f)$ and $f(x) \neq 0$, then there exists $g \in Q$ such that fg is not quasicontinuous at x.

Proof. Because $f \in Q$ and $x \notin C(f)$, there exists $\varepsilon > 0$ such that $x \in Cl(\mathrm{Int}(\{t \in X : |f(t) - f(x)| > \varepsilon\}))$, where Int denotes the interior operation. We can assume that $\varepsilon < |f(x)|/2$. Let us put

$$g(u) = \begin{cases} c \text{ if } u = x \quad \text{or } (u \in Cl(\mathrm{Int}(\{t \in X : |f(t) - f(x)| \geq \varepsilon\})) \\ \qquad\qquad\qquad \text{and } |f(u) - f(x)| \geq \varepsilon) \\ 1/f(u) \qquad \text{otherwise} \end{cases}$$

where $c > 0$ is a number such that $cf(x) \neq 1$. Obviously g is quasicontinuous at each point $u \in Cl(\mathrm{Int}(\{t \in X : |f(t) - f(x)| \geq \varepsilon\}))$ such that $|f(u) - f(x)| \geq \varepsilon$. Because $x \in Cl(\mathrm{Int}(\{t \in X : |f(t) - f(x)| > \varepsilon\}))$, it is also quasicontinuous at x. Therefore g is quasicontinuous at each point u such that $g(u) = c$.

Let $u \neq x$ be a point at which $|f(u) - f(x)| < \varepsilon$. Let $\eta > 0$ be a number such that $|f(x)|^2\eta/4 < \min(f(u) - f(x) + \varepsilon, f(x) - f(u) + \varepsilon)$ and let $U \in T$ be a neighborhood of u. Since $f \in Q$, there is a nonempty set $V \in T$ such that $V \subset U$ and $|f(t) - f(u)| < |f(x)|^2\eta/4$ for every $t \in V$. Because $x \notin C(f)$, we can assume that $x \notin V$. We have $V \subset \{t \in X : |f(t) - f(x)| < \varepsilon\}$ and for every $t \in V$,

$$\begin{aligned} |g(t) - g(u)| &= |1/f(t) - 1/f(u)| = |f(u) - f(t)|/|f(t)f(u)| \\ &< 4|f(u) - f(t)|/|f(x)|^2 \\ &< |f(x)|^2\eta/|f(x)|^2 = \eta. \end{aligned}$$

Therefore g is quasicontinuous at u.

Analogously we show the quasicontinuity of the function g at points $u \notin Cl(\mathrm{Int}(\{t \in X : |f(t) - f(x)| \geq \varepsilon\}))$ at which $|f(u) - f(x)| = \varepsilon$. Hence $g \in Q$. But for $u \neq x$ either $fg(u) = 1$ or $fg(u) \leq c(f(x) - \varepsilon)$ or $fg(u) \geq c(f(x) + \varepsilon)$, so fg is not quasicontinuous at any point x for which $fg(x) = cf(x)$. This completes the proof.

THEOREM 1. Let $f \in Q$. If there exists a nonempty set $U \in T$ such that $A = \{u \in U : f(u) = 0\} \neq \emptyset$ and $f(u) \neq 0$ for every point $u \in C(f) \cap Cl\,U$, then there exists a function $g \in Q$ such that $fg \notin Q$.

Proof. Because $f \in Q$, and $f(u) \neq 0$ at each point $u \in C(f) \cap Cl\,U$, $Cl(\{u \in Cl\,U : f(u) = 0\})$ is nowhere dense. Let

$$B(x) \;=\; \{y \in \mathsf{R} : \text{ for every } \varepsilon > 0,$$

$$x \in Cl(\text{Int}(\{u : |f(u) - y| < \varepsilon\}))$$

for $x \in X$. Let $A_1 = \{u \in A : B(u) - \{0\} \neq \emptyset\}$. For every $u \in A_1$ let $a(u) \in B(u) - \{0\}$ be fixed. We define the function g as follows:

$$g(x) = \begin{cases} 1 & \text{for each } \; x \in Cl(X - ClU) \\ 1/f(x) & \text{for each } \; x \in Cl\,U - Cl(X - Cl\,U) \text{ with } f(x) \neq 0 \\ 1/a(x) & \text{for each } \; x \in A_1 - Cl(X - Cl\,U) \\ 0 & \text{otherwise.} \end{cases}$$

