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INTERSECTION CONDITIONS FOR SOME DENSITY AND I-DENSITY LOCAL SYSTEMS

1. Introduction

By a local system we mean a family $\mathbf{S} = \{\mathbf{S}(x);\ x \in R\}$ such that $\mathbf{S}(x)$ is a nonempty collection of subsets of the real line, with the following properties:

(i) $\{x\} \notin \mathbf{S}(x)$,

(ii) if $S \in \mathbf{S}(x)$, then $x \in S$,

(iii) if $S \in \mathbf{S}(x)$ and $S' \supset S$, then $S' \in \mathbf{S}(x)$,

(iv) if $S \in \mathbf{S}(x)$ and $\delta > 0$, then $S \cap (x-\delta, x+\delta) \in \mathbf{S}(x)$.

Let $\mathbf{S}$ be a local system. We say that a function f is ($\mathbf{S}$)-continuous at the point x_o if, for each $\varepsilon > 0$, the set $\{x\ ;\ |f(x) - f(x_o)| < \varepsilon\}$ belongs to the family $\mathbf{S}(x_o)$.

The notion of a local system is the basis for considerations in B. Thomson's book [T2]. This notion makes it possible to unify the way of formulating and demonstrating many properties related to generalized limits, continuity and derivatives.

The proofs of the majority of theorems in [T2] are based on the observation that the local system considered satisfies some "intersection condition" or some "porosity condition". In this paper we shall study "intersection conditions" for several local systems related to the approximate system $\mathbf{S}_{ap}$ (see [T2], p. 22) and to its category analogue which was introduced by W. Wilczyński. We start with the following definition

DEFINITION 1. We say that a local system $\mathbf{S}$ satisfies an intersection condition of the form "$S_x \cap S_y \neq \emptyset$" ("$S_x \cap S_y \cap (x,y) \neq \emptyset$", etc.) if, for each choice of sets $\{S_x;\ x \in R\}$ with $S_x \in \mathbf{S}(x)$, there is a positive function δ on R such that $S_x \cap S_y \neq \emptyset$ ($S_x \cap S_y \cap (x,y) \neq \emptyset$, etc.) whenever $0 < y - x < \min\{\delta(x),\delta(y)\}$.

The intersection conditions considered most frequently in Thomson's book and other papers are those of the form "$S_x \cap S_y \neq \emptyset$", "$S_x \cap S_y \cap (x,y) \neq \emptyset$", "$S_x \cap S_y \cap [x,y] \neq \emptyset$" and of the form "$S_x \cap S_y \cap [x - \lambda(y - x),x] \neq \emptyset$ or/and $S_x \cap S_y \cap [y,y + \lambda(y - x)] \neq \emptyset$", where $\lambda \geq 1$ is a parameter. The reader who is interested in the applications of the intersection conditions can find them in [T2]. We cite here only one example of such an application.

THEOREM A ([T2], Theorem 33.1). If a local system $\mathbf{S}$ satisfies an intersection condition of the form "$S_x \cap S_y \neq \emptyset$", then every ($\mathbf{S}$)-continuous function is in the first class of Baire.

DEFINITION 2. We say that a local system $\mathbf{S}$ satisfies a strong intersection condition of the form "$S_x \cap S_y \neq \emptyset$" ("$S_x \cap S_y \cap (x,y) \neq \emptyset$", etc.) if, for any $x \in R$ and $S \in \mathbf{S}(x)$ there is a positive number $\delta(x,S)$ such that $S_x \cap S_y \neq \emptyset$ ($S_x \cap S_y \cap (x,y) \neq \emptyset$, etc.) whenever $S_x \in \mathbf{S}(x)$, $S_y \in \mathbf{S}(y)$ and $0 < y - x < \min\{\delta(x,S_x),\delta(y,S_y)\}$.

The strong intersection condition is related to the "essential radius condition ([Z], p. 321) and "O'Malley's condition"

([T1], p. 292). Evidently, if $\mathbf{S}$ satisfies a strong intersection condition of any form, then it satisfies an intersection condition of the same form.

The theorem whose proof (in the implicit form) can be found in the paper by E. Łazarow and W. Wilczyński may serve as an example of an application of a strong intersection condition.

THEOREM B ([ŁW], Theorem 2). If a local system $\mathbf{S}$ satisfies a strong intersection condition of the form "$S_x \cap S_y \cap (x,y) \neq \emptyset$" and f is a finite ($\mathbf{S}$)-derivative, then f is a selective derivative.

2. Density systems

If E is a measurable subset of the real line, then $|E|$ denotes the Lebesgue measure of E. By a right upper (lower) density of E at a point x we mean

$$d^+(E,x) = \limsup_{h\to 0^+} \frac{|E \cap (x,x+h)|}{h}$$

$$\left(d_+(E,x) = \liminf_{h\to 0^+} \frac{|E \cap (x,x+h)|}{h}\right).$$

In the same way we define the left densities.

In this paragraph we shall study intersection conditions for local systems $\mathbf{S}_o^+, \mathbf{S}^+$ and $\mathbf{S}$ which we define in the following way:

$A \in \mathbf{S}_o^+(x) \iff x \in A$ and there is a measurable set $E \subset A$ such that $d^+(E,x) = 1$ and $d_+(E,x) > 0$,

$A \in \mathbf{S}^+(x) \iff x \in A$ and there is a measurable set $E \subset A$ such that, for each measurable $F \in \mathbf{S}_o^+(x)$, we have $E \cap F \in \mathbf{S}_o^+(x)$,

$\mathbf{S}(x) = \mathbf{S}^+(x) \cap \mathbf{S}^-(x)$.

