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Tudor Zamfirescu, Abteilung Mathematik, Universität Dortmund, August - Schmidt-Straße, D-4600 Dortmund.

Porosity in Convexity

§1 Introduction.

The Baire category theorem was for a long time and continues to be an important tool in analysis used for distinguishing between "small" and "big" sets or simply for proving existence theorems. In geometry, the first use of the Baire category theorem in the space of all convex bodies (endowed with the Hausdorff metric) seems to have been in V. Klee's paper [18] published in 1959. For an up-to-date survey of generic results in convex geometry, see [47].

The notion of first Baire category is based on the notion of a nowhere dense set. The latter was strengthened by E. Dolzhenko [8] in 1967, who introduced the notion of set porosity, a notion essentially already known to Denjoy. After Dolzhenko's introduction, both porosity and the related σ-porosity were used to strengthen theorems involving Baire categories, once again in analysis first. It was natural to think about using porosity to extend results in convex geometry as well. It is the intention of this paper to present a survey of such results, which have all appeared during the past few years. This line of research has in fact just started, many questions are still open, and much work remains to be done.

§2 Definitions and Notation.

A set in a topological space is called *nowhere dense* if its closure has an empty interior. A countable union of nowhere dense sets is said to be of *first category*. If a set is not of first category, then it is of *second category*. A topological space in which each open set is of second category is called a *Baire space*. A set in a Baire space is called *residual* if its complement is of first category.

In the course of this paper we shall meet several Baire spaces: the space $\mathfrak{R}^d$ with the Euclidean distance, the space $\mathbf{K}$ of all compact sets in $\mathfrak{R}^d$ with the Hausdorff distance, its subspaces T and $\mathbf{K}^*$ of all starshaped sets (always presumed compact) and of all compact convex sets respectively, the subspaces of $\mathbf{K}$ of all convex bodies and of all convex surfaces (i.e. d-dimensional members of $\mathbf{K}^*$ and their boundaries respectively), any convex surface with its intrinsic metric.

We say that *most*, or *typical*, elements of a Baire space enjoy a certain property if those not enjoying it form a set of first category, i.e. if those enjoying it form a residual set.

In a metric space (X, ρ) we call a set M *porous at* $x \in X$ if there is a positive number α such that for any positive number ϵ, there is a point y in the open ball $B(x, \epsilon)$ with center x and radius ϵ such that

$$(*) \qquad B(y, \alpha\rho(x, y)) \cap M = \emptyset.$$

A set which is porous at all points of X is simply called *porous* [8]*. If for some $x \in X$ the above number α can be chosen as close to 1 as we wish, the set M is called *strongly porous at* x. A set which is strongly porous at every point of X is said to be *strongly porous.*

A directionally stronger version of porosity was proposed by Agronsky and Bruckner [1]: A set $M \subset X$ is *totally porous at* $x \in M$ if there is a positive number α such that for any $\epsilon > 0$ and for any $z \neq x$, there is a point $y \in B(x, \epsilon)$ such that

$$B(y, \alpha\rho(x, y)) \subset B(z, \rho(x, z))$$

and $(*)$ hold. A set which is totally porous at each of its points is called *totally porous.* Much stronger along the same line is the following notion: A set $M \subset X$ is *hyperporous at* $x \in M$ if there is a positive α such that for any $z \neq x$, there

* Editor's note: This is different from the notion of porosity used by real analysts where a set is porous if it is porous at each of its points.

such that (*) holds. The *hyperporosity* of M is defined analogously.

A countable union of porous sets is called $\sigma-porous$. We say that *nearly all* (real analysts say *virtually all*) elements of a metric Baire space have a certain property if those which do not enjoy it form a σ-porous set [36]. Every porous set in $\Re^d$ is of measure zero, by Lebesgue's density theorem. Therefore, any σ-porous set in $\Re^d$ is both of the first Baire category and of measure zero. Thus, σ-porosity is a convincing smallness attribute and has the advantage of being available in spaces like $\mathbf{K}^*$ with no geometrically meaningful Borel measure (see C. Bandt and G. Baraki [4], Theorem 3).

