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A DESCRIPTIVE CHARACTERIZATION OF THE GENERALIZED RIEMANN INTEGRAL

A function f is Denjoy-Perron integrable on $[a,b]$ if and only if there exists an ACG_* function F on $[a,b]$ such that $F' = f$ almost everywhere on $[a,b]$. In this paper we present a similar result (Theorem 4) for the generalized Riemann integral using a different notion of absolute continuity. See also the paper by C. Seng in this volume.

We will assume familiarity with the definitions of the Denjoy-Perron and generalized Riemann integrals. Throughout this paper $\mathcal{P}$ will denote a finite collection of non-overlapping tagged intervals in $[a,b]$. For $\mathcal{P} = \{(t_i, [c_i, d_i]) : 1 \le i \le N\}$, we will write

$$f(\mathcal{P}) = \sum_{i=1}^{N} f(t_i)(d_i - c_i), \quad F(\mathcal{P}) = \sum_{i=1}^{N} \big(F(d_i) - F(c_i)\big), \quad \text{and} \quad \mu(\mathcal{P}) = \sum_{i=1}^{N} (d_i - c_i).$$

This is an abuse of notation, but it is quite convenient. Let δ be a positive function defined on $[a,b]$. We say that $\mathcal{P}$ is subordinate to δ if $[c_i, d_i] \subset (t_i - \delta(t_i), t_i + \delta(t_i))$ for each i and that $\mathcal{P}$ is subordinate to δ on $[a,b]$ if in addition $\mathcal{P}$ is a partition of $[a,b]$. Given a set E and a point t, let $\rho(t,E)$ be the distance from t to E, $\mathcal{C}E$ be the complement of E, and $\overline{E}$ be the closure of E.

DEFINITION 1: Let $F : [a,b] \to R$ and let $E \subset [a,b]$. The function F is AC_δ on E if for each $\epsilon > 0$ there exist a positive number η and a positive function δ on E such that $|F(\mathcal{P})| < \epsilon$ whenever $\mathcal{P}$ is subordinate to δ, all of the tags of $\mathcal{P}$ are in E, and $\mu(\mathcal{P}) < \eta$. The function F is ACG_δ on E if E can be written as a countable union of sets on each of which the function F is AC_δ.

LEMMA 2: Suppose that $F : [a,b] \to R$ is ACG_δ on $[a,b]$ and let $E \subset [a,b]$. If $\mu(E) = 0$, then for each $\epsilon > 0$ there exists a positive function δ on E such that $|F(\mathcal{P})| < \epsilon$ whenever $\mathcal{P}$ is subordinate to δ and all of the tags of $\mathcal{P}$ are in E.

PROOF: Let $E = \cup_n E_n$ where the E_n's are disjoint and F is AC_δ on each E_n. Let $\epsilon > 0$. For each n there exist a positive function δ_n on E_n and a positive number η_n such that $|F(\mathcal{P})| < \epsilon/2^n$ whenever $\mathcal{P}$ is subordinate to δ_n, all of the tags of $\mathcal{P}$ are in E_n, and $\mu(\mathcal{P}) < \eta_n$. For each n choose an open set O_n such that $E_n \subset O_n$ and $\mu(O_n) < \eta_n$. Let $\delta(t) = \min\{\delta_n(t), \rho(t, \mathcal{C}O_n)\}$ for $t \in E_n$. Suppose that $\mathcal{P}$ is subordinate to δ and that all of the tags of $\mathcal{P}$ are in E. Let $\mathcal{P}_n$ be the subset of $\mathcal{P}$ that has tags in E_n. Note that $\mu(\mathcal{P}_n) < \eta_n$ and compute

$$|F(\mathcal{P})| \le \sum_n |F(\mathcal{P}_n)| < \sum_n \epsilon/2^n < \epsilon.$$

This completes the proof.

LEMMA 3: Suppose that $f:[a,b]\to R$ and let $E\subset[a,b]$. If $\mu(E)=0$, then for each $\epsilon>0$ there exists a positive function δ on E such that $|f(\mathcal{P})|<\epsilon$ whenever $\mathcal{P}$ is subordinate to δ and all of the tags of $\mathcal{P}$ are in E.

