BOREL MEASURABLE SELECTIONS

AND

APPLICATIONS OF THE BOUNDEDNESS PRINCIPLE

by R. Daniel Mauldin and Glen A. Schlee

Department of Mathematics

University of North Texas

Denton, Texas 76203-5116

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This paper is mainly an expanded version of the talk given by Mauldin during the real analysis conference at Michigan State University, June 14 – 17, 1989. The results of section 6 were presented by Schlee.

We wish to promote some classical and modern techniques in descriptive set theory and by the way present some selection theorems and a group of four unsolved problems.

As a starting point let $B \subset X \times Y$ and assume that each X–fiber of B, B_x is countable. By the axiom of choice, B can be expressed as the union of countably many graphs. It is a fact that if B is a Borel set (or analytic) set, then B can be expressed as the union of countably many Borel (analytic) graphs. The descriptive set theoretic techniques exposited here lie at the heart of the proofs of these facts. These techniques and the analysis of sets with countable sections form the first five sections of this paper.

In section 6, we extend some of the results obtained for sets with countable fibers to sets with compact or σ–compact fibers. We state the definitive result of Saint–Raymond, reprove a crucial part of the argument in terms of the boundedness principle and state some unsolved problems.

In sections 7 and 8, we discuss the possibility of filling up Borel sets with uncountable fibers by disjoint Borel graphs or even disjoint Borel isomorphisms and state more unsolved problems.

A fundamental tool in descriptive set theory is the first separation principle of Souslin. A less well known, but very useful tool, is Novikov's generalized first separation principle. Novikov's theorem and the modern version of the Lusin–Sierpinski index theorem: the boundedness principle for monotone coanalytic operators (or, equivalently, analytic derivations) are the basis for several deep results in selection theory. We hope to exposit their usefulness here. We also want to report on a recent result of Saint-Raymond and Debs concerning 1-1 selections that is quite intriguing. We refer to the reader the general survey of Wagner [23, 24] and the article of Levi [11] for a listing of results in the field.

1. <u>The setting and separation principles</u>. Let X and Y be Polish spaces (separable topological spaces with a compatible complete metric). Let ρ be a metric for $X \times Y$. In addition, let $\mathscr{B}(X \times Y)$ and $\mathscr{A}(X \times Y)$ denote the collection of Borel and analytic sets of $X \times Y$ respectively, let $\mathscr{G}$ denote the collection of all Borel graphs in $X \times Y$, and let $\mathscr{G}_\sigma$ denote the collection of all countable unions of elements in $\mathscr{G}$. Given $E \subset X \times Y$ and $x \in X$, we denote by E_x the set $\{ y \mid (x,y) \in E \}$. Let $\mathscr{C}$ denote the collection $\{ K \in (X \times Y) \mid \forall x\ K_x$ is compact $\}$, and let $\mathscr{C}_\sigma$ denote the collection of all countable unions of elements in $\mathscr{C}$. Also, by $\mathscr{K}(X)$, we denote the space of compact subsets of X given the exponential topology.

First Separation Principle. (Souslin, 1917 [22]) Let A and E be disjoint analytic subsets of a Polish space X. Then there are disjoint Borel sets B and D such that $A \subset B$ and $E \subset D$.

For Novikov's generalized first separation theorem two different types of proofs have been given: one, in the original style of Novikov [10,p.510] and the other by Saint-Raymond [20].

Novikov's Generalized First Separation Principle. (Novikov, 1934 [18]) If $\{A_n\}_{n=1}^{\infty}$ is a sequence of analytic subsets of a Polish space X with $\bigcap_{n=1}^{\infty} A_n = \emptyset$, then there is a sequence $\{B_n\}_{n=1}^{\infty}$ of Borel subsets of X such that $B_n \supset A_n$ and $\bigcap_{n=1}^{\infty} B_n = \emptyset$.

2. Sets with countable sections and preliminary results. Our first two theorems apply the separation principles to sets with countable sections consisting of no more than one point or else isolated points.

Theorem 1. Let A be an analytic graph in $X \times Y$. Then $A \subset G \in \mathcal{G}$. In other words, every function φ from $E \subset X$ into Y with φ (= Gr φ) an analytic subset of $X \times Y$ may be extended to a Borel measurable map $\tilde{\varphi}$ from a Borel set $\tilde{E} \supset E$ into Y.