The definitions of $\mathbf{S}$ and $\mathbf{S}^+$ were introduced by D.N. Sarkhel and A.K. De ([SD]). In their terminology, $A \in \mathbf{S}(x)$ $(A \in \mathbf{S}^+(x))$ if and only if $R \setminus A$ is sparse at x (sparse at x on the right). The properties of sparse sets are also examined in [F1].

Evidently, $\mathbf{S}^+(x) \subset \mathbf{S}_o^+(x)$. Example 3 from [F1] implies that this inclusion is strict. It is also obvious that if x is a right density point of a set A, then $A \in \mathbf{S}^+(x)$. Example 3.1 from [SD] shows that there is a measurable set $A \in \mathbf{S}^+(x)$ such that x is not a right density point of A.

In M. Sinharoy's paper [S] there is a proof of the theorem below. We repeat it here because the original theorem of Sinharoy is formulated differently and includes some additional considerations which obscure the essence of the relationships we are interested in.

THEOREM 1 ([S], Theorem 2). The local system $\mathbf{S}_o^+$ satisfies a strong intersection condition of the form "$S_x \cap S_y \neq \emptyset$".

P r o o f. Let $x \in R$ and $S \in \mathbf{S}_o^+(x)$. We must define $\delta(x,S)$. We may assume that S is measurable. There are two real numbers $r_x \in (0,\frac{1}{2})$ and $t_x \in (x,x+\frac{1}{2})$ such that

(1) $$\frac{|S \cap (x,t)|}{|(x,t)|} > 2r_x \quad \text{for} \quad t \in (x,x+1),$$

(2) $$\frac{|S \cap (x,t_x)|}{|(x,t_x)|} > 1 - r_x^2.$$

Put $\delta(x,S) = r_x^2(t_x - x)$.

Let $S_x \in \mathbf{S}(x)$, $S_y \in \mathbf{S}(y)$ and $0 < y - x < \min\{\delta(x,S_x), \delta(y,S_y)\}$. We must show that $S_x \cap S_y \neq \emptyset$. We consider two cases:

(a) $r_x \leq r_y$. From (2) it follows that

$$\frac{|S_x \cap (x,t_x)|}{|(x,t_x)|} > 1 - r_x^2 > 1 - r_x.$$

Moreover, $t_x - y = (t_x - x) - (y - x) > (t_x - x) - \delta(x,S_x) = (t_x - x)(1 - r_x^2) > \frac{1}{2}(t_x - x)$. Hence $t_x - x < 2(t_x - y)$ and (1) guarantees

$$\frac{|S_y \cap (x,t_x)|}{|(x,t_x)|} \geq \frac{|S_y \cap (y,t_x)|}{2|(y,t_x)|} > r_y.$$

From the above inequalities we obtain $\dfrac{|S_x \cap S_y \cap (x,t_x)|}{|(x,t_x)|} > (1 - r_x) + r_y - 1 \geq 0$ and, consequently, $S_x \cap S_y \neq \emptyset$.

b) $r_x \geq r_y$. From (1) it follows that

$$\frac{|S_x \cap (x,t_y)|}{|(x,t_y)|} > 2r_x.$$

As $t_y - x = (t_y - y) + (y - x) < (t_y - y) + \delta(y,S_y) = (t_y - y)(1 + r_y^2)$, therefore

$$\frac{|S_y \cap (x,t_y)|}{|(x,t_y)|} \geq \frac{1}{1 + r_y^2} \frac{|S_y \cap (y,t_y)|}{|(y,t_y)|}$$

$$> \frac{1 - r_y^2}{1 + r_y^2} > 1 - r_y.$$

Thus, from the above inequalities we conclude that

$$\frac{|S_x \cap S_y \cap (x,t_y)|}{|(x,t_y)|} > (1 - r_y) + r_x - 1 \geq 0$$

and, consequently, $S_x \cap S_y \neq \emptyset$.

COROLLARY. The local system S^+ satisfies a strong intersection condition of the form "$S_x \cap S_y \neq \emptyset$".

From Theorem 1 it follows that the local system S satisfies an intersection condition of the form "$S_x \cap S_y \neq \emptyset$". We shall show that this system satisfies none of the remaining intersection conditions mentioned in the introduction. To do this, we present an example which will also be used in the sequel.

EXAMPLE 1. Let λ, x be real numbers and $\lambda \geq 0$. Put

$$S_x = \begin{cases} \{x\} \cup \bigcup_{n=1}^{\infty} (x + x^{(n+1)^2}, x + x^{n^2+1}) \cup \bigcup_{n=1}^{\infty} (x - x^{n^2+1}, x - x^{(n+1)^2}); \\ \qquad x \in (0, \frac{1}{4(1+\lambda)}), \\ (x - 1, x + 1); \quad x \notin (0, \frac{1}{4(1+\lambda)}). \end{cases}$$

We show that, for the family $\{S_x;\ x \in R\}$, <u>there is no</u> positive function δ such that

$$(*) \qquad S_x \cap S_y \cap [x - \lambda(y - x), x + \lambda(y - x)] \neq \emptyset \quad \text{whenever}$$

$$0 < y - x < \min\{\delta(x), \delta(y)\}.$$

Suppose the contrary, i.e. there is a positive function δ for which condition (*) holds. We can assume that $\delta(x) < \frac{1}{4(1+\lambda)}$ for every x. Put $E_{ni} = (\frac{i}{n}, \frac{i+1}{n}] \cap \{x;\ \frac{1}{n} < \delta(x) \leq \frac{1}{n-1}\}$. Then $(0, \frac{1}{4(1+\lambda)}) \subset \bigcup_{n=1}^{\infty} \bigcup_{i=0}^{\infty} E_{ni}$. Hence there are two integers n and i such that $|E_{ni} \cap (0, \frac{1}{4(1+\lambda)})|^* > 0$. ($|\cdot|^*$ denotes the outer Lebesgue measure). Let $x < \frac{i+1}{n}$ be a point of outer density 1 of E_{ni}, belonging to $E_{ni} \cap (0, \frac{1}{4(1+\lambda)})$.