PROOF: For each positive integer n, let $E_n=\{t\in E: n-1\le|f(t)|<n\}$ and let $\epsilon>0$. For each n choose an open set O_n such that $E_n\subset O_n$ and $\mu(O_n)<\epsilon/n2^n$. Let $\delta(t)=\rho(t,\mathcal{C}O_n)$ for $t\in E_n$. Suppose that $\mathcal{P}$ is subordinate to δ and that all of the tags of $\mathcal{P}$ are in E. Let $\mathcal{P}_n$ be the subset of $\mathcal{P}$ that has tags in E_n and compute

$$|f(\mathcal{P})|\le\sum_n|f(\mathcal{P}_n)|<\sum_n n\mu(O_n)<\sum_n\epsilon/2^n<\epsilon.$$

This completes the proof.

THEOREM 4: A function $f:[a,b]\to R$ is generalized Riemann integrable on $[a,b]$ if and only if there exists an ACG_δ function F on $[a,b]$ such that $F'=f$ almost everywhere on $[a,b]$.

PROOF: Suppose first that f is generalized Riemann integrable on $[a,b]$ and let $F(t)=\int_a^t f$. Then (see [1] for instance) $F'=f$ almost everywhere on $[a,b]$. For each positive integer n, let $E_n=\{t\in[a,b]: n-1\le|f(t)|<n\}$. Fix n and let $\epsilon>0$. Since f is generalized Riemann integrable on $[a,b]$, there exists a positive function δ on $[a,b]$ such that $|f(\mathcal{P})-F(\mathcal{P})|<\epsilon$ whenever $\mathcal{P}$ is subordinate to δ on $[a,b]$. Let $\eta=\epsilon/n$. Suppose that $\mathcal{P}$ is subordinate to δ, all of the tags of $\mathcal{P}$ are in E_n, and $\mu(\mathcal{P})<\eta$. Then using Henstock's Lemma, we obtain

$$|F(\mathcal{P})|\le|F(\mathcal{P})-f(\mathcal{P})|+|f(\mathcal{P})|<\epsilon+n\eta=2\epsilon.$$

Hence, the function F is AC_δ on E_n and it follows that F is ACG_δ on $[a,b]$.

Now suppose that there exists an ACG_δ function F on $[a,b]$ such that $F'=f$ almost everywhere on $[a,b]$. Let $E=\{t\in[a,b]: F'(t)\ne f(t)\}$. Let $\epsilon>0$. For each $t\in[a,b]-E$ choose $\delta(t)>0$ so that $|F(s)-F(t)-f(t)(s-t)|<\epsilon|s-t|$ whenever $|s-t|<\delta(t)$. By the previous two lemmas, we can define $\delta(t)>0$ on E so that $|f(\mathcal{P})|<\epsilon$ and $|F(\mathcal{P})|<\epsilon$ whenever $\mathcal{P}$ is subordinate to δ and all of the tags of $\mathcal{P}$ are in E. This defines a positive function δ on $[a,b]$. Suppose that $\mathcal{P}$ is subordinate to δ on $[a,b]$. Let $\mathcal{P}_E$ be the subset of $\mathcal{P}$ that has tags in E and let $\mathcal{P}_d=\mathcal{P}-\mathcal{P}_E$. We then have

$$|f(\mathcal{P})-F(\mathcal{P})|\le|f(\mathcal{P}_d)-F(\mathcal{P}_d)|+|f(\mathcal{P}_E)|+|F(\mathcal{P}_E)|<\epsilon(b-a)+\epsilon+\epsilon.$$

Therefore, the function f is generalized Riemann integrable on $[a,b]$ and $\int_a^b f=F(b)-F(a)$.

We conclude this paper by giving a proof that a function is ACG_δ on $[a,b]$ if and only if it is ACG_* on $[a,b]$. Let $\omega(F,[c,d])$ denote the oscillation of F on $[c,d]$.

THEOREM 5: If F is ACG_* on $[a,b]$, then F is ACG_δ on $[a,b]$.

