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Functions with all singular sets of Hausdorff dimension bigger than one

Introduction. To obtain an integration process which leads to a very general divergence theorem W. F. Pfeffer [P] introduced the c-integral. The domain of this integration is the family of sets with bounded variation (BV sets), [Fe], [G]. During the definition of the $c-$integral first an averaging process, called $v-$integral, is defined on BV sets. Then using an extension method due to Mařik the $v-$integral is extended to the $c-$integral. The extension is necessary because the v-integral is not additive. In fact there exists a BV set $H \subset [0,1]^2$ and a function f defined on $[0,1]^2$ such that f is $v-$integrable on H, and $[0,1]^2 \setminus H$ but not on $[0,1]^2$.

To keep the notation as simple as possible and to avoid technical difficulties instead of BV sets we shall use BVS sets, that is, unions of finitely many squares and points. In this case it is obvious what we mean by the perimeter and the essential boundary of a BVS set.

Since the structure of the BVS sets is quite simple, our example might be useful in other generalizations of the Lebesgue integral.

When dealing with generalized integrals, one has to find out whether Riemann type sums for f can be well approximated by a suitable primitive function. In the definition of the $v-$integral a thin set, that is a set of small Hausdorff dimension, is dropped and one has to check the accuracy of the above approximation modulo thin sets. This motivates our definition of singular sets (Definition 2). For a given function f defined on a set A one can ask, whether there are sets S such that the singular behavior of f is concentrated on these sets, that is, Riemannian sums on $A \setminus S$ can be approximated by a suitable primitive. Plainly $S = A$ is always a good singular set. In the definitions of generalized integrals one has to find small singular sets, but as our Theorem demonstrates this is not always possible.

Although we do not discuss the $v-$integral in this paper, we remark that it is obvious that the $v-$integral of f can be evaluated on $[0,1]^2 \setminus H$. From our Theorem it follows that this averaging process does not integrate f on $[0,1]^2$. To show that this averaging process integrates f on H one have to state and prove a version of Lemma 1 that is valid for BV sets instead of squares. There is another possibility, namely, preserving the essential properties of our example one can modify the definition of f as it was done by W. F. Pfeffer in [P]. In that case one can apply Theorem 5.19. of [P] to show that f is $v-$integrable.

We also want to remark that modifying our construction (using grids different from the ternary, taking a countable union etc.) one can obtain examples where the Hausdorff dimension of all the singular sets equals two.

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Preliminaries. In this paper we work in the Euclidean plane $\mathbf{R}^2$. If a set A consists of countably many rectangles then we denote by $|A|$, $||A||$, and $d(A)$ respectively the Lebesgue measure, the perimeter, and the diameter of A. A BVS set consists of finitely many squares and points. We define the regularity of a BVS set A by

$$r(A) = \begin{cases} \frac{|A|}{d(A)||A||} & \text{if } d(A)||A|| > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 1. Suppose that the function f is defined on the square J of side a. The square J is divided into 2^{2m} subsquares of sides $a \cdot 2^{-m}$, denoted by K_k $(k = 1, \dots, 2^{2m})$ where the indices k are chosen so that if two squares K_{k_1} and K_{k_2} have a common side then one of the indices k_1, k_2 is odd and the other is even, that is, if we color the squares

K_k by black or white according to the parity of k then we get a $2^m \times 2^m$ chessboard. Suppose also that for $x \in int(K_k)$, $f(x) = c$ if $k = 2\ell$ ($\ell = 1, ..., 2^{2m-1}$) and $f(x) = -c$ if $k = 2\ell - 1$ ($\ell = 1, ..., 2^{2m-1}$).