We shall also use the following notion of dimension. We say that the set $M \subset \Re^d$ has *local dimension* $k \leq d$ *at* $x \in M$ if k is the smallest integer for which there is an affine k-dimensional subspace $A \subset \Re^d$ through x such that for any $\epsilon > 0$ there is a $\nu > 0$ for which

$$M \cap B(x, \nu) \ \subset \ B(A + \epsilon\nu U),$$

where the "U" denotes the unit ball and the "+" Minkowski addition. We write $ldim_x M$ for the local dimension of M at x. Then the *local dimension* of M is defined by

$$ldim\ M \ = \ \sup_{x \in M} ldim_x M.$$

For $k = 0$, i.e. A is a single point, these definitions make sense in any metric space.

The following implications are easily checked: Every set $M \subset \Re^d$ with $ldim_x M < d$ is totally and strongly porous at x. If X is a connected metric space, $M \subset X$, and $ldim_x M = 0$, then M is strongly porous at x; if $x \in M$ has a neighborhood which is convex in Menger's sense, $M \subset X$, and $ldim_x M = 0$, then M is hyperporous at x.

§3 Smoothness and Strict Convexity of Convex Surfaces.

A convex surface is said to be *smooth* if at each of its points it has a unique supporting hyperplane, and *strictly convex* if it contains no line segment.

Any convex surface is smooth a.e. with respect to the (d-1)-dimensional Hausdorff measure (see [21],[2]). Klee's result mentioned in the introduction asserts that most convex surfaces are smooth and strictly convex. We have the following stronger result which uses the notion of porosity.

THEOREM 1. ([35]) *Nearly all convex surfaces are smooth and strictly convex.*

Not yet investigated is the question whether nearly all convex curves on a smooth convex surface in $\mathfrak{R}^3$ are smooth, as is true for most of them (see [37] for definitions, the result and an analogous problem about strong convexity).

We conjecture that Theorem 1 can be strengthened via hyperporosity in the case of smoothness, but not in the case of strict convexity.

§4 Conjugate Points on Convex Surfaces.

Any shortest path between two points of a convex surface in $\mathfrak{R}^d$ is called a *segment*. A curve which is locally a segment is called a *geodesic* (see [7], p.77 for a precise definition). Two points of a convex surface which are joined by more than one segment are called *conjugate*. A point of a convex surface which is not an interior point of any segment is called an *endpoint* of the surface.

Concerning the conjugate points of a convex surface in $\mathfrak{R}^3$ we have the following result.

THEOREM 2. ([46]) *Let S be any convex surface in $\mathfrak{R}^3$ and $x \in S$. Then nearly all points of S are not conjugate to x.*

In higher dimensions the results now available leave much room for improvement. Only in case the convex surface $S \subset \mathfrak{R}^d$ is typical [11] or class C^3 [47] is it known that for any point $x \in S$ most points of S are not conjugate to

x. On the other hand, it is known that on a typical convex surface $S \subset \mathfrak{R}^d$ and for any $x \in S$, the set of points conjugate to x is dense in S (see [11], [46]).

Not yet investigated is the possibility of extending the following noteworthy results of P. Gruber [12] and the author [32], [45] by using porosity:

THEOREM 3. ([32]) *On most convex surfaces in $\mathfrak{R}^d$ most points are endpoints.*

THEOREM 4. *On most convex surfaces in $\mathfrak{R}^3$*

(i) there is no closed geodesic ([12])

(ii) there are arbitrarily long geodesics without self-intersections ([45]).

§5 Normals to Convex Surfaces.

All convex surfaces considered in this section will be smooth, as most of them are. A *normal* to a convex surface is a line passing through a point of the surface, orthogonal to the supporting hyperplane at that point. For any usual surface the points lying on infinitely many normals are exceptional. However, this is not true for typical convex surfaces; on them most points of $\mathfrak{R}^d$ lie on infinitely many normals ([31], [34]).