We shall prove that $g \in Q$. Obviously g is quasicontinuous at each point $x \in Cl(X - Cl\,U)$. Let $x \in Cl\,U - Cl(X - Cl\,U)$ be a point at which $f(x) \neq 0$, let V be an open neighborhood of x and let $\varepsilon > 0$. Since $f \in Q$, there is a nonempty open set $W \subset V - Cl(X - Cl\,U) - Cl\,A$ such that

$$|f(u) - f(x)| < \min(|f(x)|/2, (|f(x)|^2/2)\varepsilon)$$

for every $u \in W$. We have for all $u \in W \subset V$,

$$\begin{aligned} |g(u) - g(x)| &= |1/f(u) - 1/f(x)| = |f(u) - f(x)|/|f(u)||f(x)| \\ &< (|f(x)|^2 \cdot \varepsilon/2)/(|f(x)|^2/2) = \varepsilon \end{aligned}$$

and g is quasicontinuous at x.

Now let $x \in A_1 - Cl(X - Cl\,U)$, let V be an open neighborhood of x and let $\varepsilon > 0$. Since $a(x) \in B(x)$,

$$x \in Cl(\text{Int}(\{u \in X : |f(u) - a(x)| < \min(|a(x)|/2, |a(x)|^2\varepsilon/2)\})).$$

There exists a nonempty open set $W \subset V - Cl(X - Cl\,U) - Cl\,A$ such that $|f(u) - a(x)| < \min(|a(x)|/2, |a(x)|^2\varepsilon/2)$. We have for $u \in W$

$$\begin{aligned} |g(u) - g(x)| &= |1/f(u) - 1/a(x)| = |f(u) - a(x)|/|f(u)||a(x)| \\ &< |a(x)|^2\varepsilon/(2|a(x)|^2/2) = \varepsilon \end{aligned}$$

and g is quasicontinuous at x.

Suppose that $g(x) = 0$. Let V be an open neighborhood of x and let $\varepsilon > 0$. Observe that in this case $B(x) - \{0\} = \emptyset$. Let $a > 0$ be such that $1/a < \varepsilon/2$. We will show that there is a point $u \in V$ such that $|f(u)| > a$. Indeed, if $|f(t)| \leq a$ for every $t \in V$, then for each $\delta > 0$ ($\delta < a$) and for each $y \in [-a, a] - (-\delta, \delta)$ there is an open neighborhood $W(y)$ of x and a positive number $\eta(y)$ such that

$$\text{Int}(\{t \in X : |f(t) - y| < \eta(y)\}) \cap W(y) = \emptyset.$$

Because f is quasicontinuous,

$$(*) \qquad \{t \in X : |f(t) - y| < \eta(y)\} \cap W(y) = \emptyset.$$

There are $y_1, y_2, \dots, y_n \in [-a,a] - (-\delta,\delta)$ such that $\bigcup_{i=1}^{n}(y_i - \eta(y_i), y_i + \eta(y_i)) \supseteq [-a,a] - (-\delta,\delta)$. Put $W_0 = \bigcap_{i=1}^{n} W(y_i)$ and let $u \in W_0$.

If $f(u) \in [-a,a] - (-\delta,\delta)$, then there is $i_0 \le n$ such that

$$f(u) \in (y_{i_0} - \eta(y_{i_0}), y_{i_0} + \eta(y_{i_0})),$$

contradicting $(*)$. So $|f(u)| < \delta$ for every $u \in W_0$ and f is continuous at x, contrary to $f(x) = 0$ and $x \in Cl\,U$. So there exists a point $u \in V$ such that $|f(u)| > a$. But f is quasicontinuous at u. Therefore there is a nonempty open set $W \subset V$ such that $|f(t)| > a$ for every $t \in W$. Consequently $|g(t) - g(x)| = 1/|f(t)| < 1/a < \varepsilon$ for every $t \in W$ and g is quasicontinuous at x. So $g \in Q$. But

$$fg(x) = \begin{cases} f(x) & \text{for each } \; x \in Cl(X - Cl\,U) \\ 1 & \text{for each } \; x \in Cl\,U - Cl(X - Cl\,U) - A \\ 0 & \text{for each } \; x \in A, \end{cases}$$

or fg is not quasicontinuous at any point $x \in U \cap A$. This completes the proof.