Then for any square $S \subset J$ we have

$$\frac{|\int_S f|}{||S||} \leq \frac{c}{4} \cdot \frac{a}{2^m}.$$

Proof of Lemma 1. We may suppose that $J = [0, a]^2$ and $S = [x_0, x_0+b] \times [y_0, y_0+b]$. Then from the definition of f it follows that $f(x + \frac{a}{2^m}, y) = -f(x, y)$, $|f(x, y)| = c$ for $(x, y), (x + \frac{a}{2^m}, y) \in \cup_{k=1}^{2^{2m}} int(K_k)$. Thus

$$|\int_{x_0}^{x_0+b} f(x, y) d(x)| = |g(y)| < c \cdot \frac{a}{2^m}.$$

Again from the definition of f it follows that $g(y + \frac{a}{2^m}) = -g(y)$ and $|g(y)| = c' < c\frac{a}{2^m}$ for almost every y such that $y_0 < y < y_0 + b - \frac{a}{2^m}$. Thus

$$|\int_{y_0}^{y_0+b} g(y) dy| < c' \frac{a}{2^m} < c(\frac{a}{2^m})^2.$$

If $b \geq \frac{a}{2^m}$ then $||S|| \geq 4\frac{a}{2^m}$ and hence

$$\frac{|\int_S f|}{||S||} \leq \frac{c(\frac{a}{2^m})^2}{4\frac{a}{2^m}} = \frac{c}{4} \cdot \frac{a}{2^m}.$$

If $b \leq \frac{a}{2^m}$ then $|\int_S f| \leq c \cdot |S| = c \cdot b^2$ and

$$\frac{|\int_S f|}{||S||} \leq \frac{c \cdot b^2}{4 \cdot b} = \frac{cb}{4} \leq c\frac{a}{2^m}.$$

q.e.d.

Definition 1. Suppose that $H \subset [0, 1]^2$ is an open set. We say that f is $\mathcal{BVSF}$ on H, ($f \in \mathcal{BVSF}(H)$), if f is defined on $[0, 1]^2$, $f(x) = 0$ for $x \in [0, 1]^2 \setminus H$, f is Lebesgue integrable on every BVS subset of H and for every $\epsilon > 0$ there is an $\eta > 0$ such that $|\int_B f| < \epsilon$ for each $B \in BVS$, $B \subset H$ with $|B| < \eta$ and $||B|| < \frac{1}{\epsilon}$.

This $\mathcal{BVSF}$ property is roughly equivalent to the fact that the (generalized) integral function of f is continuous in the BVS sense.

Definition 2. Suppose that $H \subset [0,1]^2$ is an open set and $f \in \mathcal{BVSF}(H)$. A set A is a regular set for f, if there is a $\delta(x) > 0$ gauge function defined on $[0,1]^2$ such that

$$\sum_{j=1}^{\infty} \left| f(x_j)|B_j| - \int_{B_j} f(x_j) \right| < \infty \tag{1}$$

for every sequence B_j of pairwise disjoint BVS sets such that $x_j \in A$, $x_j \in B_j$, $d(B_j) < \delta(x_j)$, $r(B_j) > \epsilon > 0$ and f is Lebesgue integrable on B_j. A set T is a singular set for f, $(T \in S(f))$, if $[0,1]^2 \setminus T$ is regular.

Theorem. There exists a real number $\gamma > 1$, an open set $H \subset [0,1]^2$ of finite perimeter and a function $f \in \mathcal{BVSF}(H)$ such that the Hausdorff dimension of every singular set of f is bigger than γ.

Proof. First we shall do a Cantor triadic set like construction in the plane, that is at each step we remove the middle $\frac{1}{3} \times \frac{1}{3}$ open square of the former ones. Put $I_{1,0} = [0,1]^2$. Then $I_{1,0} \setminus (\frac{1}{2} - \frac{1}{6}, \frac{1}{2} + \frac{1}{6})^2$ can be divided into 8 closed squares each of sides $\frac{1}{3}$. We denote these squares by $I_{j,1}$ $(j = 1, \ldots, 8)$. If the squares $I_{j,m}$ $(j = 1, \ldots, 8^m)$ of sides $\frac{1}{3^m}$ are given then remove the middle subsquare of sides $\frac{1}{3^{m+1}}$ from each of these squares and take the remaining 8^{m+1} squares of sides $\frac{1}{3^{m+1}}$ and denote them by $I_{j,m+1}$ $(j = 1, \ldots, 8^{m+1})$.