Let $\Psi(x)$ be the set of directions (unit vectors) of all normals to a given convex surface passing through $x \in \mathfrak{R}^d$. It was natural to ask more about the structure of this set, besides its being infinite. I. Bárány and T. Zamfirescu provided an answer :

THEOREM 5. ([5]) *Let $Z \subset \mathfrak{R}^d$ be a countable set. For most co nvex surfaces the following holds: For any point $x \in Z$ the set $\Psi(x)$ is perfect and for any point $x \in \mathfrak{R}^d$ the set $\Psi(x)$ is porous in S^{d-1}.*

Recently M. Laczkovich [20] succeeded in showing that for most convex curves, most points in $\mathfrak{R}^2$ lie on uncountably many normals. Unfortunately, his proof is not extendable to higher dimensions. To date, this result has not been extended to $d \geq 3$ and Theorem 5 is not known to be true for nearly all convex surfaces.

§6 Diameters of Convex Bodies.

A *diameter* of a convex body $K \subset \Re^d$ is a chord of K such that K admits parallel supporting hyperplanes at its endpoints. From a result of A. Kosiński [19] it follows that every convex body has a point lying on at least three diameters. In the typical case many more diameters meet together. For most convex bodies $K \subset \Re^d$, most points of K lie on infinitely many diameters ([33], [5]).

Let $\Phi(x)$ be the set of all directions of diameters of K passing through $x \in K$. Again, it was natural to further investigate the set $\Phi(x)$.

THEOREM 6. ([5]) *Let $Z \subset \Re^d$ be countable. For most convex bodies $K \subset \Re^d$ the following is true: At each point $x \in Z \cap K$ the set $\Phi(x)$ is perfect and at each point $x \in K$ the set $\Phi(x)$ is porous in S^{d-1}.*

In 1965 P. C. Hammer [17] raised the question whether there is a convex body K with an interior point z such that the set $R(z)$ of all ratios into which z divides the various diameters through z is uncountable. A.S. Besicovitch and T. Zamfirescu [6] answered the question by providing such a convex body and an appropriate interior point. In fact, this is a generic property:

THEOREM 7. ([5]) *If $Z \subset \Re^d$ is countable, then for most convex bodies $K \subset \Re^d$, at each point $x \in Z \cap K$ the set $R(x)$ is uncountable.*

Let M_α (respectively T_α) be the set of all interior points of the convex body $K \subset \Re^d$ lying on at least (respectively exactly) α diameters. Generic connectivity properties of M_α have been established: For most convex bodies in $\Re^2$, M_α is connected for any at most countable α and T_α is totally disconnected for any finite α [33]. Until now no attempt has been made to improve these results using porosity.

§7 The Nearest Point Mapping.

Let $K \subset \Re^d$ be a compact set. We shall consider the nearest point mapping p_K defined on $\Re^d$ as the multivalued function

$$p_K(x) \;=\; \{y \in K : \; ||x-y|| = min_{z\in K}||x-z||\}.$$

In particular, the set K can be convex. In that case (and only in that case) the function p_K is single valued everywhere. A natural question to ask is about the proportion between the set on which p_K is single valued and the set on which p_K is properly multiple valued. The answers are known from both the measure theoretic and Baire category points of view, and these answers agree: For any compact set $K \in \mathbf{K}$, p_K is not single valued on a set which is both of measure zero and of the first category [24].

THEOREM 8. ([44]) *For any compact set $K \in \mathbf{K}$ the nearest point mappi ng p_K is single valued at nearly all points of $\mathfrak{R}^d$.*

It is known that for most compact sets K the mapping p_K is not single valued at densely many points [44], but it remains unsettled whether this is true for nearly all compact sets K as well.