PROOF: It is sufficient to prove that F is AC_δ on a closed set E if F is AC_* on E. Without loss of generality, we may assume that $a,b\in E$. Let $[a,b]-E=\cup_k(a_k,b_k)$ and let $\epsilon>0$. Since F is AC_*

on E, there exists a positive integer K such that $\sum_K^\infty \omega(F,[a_k,b_k]) < \epsilon$. Let $A = \bigcup_{k=1}^{K-1}\{a_k,b_k\}$. Since F is continuous, there exists a positive function δ_1 on A such that $|F(\mathcal{P})| < \epsilon$ whenever $\mathcal{P}$ is subordinate to δ_1 and all of the tags of $\mathcal{P}$ are in A. Let

$$\delta(t) = \begin{cases} \delta_1(t), & \text{if } t \in A; \\ \rho(t,A), & \text{if } t \in E - A; \end{cases}$$

and choose $\eta > 0$ so that $\sum_i \omega(F,[c_i,d_i]) < \epsilon$ whenever $\{[c_i,d_i]\}$ is a finite collection of non-overlapping intervals that have endpoints in E and satisfy $\sum_i (d_i - c_i) < \eta$. Suppose that $\mathcal{P}$ is subordinate to δ, all of the tags of $\mathcal{P}$ are in E, and $\mu(\mathcal{P}) < \eta$. We may assume that all of the tags are endpoints. Let $\mathcal{P}_a$ be the subset of $\mathcal{P}$ that has tags in A, let $\mathcal{P}_0$ be the subset of $\mathcal{P} - \mathcal{P}_a$ for which both endpoints belong to E, let $\mathcal{P}_1$ be the subset of $\mathcal{P} - \mathcal{P}_a$ for which the left endpoint does not belong to E, and let $\mathcal{P}_2$ be the subset of $\mathcal{P} - \mathcal{P}_a$ for which the right endpoint does not belong to E. Let $\mathcal{P}_0 = \{(t_i,[c_i,d_i])\}$ and compute

$$|F(\mathcal{P}_0)| \le \sum_i |F(d_i) - F(c_i)| \le \sum_i \omega(F,[c_i,d_i]) < \epsilon.$$

Let $\mathcal{P}_1 = \{(s_j,[u_j,v_j])\}$. For each j there exists a unique $k_j \ge K$ such that $a_{k_j} < u_j < b_{k_j}$. Hence,

$$\begin{aligned} |F(\mathcal{P}_1)| \le \sum_j |F(v_j) - F(u_j)| &\le \sum_j |F(v_j) - F(b_{k_j})| + \sum_j |F(b_{k_j}) - F(u_j)| \\ &\le \sum_j \omega(F,[b_{k_j},v_j]) + \sum_j \omega(F,[a_{k_j},b_{k_j}]) \\ &< \epsilon + \epsilon = 2\epsilon. \end{aligned}$$

Similarly $|F(\mathcal{P}_2)| < 2\epsilon$. We thus have

$$|F(\mathcal{P})| \le |F(\mathcal{P}_a)| + |F(\mathcal{P}_0)| + |F(\mathcal{P}_1)| + |F(\mathcal{P}_2)| < \epsilon + \epsilon + 2\epsilon + 2\epsilon = 6\epsilon.$$

Hence, the function F is AC_δ on E.

Recall that a function satisfies condition (N) if it maps sets of measure zero to sets of measure zero. It is well-known (see [2]) that a continuous BVG_* function is ACG_* if and only if it satisfies condition (N).

THEOREM 6: If F is ACG_δ on $[a,b]$, then F is ACG_* on $[a,b]$.

PROOF: It is easy to verify that the function F is continuous on $[a,b]$. Let $[a,b] = \cup_j B_j$ where F is AC_δ on each B_j. It is sufficient to prove that F is BVG_* and satisfies condition (N) on each B_j. To this end, fix j and let $E = B_j$.