We also put $J_{1,0} = (\frac{1}{2} - \frac{1}{2 \cdot 12}, \frac{1}{2} + \frac{1}{2 \cdot 12})^2$, that is the centers of $J_{1,0}$ and $I_{1,0}$ coincide and the sides of $J_{1,0}$ are of length $\frac{1}{12}$. For $j = 1, \ldots, 8^m$ denote by $J_{j,m}$ the open square with length of sides 12^{-m-1} and concentric with $I_{j,m}$. Obviously $J_{j,m}$ will be a subset of the middle square removed from $I_{j,m}$ at the $(m+1)$'st step of the above definition of $I_{j,m+1}$ $(j = 1, \ldots, 8^{m+1})$.

Put $H = \cup_{j,m} J_{j,m}$.

Plainly

$$\sum_{j,m} ||J_{j,m}|| = \sum_{m=0}^{\infty} \sum_{j=1}^{8^m} ||J_{j,m}|| = \sum_{m=0}^{\infty} 8^m \cdot 4 \cdot 12^{-m-1} < \infty,$$

thus the perimeter of H is finite.

Divide $J_{j,m}$ into 2^{2m} subsquares of sides $12^{-m-1} 2^{-m}$, denote them by $K_{j,m,k}$ where $k = 1, \ldots, 2^{2m}$. Choose the indices k so that if two squares K_{j,m,k_1} and K_{j,m,k_2} have a common side then one of the indices k_1, k_2 is odd and the other is even, that is, if we color the squares $K_{j,m,k}$ by black or white according to the parity of k then we get a $2^m \times 2^m$ chessboard.

For $x \in int(K_{j,m,k})$ put $f(x) = 2 \cdot 12^{2m+2} \cdot 8^{-m}$ if $k = 2\ell$ ($\ell = 1, ..., 2^{2m-1}$) and $f(x) = -2 \cdot 12^{2m+2} \cdot 8^{-m}$ if $k = 2\ell - 1$ ($\ell = 1, ..., 2^{2m-1}$), otherwise put $f(x) = 0$.

First we prove that $f \in \mathcal{BVSF}(H)$. It is obvious that f is defined on $[0,1]^2$ and $f(x) = 0$ for $x \in [0,1]^2 \setminus H$. Since every BVS subset of H can be covered by finitely many squares of the form $J_{j,m}$ the function f is Lebesgue integrable on every BVS subset of H. If $S \subset J_{j,m}$ is a square then from Lemma 1 (with $a = 12^{-m-1}$, $c = 2 \cdot 12^{2m+2} \cdot 8^{-m}$) it follows that

$$\frac{|\int_S f|}{||S||} \le \frac{2 \cdot 12^{2m+2} \cdot 8^{-m} \cdot 12^{-m-1}}{4 \cdot 2^{-m}} = \frac{3^m}{4^m} \cdot 6$$

thus

$$|\int_S f| \le 6 \cdot (\frac{3}{4})^m ||S||.$$

For a given $\epsilon > 0$ choose M such that

$$6 \cdot (\frac{3}{4})^M \cdot \frac{1}{\epsilon} < \frac{\epsilon}{2}.$$

Since f is bounded on $\cup_{m=0}^{M-1} \cup_{j=1}^{8^m} J_{j,m}$ we can choose $\eta > 0$ such that if

$$B' \subset \cup_{m=0}^{M-1} \cup_{j=1}^{8^m} J_{j,m} \qquad \text{and} \qquad |B'| < \eta$$

then $|\int_{B'} f| < \frac{\epsilon}{2}$. If $|B| < \eta$, and $B \subset H$ is a BVS set then $B = B' \cup B''$ where B', B'' are BVS sets and $B' \subset \cup_{m=0}^{M-1} \cup_{j=1}^{8^m} J_{j,m}$ and $B'' \subset \cup_{m=M}^{\infty} \cup_{j=1}^{8^m} J_{j,m}$. Thus