Proof. Let $E = \mathrm{proj}_X(A)$ and $\varphi\colon E \to Y$ with Gr $\varphi = A$ (of course, E is an analytic subset of X). Note that φ is relatively Borel measurable. If U is open in Y, then

$$\varphi^{-1}(U) = \pi_X((X \times U) \cap A)$$
$$\text{and } \varphi^{-1}(Y \setminus U) = \pi_X(X \times (Y \setminus U) \cap A).$$

The first separation principle implies there is a Borel subset B of Y such that $\varphi^{-1}(U) \subset B$ and $B \cap \varphi^{-1}(Y \setminus U) = \emptyset$. Thus, $\varphi^{-1}(U) = B \cap E$. Consequently, by an extension theorem of Kuratowski [10,p.434], there is a Borel set $D \supset E$ and a Borel measurable map $\hat{\varphi}\colon D \to E$. Let $G = \mathrm{Gr}\ \hat{\varphi}$. Q.E.D.

In order to generalize the first theorem, we use Novikov's generalized first separation principle.

Theorem 2. If $A \in \mathscr{A}(X\times Y)$, $\forall x$ A_x consists of isolated points, then $A \subset G \in \mathscr{G}_\sigma$.

Proof. Let $\{V_n\}_{n=1}^\infty$ be a base for the topology on Y. For each n, let

$$T_n = \{ x \mid \mathrm{card}(V_n \cap A_x) \geq 2 \}.$$

Each T_n is analytic, since

$$T_n = \bigcup \left[\pi_X(S_n \cap (X\times V_m)) \cap \pi_X(S_n \cap (X\times V_p))\right]$$

where $S_n = (X\times V_n) \cap A$ and where the union is over all pairs (m,p) such that $V_m \cap V_p = \emptyset$. Next, for $n \geq 1$, let

$$Z_n = [(T_n\times Y)\cap A] \cup [(X\times(Y\setminus V_n))\cap A]$$

Each Z_n is clearly analytic and $\bigcap_{n=1}^\infty Z_n = \emptyset$. By Novikov's separation principle, there are Borel sets B_n such that $\bigcap_{n=1}^\infty B_n = \emptyset$ and for each n, $Z_n \subset B_n$. For each n, let

$$A_n = [(X\times Y)\setminus B_n] \cap A.$$

Note that each A_n is analytic and for each x, $\mathrm{card}(A_{nx}) \leq 1$. Thus, by Theorem 1, for each n there is $G_n \in \mathscr{G}$ such that $A_n \subset G_n$. Also note that $A = \bigcup_{n=1}^\infty A_n$. Therefore, $A \subset \bigcup_{n=1}^\infty G_n \in \mathscr{G}_\sigma$. Q.E.D.

3. Operators and the boundedness principle. In order to continue a deeper analysis of sets with countable sections we need a powerful tool. We use the *boundedness principle* for analytic derivations or monotone coanalytic operators. Let us define what this means and recall the boundedness principle. The theory of these operators as presented here is fully developed in [4]. A treatment of analytic derivations is given in [6].

By an operator on X, we mean a map from the power set $\mathscr{P}(X)$ to $\mathscr{P}(X)$. An operator Γ is said to be *monotone* if for any $K \subset M \subset X$, $\Gamma(K) \subset \Gamma(M)$. The dual operator D of an operator Γ on X is defined by

$$D(A) = X \setminus \Gamma(X \setminus A).$$

Let $A \subset X$ and let Γ be an operator on X. We define

$$\Gamma^0(A) = A,$$

$$\Gamma^{\alpha+1}(A) = \Gamma(\Gamma^{\alpha}(A)) \quad \text{for all ordinals } \alpha,$$

$$\Gamma^{\lambda}(A) = \bigcup_{\alpha<\lambda} \Gamma^{\alpha}(A) \quad \text{for limit ordinals } \lambda.$$

The set $C(\Gamma;A) = \bigcup_{\alpha} \Gamma^{\alpha}(A)$ where the union is over the set of all ordinals is called the closure of Γ on A. For some ordinal $\alpha < \mathrm{card}(X)^+$, $\Gamma^{\alpha+1}(A) = \Gamma^{\alpha}(A) = \mathrm{Cl}(\Gamma;A)$, and we denote the least such ordinal by $|\Gamma;A|$. Also, we let $|\Gamma| = |\Gamma;\emptyset|$, and we let $\mathrm{Cl}(\Gamma) = \mathrm{Cl}(\Gamma;\emptyset)$.