Example 1. Let X be the interval $[0,1]$ and let $f(x) = x + |\sin(1/x)|$ for all $x \in (0,1)$ and $f(0) = f(1) = 0$.

The topology on X is the one for which the sets $[0,r)$, $0 < r < 1$, form a base of neighborhoods of 0, the sets $(x-r, x+r) \cap (0,1)$, $r > 0$, form a base of neighborhoods of x for $x \in (0,1)$; and the sets $\{1\} \cup ((0,r) \cap \{u \in X : |f(u) - u| < u\})$, $0 < r < 1$, form a base of neighborhoods of 1. Note that $f \in Q$ and f is continuous at each point $x \in X - \{0\}$. There is an open neighborhood $V \subset [0,1)$ of 0 such that $f(u) \neq 0$ for each $u \in V - \{0\}$. (It is obvious that $f(1) = 0$ and $1 \in Cl\,V - \{0\}$.) Let $g \in Q$. It follows from Remark 2 that fg is quasicontinuous at each point $x \in (0,1]$. We shall show also that it is quasicontinuous at 0. Let $\varepsilon > 0$ and let U be an open neighborhood of 0. Because $fg(1) = 0$ and fg is qusicontinuous at 1, for every $V = \{1\} \cup ((0,r) \cap \{u \in X : |f(u) - u| < u\})$ such that $(0,r) \subset U$ there is an open nonempty set $W \subset V$ with $|fg(u) - fg(1)| < \varepsilon$ for every $u \in W$. Because $fg(1) = fg(0)$, the proof is complete.

Example 2. Let $X = \mathbf{R}^2$ and let T be a topology on X such that: if $(x,y) \neq (0,0)$, then U belongs to a base of neighborhoods of (x,y) iff U is Euclidean open and $(0,0) \notin U$.; $(U_n)_{n=1}^{\infty}$ is a base of neighborhoods of $(0,0)$ iff

$$U_n = \{(x,y) \in R^2 : x = r\cos\varphi,\ y = r\sin\varphi,\ 0 \le r < 1,\ 0 \le \varphi \le 2\pi/n\}.$$

For $n = 2, 3, \ldots$ let K_n be the closed ball with center $A_n = (\cos 2\pi/n, \sin 2\pi/n)$ and with radius $r_n = \text{dist}(A_n, A_{n+1})/8$. Put

$$f(x,y) = \begin{cases} 0 & \text{if} \quad (x,y) = (0,0) \\ \text{dist}((x,y), A_n)/r_n & \text{if} \quad (x,y) \in K_n \ \ (n = 2, 3, \ldots) \\ 1 & \text{otherwise.} \end{cases}$$

Observe that f is quasicontinuous at each point $(x,y) \in X$ and continuous at each point A_n $(n = 2, 3, \ldots)$. Moreover $(0,0)$ is a discontinuity point of f, $f(0,0) = 0$ and for every neighborhood U_n of $(0,0)$, $f(x,y) \neq 0$ for $(x,y) \in U_n - \{(0,0)\}$ and $f(A_n) = 0$ and $A_n \in Cl\, U_n$. So for every open neighborhood V of $(0,0)$ there is a continuity point $(x,y) \in Cl\, V$ of f at which $f(x,y) = 0$. Define

$$g(x,y) = \begin{cases} 1/f(x,y) & \text{if} \quad (x,y) \in U_1 \cap K_n \quad (n = 2, 3, \ldots) \\ 1 & \text{otherwise} \end{cases}$$

Then $g \in Q$ and fg is not quasicontinuous at $(0,0)$.

Remark 5. Let $M_b(Q) = \{f : X \to \mathsf{R};$ for every bounded function $g \in Q$, $fg \in Q\}$. Then $f \in M_b(Q)$ iff $f \in Q$ and for each discontinuity point x of f we have $f(x) = 0$.

This remark is obvious.

Remark 6. Example 1 shows that the requirement in Theorem 1 that $f(u) \neq 0$ at all points $u \in C(f) \cap Cl\, U$ cannot be relaxed. Example 2 shows that the existence of points $u \in C(f) \cap f^{-1}(0) \cap Cl\, U$ for every neighborhood U of x in Theorem 1 is not a sufficient condition for the quasicontinuity of fg at x with $g \in Q$.

REFERENCES

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