Since F is bounded on $[a,b]$ and AC_δ on E, there exist a positive number M and a positive function δ on E such that $\sum_i |F(d_i) - F(c_i)| < M$ whenever $\mathcal{P} = \{(t_i,[c_i,d_i])\}$ is subordinate to δ and all of its tags are in E. For each positive integer n, let $E_n = \{t \in E : \delta(t) \ge 1/n\}$ and note that $E = \cup_n E_n$. Fix n and for each integer i, let $E_n^i = E_n \cap [i/n,(i+1)/n)$. Let $\{[c_k,d_k]\}$ be a finite collection of non-overlapping intervals that have endpoints in E_n^i. For each k choose points s_k, t_k in $[c_k, d_k]$ such that $|F(t_k) - F(s_k)| = \omega(F, [c_k, d_k])$. The tagged intervals $(c_k, [c_k, s_k])$ and $(c_k, [c_k, t_k])$ are subordinate to δ and have tags in E. Hence,

$$\sum_k \omega(F, [c_k, d_k]) \leq \sum_k |F(t_k) - F(c_k)| + \sum_k |F(c_k) - F(s_k)| < 2M.$$

This shows that the function F is BV_* on E_n^i. It follows easily that F is BVG_* on E.

Now let A be a subset of E with $\mu(A) = 0$ and let $\epsilon > 0$. Since F is AC_δ on E, there exist a positive function δ on E and a positive number η such that $\sum_i |F(d_i) - F(c_i)| < \epsilon/3$ whenever $\mathcal{P} = \{(t_i, [c_i, d_i])\}$ is subordinate to δ, all of the tags of $\mathcal{P}$ are in E, and $\mu(\mathcal{P}) < \eta$. Let G be an open set such that $A \subset G$ and $\mu(G) < \eta$. Define a positive function δ_1 on E by

$$\delta_1(t) = \begin{cases} \delta(t), & \text{if } t \in E - A; \\ \min\{\delta(t), \rho(t, \mathcal{C}G)\}, & \text{if } t \in A. \end{cases}$$

Suppose that F is not constant on $[c,d]$ if $A \cap (c,d) \neq \emptyset$. Let

$$\mathcal{I} = \bigcup_{t \in A} \{F([u,v]) : t - \delta_1(t) < u < t < v < t + \delta_1(t)\}.$$

Since F is continuous, the collection $\mathcal{I}$ is a Vitali cover of $F(A)$. By the Vitali Covering Lemma, there exists a finite collection $\{[u_i, v_i] : 1 \leq i \leq N\}$ of disjoint intervals such that $\mu^*(F(A)) \leq \sum_i \mu(F([u_i, v_i])) + \epsilon/3$. For each i, choose $t_i \in A$ such that $t_i - \delta_1(t_i) < u_i < t_i < v_i < t_i + \delta_1(t_i)$ and $m_i, M_i \in [u_i, v_i]$ such that $\mu(F([u_i, v_i])) = F(M_i) - F(m_i)$. Let J_i be the interval with endpoints t_i, m_i and let K_i be the interval with endpoints t_i, M_i. The collections $\{(t_i, J_i) : 1 \leq i \leq N\}$ and $\{(t_i, K_i) : 1 \leq i \leq N\}$ are subordinate to δ and have tags in $A \subset E$, and the sum of the lengths of the intervals in each collection is less than η. Therefore,

$$\mu^*(F(A)) \leq \sum_i \mu(F([u_i, v_i])) + \epsilon/3 \leq \sum_i |F(M_i) - F(t_i)| + \sum_i |F(t_i) - F(m_i)| + \epsilon/3 < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we conclude that $\mu^*(F(A)) = 0$.

To complete the proof, we must consider the case in which the function F is constant on some interval $[c,d]$ for which $A \cap (c,d) \neq \emptyset$. Let $\{I_n\}$ be the sequence of all open intervals in (a,b) with rational endpoints such that $A \cap I_n \neq \emptyset$ and F is constant on $\overline{I_n}$. Let $A_1 = A - \cup_n I_n$ and note that $A_1 \cap (c,d) \neq \emptyset$ implies that F is not constant on $[c,d]$. By the above argument $\mu^*(F(A_1)) = 0$. Since $A = \cup_n (A \cap I_n) \cup A_1$ and $\mu^*(F(A \cap I_n)) = 0$ for all n, we find that $\mu^*(F(A)) = 0$. Hence, the function F satisfies condition (N) on E.

REFERENCES

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