$$|\int_B f| \le |\int_{B'} f| + |\int_{B''} f| \le \frac{\epsilon}{2} + 6 \cdot (\frac{3}{4})^M ||B''|| \le \frac{\epsilon}{2} + 6 \cdot (\frac{3}{4})^M ||B|| \le \frac{\epsilon}{2} + 6 \cdot (\frac{3}{4})^M \frac{1}{\epsilon} < \epsilon$$

proving that $f \in \mathcal{BVSF}(H)$.

Suppose that $T \in S(f)$ and put $A = [0,1]^2 \setminus T$. Choose $\delta(x) > 0$ fulfilling the requirements of Definition 2. Put

$$C = \cap_{m=0}^{\infty} \cup_{j=1}^{8^m} I_{j,m}.$$

We say that δ is big in the square $I_{j,m}$ if there exists an $x \in I_{j,m} \cap A \cap C$ such that $\delta(x) > d(I_{j,m})$, otherwise we say that δ is small in $I_{j,m}$. Suppose that $0 < \rho' < 1$ and there are more than $\rho' 8^m$ different squares $I_{j,m}$ such that δ is big in $I_{j,m}$. Denote the set of the corresponding indices j by Φ_m, therefore $\#\Phi_m > \rho' 8^m$. Choose $x_{j,m} \in I_{j,m} \cap A \cap C$ for each $j \in \Phi_m$ such that $\delta(x_{j,m}) > d(I_{j,m})$. Denote by $I'_{j,m}$ the middle $\frac{1}{3}$ square removed from $I_{j,m}$ at the $(m+1)$'st step of the definition of the squares $I_{j,m+1}$. Since the side of $I'_{j,m}$ is $\frac{1}{3^{m+1}}$ and the side of $J_{j,m} \subset I'_{j,m}$ is 12^{-m-1} one can choose a square $L_{j,m} \subset I'_{j,m} \setminus J_{j,m}$ of side $3^{-m-3} < \frac{1}{2}(3^{-(m+1)} - 12^{-m-1})$. Put

$$B_{j,m} = \{x_j\} \cup \{L_{j,m}\} \cup \cup_{\ell=1}^{2^{2m-1}} K_{j,m,2\ell}.$$

Thus $B_{j,m}$ is a BVS set and $|B_{j,m}| > |L_{j,m}| = (\frac{1}{3^{m+3}})^2 = 3^{-2m-6}$, $d(B_{j,m}) < d(I_{j,m}) = 3^{-m}\sqrt{2}$ and $||B_{j,m}|| = \frac{4}{3^{m+3}} + 2^{2m-1}||K_{j,m,2\ell}|| = 4(3^{-m-3} + 2^{2m-1}12^{-m-1}2^{-m}) < 8 \cdot 3^{-m}$. Therefore

$$r(B_{j,m}) = \frac{|B_{j,m}|}{d(B_{j,m})||B_{j,m}||} > \frac{3^{-2m-6}}{3^{-m}\sqrt{2} \cdot 8 \cdot 3^{-m}} > 10^{-4} = \epsilon > 0.$$

Since $x_{j,m} \in C \subset [0,1]^2 \setminus H$ we have $f(x_{j,m}) = 0$. Also $f(x) = 0$ for $x \in L_{j,m}$ since $L_{j,m} \subset I'_{j,m} \setminus J_{j,m}$. Thus

$$\sum_{j \in \Phi_m} |f(x_{j,m})|B_{j,m}| - \int_{B_{j,m}} f| = \sum_{j \in \Phi_m} |\int_{B_{j,m}} f| = \sum_{j \in \Phi_m} |\sum_{\ell=1}^{2^{2m-1}} \int_{K_{j,m,2\ell}} f| =$$

$$\sum_{j \in \Phi_m} 2^{2m-1} \cdot 2 \cdot 12^{2m+2} 8^{-m} 12^{-2m-2} 2^{-2m} \geq \sum_{j \in \Phi_m} 8^{-m} \geq \rho' 8^m 8^{-m} = \rho'. \qquad (2)$$