Thus there exists $z \in (x, \frac{i+1}{n})$ which fulfils

(1) $$\frac{|(x,y) \cap E_{ni}|^*}{|(x,y)|} > \frac{7}{8} \quad \text{for} \quad y \in (x,z).$$

Let $k \geq 2$ be an integer such that $x + x^{k^2} \leq z$. From (1) it follows that there is $y \in (x + \frac{7}{8}\frac{x^{k^2}}{1+\lambda}, x + \frac{x^{k^2}}{1+\lambda}) \cap E_{ni}$. Thus we have

(2) $$y + \lambda(y - x) < x + \frac{x^{k^2}}{1+\lambda} + \lambda\frac{x^{k^2}}{1+\lambda} = x + x^{k^2},$$

(3) $$x - \lambda(y - x) > x - \frac{\lambda x^{k^2}}{1+\lambda} = x + \frac{x^{k^2}}{1+\lambda} - x^{k^2} > y - y^{k^2}.$$

Moreover,

$$y^{k^2+1} < (x + x^{k^2})^{k^2+1} = x^{k^2+1}(1 + x^{k^2-1})^{k^2+1}$$

$$< x^{k^2+1}(1 + 2^{k^2+1}x^{k^2-1}) = x^{k^2+1}(1 + 4(2x)^{k^2-1})$$

$$\leq x^{k^2+1}(1 + \frac{4}{2^3}) = \frac{3}{2}x^{k^2+1}$$

and, consequently,

(4) $$y - y^{k^2+1} > x + \frac{7}{8}\frac{x^{k^2}}{1+\lambda} - \frac{3}{2}x^{k^2+1} = x + x^{k^2}(\frac{7}{8(1+\lambda)} - \frac{3x}{2})$$

$$\geq x + x^{k^2}(\frac{7}{8(1+\lambda)} - \frac{3}{8(1+\lambda)}) > x + x^{k^2+1}.$$

Since

$$S_x \cap [x + x^{k^2+1}, x + x^{k^2}] = \emptyset,$$

$$S_y \cap [y - y^{k^2}, y - y^{k^2+1}] = \emptyset,$$

therefore (2), (3) and (4) guarantee that $S_x \cap S_y \cap [x - \lambda(y - x), y + \lambda(y - x)] \subset S_x \cap S_y \cap ((y - y^{k^2}, y - y^{k^2+1}) \cup (x + x^{k^2+1}, x + x^{k^2})) = \emptyset$.

On the other hand, from $x,y \in E_{ni}$ it follows that $0 < y - x < \frac{1}{n} < \min\{\delta(x),\delta(y)\}$. Hence by (*), we get a contradiction.

An immediate consequence of Example 1 is

THEOREM 2. There is no $\lambda \geq 0$ for which the local system $\mathbf{S}$ satisfies the parametric intersection condition:

$$"S_x \cap S_y \cap [x - \lambda(y - x), y + \lambda(y - x)] \neq \emptyset".$$

P r o o f. Let the sets S_x be defined as in Example 1. D.N. Sarkhel and A.K. De proved that, for each $c > 1$, the set $E_c = R \setminus \bigcup_{n=1}^{\infty} (c^{-n^2-1}, c^{-n^2})$ belongs to $\mathbf{S}^+(0)$ (see [SD], Example 3.1 and Theorem 3.1). Hence $S_x \in \mathbf{S}(x)$ for every x. Thus the conclusion of the theorem follows from Example 1.

We end our considerations with the following theorem which was proved in [F2].

THEOREM 3. For each $\alpha \in (0,1)$, the local system $\mathbf{S}^+$ satisfies a strong intersection condition of the form

$$"S_x \cap S_y \cap (y, y + (y - x)^{\alpha}) \neq \emptyset".$$

3. I-density systems

In the sequel, the letter I will denote the family of meager sets on the real line.

DEFINITION 3. We say that x is an I-density point of a set E having the Baire property if, for each sequence $\{t_n\}$ of real numbers tending to infinity, there exists a subsequence $\{t_{n_k}\}$ such that $\chi_{t_{n_k}(E - x) \cap [-1,1]} \xrightarrow[k\to\infty]{} \chi_{[-1,1]}$ I-a.e. (i.e. the set of point for which the convergence does not hold is of the first category).

If, in the preceding definition, one replaces the interval $[-1,1]$ by $[0,1]$ ($[-1,0]$), then one obtains the definition of a right (left) I-density point of E.

DEFINITION 4. Let E be a set having the Baire property.

(a) We say that x is a right upper I-density point of E if there exists a sequence $\{t_n\}$ of real numbers tending to infinity with $\chi_{t_n(E - x) \cap [0,1]} \xrightarrow[n\to\infty]{} \chi_{[0,1]}$ I-a.e. We denote this point by $d_I^+(E,x) = 1$. Otherwise we write $d_I^+(E,x) < 1$.