The differentiability properties of the nearest point mapping are also of considerable interest. As E. Asplund proved in [3], p_K is not only single valued a.e., but also Fréchet differentiable a.e. From the viewpoint of Baire category the situation changes: For most convex bodies $K \subset \mathfrak{R}^d$, p_K has no Fréchet derivative at most points $y \notin K$ and, for most planar convex bodies, p_K has no directional derivative in any nonnormal direction, at most points $y \notin K$ [43]. Once again we do not know whether "most" can be replaced by "nearly all" in the preceeding result.

§8 Starshaped Sets.

Several generic results on starshaped sets have been established by P. Gruber [16] and the author [40]. Among these we mention:

THEOREM 9. ([40]) *Most starshaped sets $T \in \mathbf{T}$ have a kernel consisting of a single point $k(T)$ and are not locally connected at any point different from $k(T)$.*

THEOREM 10. ([16],[40]) *For most starshaped sets $T \in \mathbf{T}$, the set of directions*

$$\{||x - k(T)||^{-1}(x - k(T)) : \ x \in T\backslash\{k(T)\}\}$$

is dense, uncountable and of the first Baire category in S^{d-1}.

THEOREM 11. ([16]) *Most starshaped sets in $\mathbf{T}$ have Hausdorff dimension 1 and non-σ-finite 1-dimensional Hausdorff measure.*

Concerning porosity, the generic aspect is given by the following theorem.

THEOREM 12. ([16],[40],[48]) *Most starshaped sets $T \in \mathbf{T}$ are nowhere dense but not porous at $k(T)$. They have $ldim_x T = 1$ (whence they are totally and strongly porous) at any point $x \neq k(T)$.*

No results on nearly all starshaped sets have been obtained thus far. In [41] several generic properties of starshaped sets with kernels of positive dimension were discovered but there were no porosity results among these.

§9 Compact Sets.

Typical compact sets in $\mathfrak{R}^d$ are thin: standard arguments show they are both measure zero and first category. This suggests that they might even be porous. This is indeed so, as is revealed by the following theorem of P. Gruber which improves an earlier version by the author [36], in which the underlying complete metric space was convex in Menger's sense.

THEOREM 13. ([13]) *If X is a complete metric space, then most compact sets $K \subset X$ have local dimension 0.*

It follows from this result that most compact sets in a complete metric space X are hyperporous if X is convex and strongly porous if X is connected. Regarding nearly all compact sets, we have the following result.

THEOREM 14. ([36]) *In a Banach space, nearly all compact sets and nearly all closed bounded sets are strongly porous.*

Here we take the opportunity to mention that L. Zajíček [27] proved that in any Banach space first category and σ−porosity are truly different notions.

In a complete metric space it is known that most compact sets are Cantor sets, but no analogous result for nearly all compact sets has been discovered. Also, the following results obtained by J.A. Wieacker for typical compact sets in $\Re^d$ have not been extended via porosity: For most $K \in \mathbf{K}$, *bd conv* K is of class C^1 but not of class C^2, *ext conv* K is a Cantor set and *exp conv* K is homeomorphic to the set of all irrational numbers [26]. (Here *conv A, bd A, ext A,* and *exp A* denote the convex hull, boundary, set of extreme points, and the set of exposed points of A respectively.)

§10 Epilogue.

There are several topics in generic convexity in which the use of porosity has not been investigated. Examples include: the study of convex tomography (see A. Volčič and T. Zamfirescu [25]), the study of billiards (see P. Gruber [10]), the approximation of convex bodies by polytopes (see P. Gruber and P. Kenderov [14] and R. Schneider and J. Wieacker [23]), the description of the set of contact points with inscribed or circumscribed spheres, ellipsoids etc. (see T. Zamfirescu [30], P. Gruber [9] and A. Zucco [49]), the study of the shadow boundary (see P. Gruber and H. Sorger [15], T. Zamfirescu [39],[42]) and curvature properties of convex surfaces (see T. Zamfirescu [28],[29],[38] and R. Schneider [22]). Thus, the few achievements of the past look tiny compared with the big tasks of the future.†

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