Suppose that $0 < \rho < 1$ and there exist infinitely many different m's such that $\#\Phi_m > \rho 8^m$. Then inductively one can choose a sequence $m_1 < ... < m_k < ...$ such that $\#\Phi_{m_k} > \rho 8^{m_k}$ and $\frac{\rho}{2} 8^{m_k} > \sum_{i=1}^{k-1} 8^{m_i}$. For a fixed integer k the sets B_{j,m_k} defined above are pairwise disjoint, and for any $k' < k$, $B_{j',m_{k'}} \cap B_{j,m_k}$ is either empty or $x_{j',m_{k'}} = x_{j,m_k}$. Deleting those indices j from Φ_{m_k} for which there exists $k' < k$, $j' \in \Phi_{m_{k'}}$ such that $x_{j,m_k} = x_{j',m_{k'}}$ we obtain Φ'_{m_k}. Since we delete at most $\sum_{i=1}^{k-1} 8^{m_i} < \frac{\rho}{2} 8^{m_k}$ indices from Φ_{m_k} we get that $\#\Phi'_{m_k} > \frac{\rho}{2} 8^m = \rho' 8^m$. Since the sets B_{j,m_k}, $j \in \Phi'_{m_k}$, are disjoint we obtain from (2) that

$$\sum_{j \in \Phi'_{m_k}} |f(x_{j,m_k})|B_{j,m_k}| - \int_{B_{j,m_k}} f| > \rho' = \frac{\rho}{2}.$$

This contradicts (1) in Definition 2 since we can order x_{j,m_k} and B_{j,m_k} $j \in \Phi_{m_k}$ $k = 1, 2, \ldots$ into infinite sequences x_j and B_j such that they fulfill the requirements in Definition 2 and $\sum_{j=1}^{\infty} \left| f(x_j)|B_j| - \int_{B_j} f(x_j) \right| = \infty$. Therefore we proved that for every $\rho > 0$ there exists $M_\rho > 0$ such that if $m > M_\rho$ then δ is small in $I_{j,m}$ for more than $(1-\rho)8^m$ different indices $j \in \{1, \ldots, 8^m\}$. Denote by Ψ_m the set of these indices.

Choose sequences ρ_k and m_k $(k = 1, \ldots)$ such that $\frac{1}{8} > \rho_1 > \ldots > \rho_k > \ldots$; $m_k > M_{\rho_k}$, and for every k we have $\rho_1 8^{m_k - m_1} + \rho_2 8^{m_k - m_2} + \ldots + \rho_{k-1} 8^{m_k - m_{k-1}} + \rho_k 8^{m_k} < 8^{m_k - 1}$. For any $\ell < k$ an interval I_{j,m_ℓ} covers $8^{m_k - m_\ell}$ intervals I_{j,m_k}. Put $C_1 = \cup_{j \in \Psi_{m_1}} I_{j,m_1}$ and $\Psi'_{m_1} = \Psi_{m_1}$. Given C_{k-1} put $C_k = C_{k-1} \cap \cup_{j \in \Psi_{m_k}} I_{j,m_k}$ and denote by $\Psi'_{m_k} \subset \Psi_{m_k}$ the set of the indices j for which $I_{j,m_k} \subset C_k$. Plainly $\#\Psi'_{m_k} > (1-\rho_k)8^{m_k} - \rho_{k-1}8^{m_k - m_{k-1}} - \ldots - \rho_2 8^{m_k - m_2} - \rho_1 8^{m_k - m_1} > (1 - 8^{-1})8^{m_k} > 8^{m_k - 1}$.