(b) We say that x is a right lower I-dispersion point of E if there exists a sequence $\{t_n\}$ of real numbers tending to infinity with $\chi_{t_n(E - x) \cap [0,1]} \xrightarrow[n\to\infty]{} 0$ I-a.e. We denote this point by $d_{I^+}(E,x) = 0$. Otherwise we write $d_{I^+}(E,x) > 0$.

Evidently, $d_I^+(E,x) = 1$ if and only if $d_{I^+}(R \setminus E,x) = 0$. In the same way as in Definition 4 one can define the left - hand analogues of these notions.

The definition of an I-density point is due to W. Wilczyński (see [PWW]). The remaining definitions and notations come from [F1].

In the sequel, we shall often use the following equivalences.

REMARK 1. Let $\{E_n\}$ be a sequence of sets.

(a) $\chi_{E_n} \xrightarrow[n\to\infty]{} 1$ I-a.e. $\iff \liminf E_n$ is a residual set.

(b) $\chi_{E_n} \xrightarrow[n\to\infty]{} 0$ I-a.e. $\iff \limsup E_n$ is of the first category.

Now, we define the local systems $\bar{S}_I$, $\bar{S}_I^+$, S_I, S_I^+, S_{oI}^+. We shall study the intersection conditions for these systems.

$A \in \bar{S}_I(x) \iff x \in A$ and there is a set $E \subset A$ having the Baire property such that x is an I-density point of E.

$A \in \bar{\mathbf{S}}_I^+(x) \iff x \in A$ and there is a set $E \subset A$ having the Baire property such that x is a right I-density point of E.

$A \in \mathbf{S}_{oI}^+ \iff x \in A$ and there is a set $E \subset A$ having the Baire property such that $d_I^+(E,x) = 1$ and $d_{I^+}(E,x) > 0$.

$A \in \mathbf{S}_I^+(x) \iff x \in A$ and there is a set $E \subset A$ having the Baire property such that, for each set $F \in \mathbf{S}_{oI}^+(x)$ having the Baire property, we have $E \cap F \in \mathbf{S}_{oI}^+(x)$.

$\mathbf{S}_I(x) = \mathbf{S}_I^+(x) \cap \mathbf{S}_I^-(x)$.

In [F1] the sets which are complements of these from $\mathbf{S}_I(x)$ ($\mathbf{S}_I^+(x)$) are called I-sparse at x (I-sparse at x on the right), while, for $E \in \mathbf{S}_I(x)$, x is said to be a *I-density point of E.

Obviously, $\mathbf{S}_I^+(x) \subset \mathbf{S}_{oI}^+(x)$. Proposition 2 from [F1] guarantees that $\bar{\mathbf{S}}_I^+(x) \subset \mathbf{S}_I^+(x)$. Examples 2 and 3 from [F1] show that both the inclusions are strict.

E. Łazarow and W. Wilczyński proved the following

THEOREM 4 ([ŁW], Theorem 2). The local system $\bar{\mathbf{S}}_I$ satisfies a strong intersection condition of the form

$$\text{"}S_x \cap S_y \cap (x,y) \neq \emptyset\text{"}.$$

Using the notion of an I-density point, W. Wilczyński defined the so-called I-density topology (see [PWW]). The neighbourhoods of x in this topology are the sets from $\bar{\mathbf{S}}_I(x)$. In [Z] L. Zajíček generalized the notion of the I-density topology for an arbitrary metric space. He showed that the system which consists of all neighbourhoods of any point of the space satisfies a strong intersection condition of the form "$S_x \cap S_y \neq \emptyset$" ([Z], Theorem 3).

We shall prove a result which is stronger than Theorem 4. We need Theorem 1 from [Ł] which we shall reformulate in our notation.

LEMMA 1. If x is a right (left) I-density point of a set E having the Baire property, then, for each natural number n, there exist a natural number $k = k(n,x)$ and a positive number $\eta = \eta(n,x)$ such that, for any $h \in (0,\eta)$ and $i \in \{1,\dots,n\}$, there is $j \in \{1,\dots,k\}$ such that E is residual in $(x + (\frac{i-1}{n} + \frac{j-1}{kn})h,\ x + (\frac{i-1}{n} + \frac{j}{kn})h)$ (resp. in $(x - (\frac{i-1}{n} + \frac{j}{kn})h,\ x - (\frac{i-1}{n} + \frac{j-1}{kn})h)$).

An easy consequence of Lemma 1 is Lemma 2. A direct proof of this lemma can be found in [PWW, Lemma 1].

LEMMA 2. If x is an I-density point of a set E having the Baire property, then, for each natural number k, there exists a positive number η such that, for any $h \in (0,\eta)$ and $i \in \{-k+1,\dots,k\}$, $E \cap (x + \frac{i-1}{k}h,\ x + \frac{i}{k}h)$ is of the second category.

THEOREM 5. If $0 \le \lambda_1 < \lambda_2 \le 1$, then the local system $\bar{\mathbf{S}}_I$ satisfies a strong intersection condition of the form

$$\text{"}S_x \cap S_y \cap (x + \lambda_1(y - x),\ x + \lambda_2(y - x)) \neq \emptyset\text{"}.$$

P r o o f. Let $x \in R$ and $S \in \bar{\mathbf{S}}_I(x)$. We can assume that S has the Baire property. Let n be a natural number such that $n(\lambda_2 - \lambda_1) \ge 2$. Lemma 1 implies that there exist a natural number $k(n,x)$ and a positive number $\eta(n,x)$ such that

(1) for any $h \in (0,\eta(n,x))$ and $i \in \{-n+1,\dots,n\}$, there is $j \in \{1,\dots,k(n,x)\}$ such that S is residual in

$$(x + (\frac{i-1}{n} + \frac{j-1}{nk(n,x)})h,\ x + (\frac{i-1}{n} + \frac{j}{nk(n,x)})h).$$

From Lemma 2 it follows that, for $k_x = 2nk(n,x)$, there exists a positive number η_{k_x} such that

(2) for any $h \in (0,\eta_{k_x})$ and $i \in \{-2nk(n,x)+1,\dots,2nk(n,x)\}$, the set

$$S \cap (x + \frac{i-1}{2nk(n,x)}h,\ x + \frac{i}{2nk(n,x)}h)$$

is of the second category.

Put $\delta(x,S) = \min\{\eta(n,x),\eta_{k_x}\}$.

Let $S_x \in \bar{\mathbf{S}}_I(x)$, $S_y \in \bar{\mathbf{S}}_I(y)$ and $0 < y-x < \min\{\delta(x,S_x), \delta(y,S_y)\}$. We show that $S_x \cap S_y \cap (x + \lambda_1(y-x), x+\lambda_2(y-x)) \neq \emptyset$. By the definition of n, there is $i_o \in \{1,\dots,n\}$ with

$$(x + \frac{i_o-1}{n}(y-x),\ x + \frac{i_o}{n}(y-x))$$

$$\subset (x + \lambda_1(y-x),\ x + \lambda_2(y-x)).$$

Put $h = y - x$ and assume that $k(n,x) \le k(n,y)$ (if $k(n,x) \ge k(n,y)$, then the proof is similar).

As $h = y - x < \eta(n,x)$, condition (1) guarantees that there exists $j_o \in \{1,\dots,k(n,x)\}$ such that S_x is residual in

$$I_{j_o} = (x + (\frac{i_o-1}{n} + \frac{j_o-1}{nk(n,x)})h,\ x + (\frac{i_o-1}{n} + \frac{j_o}{nk(n,x)})h).$$

By condition (2) and $h < \eta_{k_y}$, we have that, for each $t \in \{-2nk(n,y)+1,\dots,2nk(n,y)\}$, the set

$$S \cap J_t = S \cap (y + \frac{t-1}{2nk(n,y)}h,\ y + \frac{t}{2nk(n,y)}h)$$

is of the second category.

Since $|J_t| = \frac{h}{2nk(n,y)} \le \frac{h}{2nk(n,x)} = \frac{1}{2}|I_{j_o}|$ and $y - h \le x + \frac{i_o - 1}{n}h < x + \frac{i_o}{n}h \le y$, there is $t_o \in \{-nk(n,x) + 1,\dots,0\}$ with $J_{t_o} \subset I_{j_o}$.

As $I_{j_o} \subset (x + \frac{i_o-1}{n}h, x + \frac{i_o}{n}h) \subset (x + \lambda_1(y-x), x + \lambda_2(y-x))$, we conclude that $S_x \cap S_y \cap (x + \lambda_1(y-x), x + \lambda_2(y-x)) \neq \emptyset$.

We shall now examine intersection condition for $\bar{\mathbf{S}}_I^+$.

THEOREM 6. If $0 \le \lambda_1 < \lambda_2 < \infty$, then the local system $\bar{\mathbf{S}}_I^+$ satisfies a strong intersection condition of the form

$$\text{"}S_x \cap S_y \cap (y + \lambda_1(y - x), y + \lambda_2(y - x)) \neq \emptyset\text{"}.$$

P r o o f. Let $x \in R$ and $S \in \bar{\mathbf{S}}_I^+(x)$. Without loss of generality we may assume that S has the Baire property, $\lambda_1 = \frac{i_o-1}{k}$ and $\lambda_2 = \frac{i_o}{k}$ for any natural numbers i_o and k. Put $n = k + i_o$. By Lemma 1, there exist a natural number $k(n,x)$ and a positive number $\eta(n,x)$ such that

(1) for any $h \in (0,\eta(n,x))$ and $i \in \{1,\dots,n\}$, there is $j \in \{1,\dots,k(n,x)\}$ such that S is residual in

$$(x + (\frac{i-1}{n} + \frac{j-1}{nk(n,x)})h,\ x + (\frac{i-1}{n} + \frac{j}{nk(n,x)})h).$$

From Lemma 2 it follows that, for $k_x = 2nk(n,x)$, there exists a positive number η_{k_x} such that

(2) for any $h \in (0,\eta_{k_x})$ and $i \in \{1,\dots,2nk(n,x)\}$, the set

$$S \cap (x + \frac{i-1}{2nk(n,x)}h,\ x + \frac{i}{2nk(n,x)}h)$$

is of the second category.

Put $\delta(x,S) = \frac{1}{n}\min\{\eta(n,x),\eta_{k_x}\}$.

Let $S_x \in \bar{\mathbf{S}}_I^+(x)$, $S_y \in \bar{\mathbf{S}}_I^+(y)$ and $0 < y - x < \min\{\delta(x,S_x), \delta(y,S_y)\}$. We show that $S_x \cap S_y \cap (y+\lambda_1(y-x), y+\lambda_2(y-x)) \neq \emptyset$.

Put $h = \frac{n}{k}(y - x)$. We consider two cases.

(a) $k(n,x) \le k(n,y)$.