Put $F = \cap_{k=1}^{\infty} C_k$. Obviously F is closed. Since $F \subset C_k$ for each k, δ is small in the squares I_{j,m_k} $(j \in \Psi'_{m_k})$. For any $x \in F$, $\delta(x) > 0$ and hence there exists k and j such that $x \in I_{j,m_k}$, $j \in \Psi'_{m_k}$ and $d(I_{j,m_k}) < \delta(x)$. Therefore $x \in T$, that is $F \subset T$.

Using comparable net measures [Fa] for the ternary grid we compute the Hausdorff dimension of F. Suppose that $\rho > 0$ and U_p $(p = 1, \ldots)$ is a ρ cover of F and the sets U_p belong to the ternary grid, that is, each U_p is of the form $[k \cdot 3^{-m_p}, (k+1) \cdot 3^{-m_p}] \times [\ell \cdot 3^{-m_p}, (l+1) \cdot 3^{-m_p}]$ with integers k and ℓ. Each set U_p has eight grid neighbors of sides 3^{-m_p}. Taking the intervals U_p and all of their neighbors we obtain the set $\{W_q : q = 1, 2, \ldots\}$ which is also a ρ cover of F, and obviously $\sum_q d(W_q)^\gamma \leq 9 \sum_p d(U_p)^\gamma$. If we expand the intervals U_p slightly we can choose open intervals $U'_p \supset U_p$ such that each U'_p is still covered by U_p and its neighbors. By the compactness of F there is a P such that $F \subset \cup_{p=1}^{P} U'_p$. Thus there exists a Q such that $\cup_{q=1}^{Q} W_q \supset \cup_{p=1}^{P} U'_p \supset F$. Recall that $F = \cap_{k=1}^{\infty} C_k$ and the sets $C_k \supset C_{k+1}$, $k = 1, 2, \ldots$, are compact. Hence the sets $C'_k = C_k \setminus \cup_{p=1}^{P} U'_p$ are also compact and from $\cup_{p=1}^{P} U'_p \supset F$ it follows that $\cap_{k=1}^{\infty} C'_k = \emptyset$. By Cantor's theorem there exists a natural number K such that $C'_K = \cap_{k=1}^{K} C'_k = \emptyset$. Choose a $k > K$ such that $3^{-m_k} \leq d(W_q)$, $q = 1, \ldots, Q$. Then $C'_k = \emptyset$, that is, $C_k \subset \cup_{p=1}^{P} U'_p \subset \cup_{q=1}^{Q} W_q$. Using that

$$C_k = \cup_{j \in \Psi'_{m_k}} I_{j,m_k}$$

and $\#\Psi'_{m_k} > 8^{m_k - 1}$ we obtain that $\cup_{q=1}^{Q} W_q$ covers more than $8^{m_k - 1}$ intervals I_{j,m_k}. It follows from the definition of the intervals I_{j,m_k} that an interval W_q of sides $m(q)$ can cover at most $8^{m_k-m(q)}$ of the intervals I_{j,m_k}. Thus

$$\sum_{q=1}^{Q}(d(W_q))^\gamma = \sum_{q=1}^{Q}(\sqrt{2}3^{-m(q)})^\gamma = \frac{(\sqrt{2})^\gamma}{8^{m_k}} \cdot \sum_{q=1}^{Q} 8^{m_k-m(q)}\left(\frac{8}{3^\gamma}\right)^{m(q)} \geq$$

$$\frac{(\sqrt{2})^\gamma}{8^{m_k}} \cdot 8^{m_k-1} = \frac{(\sqrt{2})^\gamma}{8}$$

whenever $\frac{8}{3^\gamma} \geq 1$, that is, $\gamma \leq \frac{\log 8}{\log 3}$. Therefore

$$\sum_p d(U_p)^\gamma \geq \frac{1}{9}\sum_q d(W_q)^\gamma \geq \frac{(\sqrt{2})^\gamma}{72}.$$

Thus we proved that every singular set T has Hausdorff dimension bigger than $\frac{\log 8}{\log 3} = \gamma$.

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