Since $h = \frac{n}{k}(y - x) < n\delta(x,S_x) \le \eta(n,x)$, condition (1) implies that there is $j_o \in \{1,\dots,k(n,x)\}$ such that S_x is residual in

$$I_{j_o} = (x + (\frac{n-1}{n} + \frac{j_o-1}{nk(n,x)})h,\ x + (\frac{n-1}{n} + \frac{j_o}{nk(n,x)})h).$$

On the other hand, we have $h < n\delta(y,S_y) \le \eta_{k_y}$. Thus, from (2) it follows that, for each $t \in \{1,\dots,2nk(n,y)\}$, the set

$$S_y \cap J_t = S_y \cap (y + \frac{t-1}{2nk(n,y)}h,\ y + \frac{t}{2nk(n,y)}h)$$

is of the second category.

As $|J_t| = \frac{h}{2nk(n,y)} \le \frac{h}{2nk(n,x)} = \frac{1}{2}|I_{j_o}|$ and $y \le x + \frac{n-1}{n}h < x + h < y + h$, there is $t_o \in \{1,\dots,2nk(n,y)\}$ with $J_{t_o} \subset I_{j_o}$. Moreover, since

$$\begin{aligned} I_{j_o} &\subset (x + \frac{n-1}{n}h,\ x + h) = \\ &= (x + \frac{k+i_o-1}{k}(y-x),\ x + \frac{k+i_o}{k}(y-x)) \\ &= (y + \lambda_1(y - x),\ y + \lambda_2(y - x)), \end{aligned}$$

we conclude that $S_x \cap S_y \cap (y + \lambda_1(y - x), y + \lambda_2(y - x)) \neq \emptyset$.

(b) $k(n,x) \geq k(n,y)$.

Since $h < \eta(n,y)$, condition (1) guarantees that there exists $j_o \in \{1,\dots,k(n,y)\}$ such that S_y is residual in

$$I_{j_o} = \left(y + \left(\frac{i_o-1}{n} + \frac{j_o - 1}{nk(n,y)}\right)h,\ y + \left(\frac{i_o-1}{n} + \frac{j_o}{nk(n,y)}\right)h\right).$$

By condition (2) and $h < \eta_{k_x}$, we have that, for each $t \in \{1,\dots,2nk(n,x\}$, the set

$$S_x \cap J_t = S_x \cap \left(x + \frac{t-1}{2nk(n,x)}h,\ x + \frac{t}{2nk(n,x)}h\right)$$

is of the second category.

As $x + h = y + \frac{i_o}{k}(y - x) = y + \frac{i_o}{n}h > y + \frac{i_o-1}{n}h > x$ and $|J_t| = \frac{h}{2nk(n,x)} \leq \frac{h}{2nk(n,y)} = \frac{1}{2}|I_{j_o}|$, there is $t_o \in \{1,\dots, 2nk(n,x)\}$ with $J_{t_o} \subset I_{j_o}$. Moreover, since

$$\begin{aligned} I_{j_o} &\subset \left(y + \frac{i_o-1}{n}h,\ y + \frac{i_o}{n}h\right) \\ &= (y + \lambda_1(y - x),\ y + \lambda_2(y - x)), \end{aligned}$$

we conclude that $S_x \cap S_y \cap (y + \lambda_1(y - x),\ y + \lambda_2(y - x)) \neq \emptyset$.

COROLLARY 1. (a) The local system $\bar{\mathbf{S}}_I^+$ satisfies a strong intersection condition of the form

"$S_x \cap S_y \cap (y,\ y + (y - x)) \neq \emptyset$".

(b) The local system $\bar{\mathbf{S}}_I$ satisfies a strong intersection condition of the form

"$S_x \cap S_y \cap (x - (y - x), x) \neq \emptyset$

and

$S_x \cap S_y \cap (y, y + (y - x)) \neq \emptyset$".

We say that a function f is I-approximately continuous on the right if it is $(\overline{\mathbf{S}}_I^+)$-continuous (see [PWW]).

COROLLARY 2. If f is I-approximately continuous on the right, then it is in the first class of Baire.

In order to investigate the local systems $\mathbf{S}_I$ and $\mathbf{S}^+_I$, we start with proving the analogues of Lemmas 1 and 2 for these systems.

LEMMA 3. If G is open and I-sparse at 0 on the right (i.e. $R \setminus G \in \mathbf{S}_I^+(0)$), then there exist a natural number k_o and a real number $\eta_o > 0$ such that, for each $h \in (0, \eta_o)$, there is $i \in \{1, \dots, k_o\}$ with $(\frac{i-1}{k_o} h, \frac{i}{k_o} h) \cap G = \emptyset$.

P r o o f. Suppose that G is an open set which does not possess the property of the lemma. We show that G is not I-sparse at 0 on the right. From our assumption it follows that there is a decreasing sequence $\{h_k\}$ such that, for every k, we have

(1) $h_k < \frac{1}{k^2} h_{k-1}$,

(2) $(\frac{i-1}{k} h_k, \frac{i}{k} h_k) \cap G \neq \emptyset$ for $i \in \{1, \dots, k\}$.

Put $b_k = \frac{1}{k-1} h_{k-1}$, $I_k = (h_k, b_k)$, $B_k = I_k \cap \bigcup_{i=1}^{\infty} ((2i-1)h_k, 2ih_k)$ and $B = \bigcup_{k=2}^{\infty} B_k$. As in [F1, Example 3] it is easy to check that $d_{I^+}(B,0) = 0$ by examining $\chi_{t_nB} \cap [0,1]$ for $t_n = \frac{1}{h_n}$.

To prove that G is not I-sparse at 0 on the right it is sufficient to show that $d_{I^+}(B \cup G,0) > 0$ (see [F1], Theorem 2), i.e. that, for each sequence $\{t_n\}$ tending to infinity,

(3) $\chi_{t_n(B \cup G)} \cap [0,1]$ does not converge to 0 I-a.e.

Let $\{t_n\}$ be a sequence tending to infinity. Obviously, it is enough to show that condition (3) is true for some subsequence of $\{t_n\}$. Let $x_n = \frac{1}{t_n}$ and let $\{h_{k_n}\}$ be a subsequence of $\{h_n\}$ for which $4h_{k_n} \le x_n < 4h_{k_n-1}$. Replacing the sequences $\{t_n\}$ and $\{h_{k_n}\}$ by subsequences (if it is necessary), we can make one of the following conditions hold:

(a) there is $M \ge 4$ such that $\frac{x_n}{h_{k_n}} \le M$ for every n,

(b) $\frac{x_n}{h_{k_n}} \xrightarrow[n\to\infty]{} \infty$ and there is $M > 0$ such that $\frac{x_n}{b_{k_n}} \le M$ for every n,

(c) $\frac{x_n}{b_{k_n}} \xrightarrow[n\to\infty]{} \infty$.

We consider each case separately.

(a) For each natural number n, we have

$$(0,1) \cap t_nB \supset (0,1) \cap t_nB_{k_n}$$
$$\supset (0,1) \cap t_n(h_{k_n}, 2h_{k_n}) = \left(\frac{h_{k_n}}{x_n}, 2\frac{h_{k_n}}{x_n}\right).$$

This means that $(0,1) \cap t_nB$ contains an interval of length $\frac{1}{M}$.

Put $a_i = \frac{i}{2M}$ for $i = 0,1,\ldots,2M$. Since each set t_nB contains at least one of intervals (a_i,a_{i+1}), there is $i_o \in \{0,\ldots,2M-1\}$ such that (a_{i_o},a_{i_o+1}) is contained in infinitely many sets t_nB. This proves condition (3).

(b) For each natural number n, we have

$$(0,\tfrac{1}{M}) \cap t_nB \supset (0,\tfrac{1}{M}) \cap t_nB_{k_n}$$
$$= (0,\tfrac{1}{M}) \cap \bigcup_{i=1}^{\infty} \left(\frac{2(i-1)h_{k_n}}{x_n}, \frac{2ih_{k_n}}{x_n}\right).$$

Thus, from $\frac{h_{k_n}}{x_n} \xrightarrow[n\to\infty]{} 0$ it follows that the set $\bigcup_{n=r}^{\infty} t_nB$ is dense in $(0,\frac{1}{M})$ for each natural r. Hence $\limsup t_nB$ is residual in $(0,\frac{1}{M})$, so (3) holds.

(c) Let p be a natural number. If n is a sufficiently large natural number, then $\frac{x_n}{b_{k_n}} > 2p$ and, consequently,

$$\left|\left(\frac{i-1}{p}\,x_n, \frac{i}{p}\,x_n\right)\right| = \frac{x_n}{p} > 2b_{k_n} = \frac{2h_{k_n-1}}{k_n-1}.$$

From (2) it follows that, for a sufficiently large n and each $i \in \{1,\ldots,p\}$, $G \cap (\frac{i-1}{p}x_n, \frac{i}{p}x_n) \neq \emptyset$. Thus $(0,1) \cap \bigcup_{n=r}^{\infty} t_nG$ is a dense open subset of $(0,1)$ for every natural r and, therefore, $\limsup t_nG$ is residual in $(0,1)$. This proves (3).

An easy consequence of Lemma 3 is

LEMMA 4. If $E \in \mathbf{S}_I^+(x)$ has the Baire property, then there exists a natural number k such that, for each $h \in (0,1)$, there is $i \in \{1,\dots,k\}$ for which E is residual in $(x + \frac{i-1}{k}h, x + \frac{i}{k}h)$.

P r o o f. We assume that $x = 0$. As E has the Baire property, $R \setminus E = G \,\Delta\, P$ where G is open and I-sparse at 0 on the right, and P is of the first category. Let k_o and η_o be the numbers guaranteed by Lemma 3 and let $n > \frac{1}{\eta_o}$ be a natural number. Put $k = 2k_o n$ and let $h_1 \in (0,1)$. Putting $h = h_1\eta_o$, we obtain $h < \eta_o$; so, by Lemma 3, there is $i \in \{1,\dots,k_o\}$ such that $(\frac{i-1}{k_o}h, \frac{i}{k_o}h) \cap G = \emptyset$. Since $\frac{h_1}{k} = \frac{h}{\eta_o k} < \frac{h}{2k_o}$, there exists $j \in \{1,\dots,k\}$ with $(\frac{j-1}{k}h_1, \frac{j}{k}h_1) \subset (\frac{i-1}{k_o}h, \frac{i}{k_o}h)$. Hence E is residual in $(\frac{j-1}{k}h_1, \frac{j}{k}h_1)$.

REMARK 2. There is an open set G such that $d_I^+(G,0) < 1$, $d_{I^+}(G,0) = 0$ and G does not satisfy the conclusion of Lemma 3. It is sufficient to put $G = \operatorname{Int} A$ where A is defined in Example 3 in [F1].

LEMMA 5. If E has the Baire property and $d_I^+(E,x) = 1$, then for each natural number k and each positive real number η, there is $h \in (0,\eta)$ such that, for each $i \in \{1,\dots,k\}$, $E \cap (x + \frac{i-1}{k}h, x + \frac{i}{k}h)$ is of the second category.

P r o o f. We assume that $x = 0$. Since $d_I^+(E,0) = 1$, there is a sequence $\{t_n\}$ tending to infinity with

$\chi_{t_nE} \cap [0,1] \xrightarrow[n\to\infty]{} \chi_{[0,1]}$ I-a.e. Suppose to the contrary that there exist a natural number k_o and a real number $\eta_o > 0$ such that, for each $h \in (0,\eta_o)$, there is $i \in \{1,\ldots,k_o\}$ for which $E \cap (\frac{i-1}{k_o}h, \frac{i}{k_o}h)$ is of the first category. Putting $h = \frac{1}{t_n}$, we find a sequence $\{i_n\}$ of numbers from the set $\{1,\ldots,k_o\}$, such that $t_nE \cap (\frac{i_n-1}{k_o}, \frac{i_n}{k_o})$ is of the first category. Let $\{k_n\}$ be an increasing sequence of natural numbers such that $\{i_{k_n}\}$ is constant, i.e. $i_{k_n} = i_o$ for every n. Then $t_{k_n}E \cap (\frac{i_o-1}{k_o}, \frac{i_o}{k_o})$ is of the first category for every n. This contradicts the assumption that $\chi_{t_nE} \cap [0,1] \xrightarrow[n\to\infty]{} \chi_{[0,1]}$ I-a.e.

THEOREM 7. The local system $\mathbf{S}_I^+$ satisfies a strong intersection condition of the form "$S_x \cap S_y \neq \emptyset$".

P r o o f. Let $x \in R$ and $S \in \mathbf{S}_I^+(x)$. We may assume that S has the Baire property. By Lemma 4, there is a natural number k_x such that

(1) for each $h \in (0,1)$, there is $i \in \{1,\ldots,k_x\}$ such that S is residual in $(x + \frac{i-1}{k_x}h,\ x + \frac{i}{k_x}h)$.

By Lemma 5, for $k = 2k_x$ and $\eta = 1$, there is $h_x \in (0,1)$ such that

(2) $S_x \cap (x + \frac{i-1}{2k_x}h_x,\ x + \frac{i}{2k_x}h_x)$ is of the second category for each $i \in \{1,\ldots,2k_x\}$.

Put $\delta(x,S) = \dfrac{h_x}{2k_x}$.

Let $S_x \in \mathbf{S}_I^+(x)$, $S_y \in \mathbf{S}_I^+(y)$ and $0 < y - x < \min\{\delta(x,S_x), \delta(y,S_y)\}$. We must show that $S_x \cap S_y \neq \emptyset$. We need consider two cases:

(a) $k_x \le k_y$. Put $h = h_y$ in (1). There exists $i_o \in \{1,\dots,k_x\}$ such that S_x is residual in $I_{i_o} = (x + \frac{i_o - 1}{k_x} h_y, x + \frac{i_o}{k_x} h_y)$.

From (2) we obtain that $S_y \cap J_i = S_y \cap (y + \frac{i-1}{2k_y} h_y, y + \frac{i}{2k_y} h_y)$ is of the second category for each $i \in \{1,\dots,2k_y\}$.

As $|J_i| = \frac{h_y}{2k_y} \le \frac{h_y}{2k_x} = \frac{1}{2}|I_{i_o}|$ and $y - x < \frac{h_y}{2k_y} \le \frac{1}{2}|I_{i_o}|$, $J_{i_1} \subset I_{i_o}$ for some $i_1 \in \{1,\dots,2k_y\}$. Thus S_x is residual in J_{i_1}, and consequently, $S_x \cap S_y \supset S_x \cap S_y \cap J_{i_1} \neq \emptyset$.

(b) $k_x \ge k_y$. Put $h = h_x$ in (1). There exists $j_o \in \{1,\dots,k_y\}$ such that S_y is residual in $I_{j_o} = (y + \frac{j_o - 1}{k_y} h_x, y + \frac{j_o}{k_y} h_x)$.

From (2) it follows that $S_x \cap J_j = S_x \cap (x + \frac{j-1}{2k_x} h_x, x + \frac{j}{2k_x} h_x)$ is of the second category for each $j \in \{1,\dots,2k_x\}$.

As $|J_j| = \frac{h_x}{2k_x} \le \frac{h_x}{2k_y} = \frac{1}{2}|I_{j_o}|$ and $(y + h_x) - (x + h_x) = y - x < \frac{h_x}{2k_x} = \frac{1}{2}|I_{j_o}|$, $J_{j_1} \subset I_{j_o}$ for some $j_1 \in \{1,\dots,2k_x\}$. Hence $S_x \cap S_y \supset S_x \cap S_y \cap J_{j_1} \neq \emptyset$.

We say that a function f is I-proximally continuous on the right if it is $(\mathbf{S}_I^+)$-continuous (see [F1]).

COROLLARY. If f is I-proximally continuous on the right, then it is in the first class of Baire.

It is unsolved if the local system S_{oI}^{+} satisfies any intersection condition (see Remark 2).

We end the paper with an analogue of Theorem 2 for the local system S_I.

THEOREM 8. There is no $\lambda \geq 0$ for which the local system S_I satisfies the parametric intersection condition:

$$"S_x \cap S_y \cap [x - \lambda(y - x), x + \lambda(y - x)] \neq \emptyset".$$

P r o o f. Let the sets S_x be defined as in Example 1. In [F1, Example 2] it was proved that, for each $c > 1$, the set $E_c = R \setminus \bigcup_{n=1}^{\infty} (c^{-n^2-1}, c^{-n^2})$ belongs to $S_I^{+}(0)$. Hence $S_x \in S_I(x)$ for every x. Thus the conclusion of the theorem follows from Example 1.

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