An operator Δ over a Polish space X is said to be Borel (or Δ^1_1) if it is defined in one of the following ways:

(a) $\Delta(K) = B$, where B is a fixed Borel subset of X;

(b) $\Delta(K) = f^{-1}(K)$, where f is a fixed Borel map from X to X;

(c) $\Delta(K) = X \setminus K$;

(d) $\Delta(K) = \Delta_1(\Delta_2(K))$, where Δ_1 and Δ_2 are previously defined Borel operators;

(e) $\Delta(K) = \bigcup_{n=1}^{\infty} \Delta_n(K)$, where the Δ_n are previously defined Borel operators.

An operator Γ over a Polish space X is analytic or Σ^1_1 (respectively coanalytic or Π^1_1) if there is a Polish space Y and a Borel operator Δ over $X \times Y$ such that for all x and K:

$$x \in \Gamma(K) \quad \text{iff} \quad (\exists y)\ (x,y) \in \Delta(K \times Y),$$

$$\text{(respectively)} \quad (\forall y)\ (x,y) \in \Delta(K \times Y).$$

Note that Γ is an analytic operator if and only if its dual is coanalytic.

Boundedness Principle for Monotone Π_1^1 *Operators.* (Cenzer and Mauldin, 1980 [4]) If Γ is a coanalytic monotone operator with closure C, on the coanalytic subset P of X, then for any analytic subset A of X with $A \subset C$, there is some countable ordinal α such that $A \subset \Gamma^{\alpha}(P)$.

By an *analytic derivation*, we mean an operator whose dual operator is monotone and coanalytic. If D is an analytic derivation, the set $\bigcap_{\alpha<\omega_1} D^{\alpha}(A)$ is called the kernel of D on A. The boundedness priniple for analytic derivations given below follows from the boundedness principle for monotone Π_1^1 operators.

Boundedness Principle for Analytic Derivations. If D is an analytic derivation on the analytic set A with kernel K, then for any coanalytic subset C of X with $K \subset C$ there is some countable ordinal β such that $D^{\beta}(A) \subset C$. In particular, if D is an analytic derivation on X with $\bigcap_{\alpha} D^{\alpha}(X) = \emptyset$, then there exists a countable ordinal β such that $D^{\beta}(X) = \emptyset$.

4. <u>Sample applications of the boundedness principle</u>. The following theorem was stated by Lusin. A proof is given in [15]. However, this theorem follows almost immediately from the boundedness principle.

Theorem 3. Let $A \in \mathscr{A}(X\times Y)$ and suppose that for every x, A_x is scattered. Then there exists some $\alpha < \omega_1$ such that for each x, $A_x^{\alpha} = \emptyset$.

Proof. Define Γ: $\mathscr{P}(X\times Y) \to \mathscr{P}(X\times Y)$ by

$$\Gamma(E) = \bigcup \{x\} \times E_x'$$

where E_x' is the αth Cantor–Bendixon derived set of E_x. Then Γ is an analytic derivation [4,p.61], and $\bigcap_{\alpha<\omega_1} \Gamma^{\alpha}(A) = \emptyset$. By the boundedness principle, there is $\alpha < \omega_1$ such that $\Gamma^{\alpha}(A) = \emptyset$. Therefore, $A_x^{\alpha} = \emptyset$ for each x. Q.E.D.

Theorem 4. Let A be an analytic subset of $\mathscr{K}(X)$ and assume each set in A is countable. Then there exists some $\alpha < \omega_1$ such that if $K \in A$, then $K^{\alpha} = \emptyset$.

Proof. Let F be a Borel measurable map of $J = \mathbb{N}^{\mathbb{N}}$ onto A. Let

$$B = \{(x,y): y \in F(x)\}.$$

Then $B \in \mathscr{B}(X\times Y)$. Applying theorem 3 to B noting the fibers of B are the elements of A, the theorem follows. Q.E.D.

Let us give another example of the use of the boundedness principle.

Theorem 5. (Bourgain [2]) Let X be a Banach space. Suppose that for each $\alpha < \omega_1$, $C(\alpha)$ can be isomorphically embedded into X, i.e., $C(\alpha) \hookrightarrow X$ $\forall\alpha < \omega_1$. Then $C([0,1]) \hookrightarrow X$.

Proof. By $C(\alpha)$ we mean the Banach space of all continuous functions on the ordinal space $\{ \beta \mid \beta \leq \alpha \}$ with the order topology. To show that $C([0,1]) \hookrightarrow X$, it suffices to show that $C(K) \hookrightarrow X$ where K is some closed uncountable subset of [0,1]. It is well known that for each $\alpha < \omega_1$, there is an order preserving homeomorphism of α onto a subset H_{α} of the rationals. This can be proven by transfinite induction.

Thus, we consider

$$A = \{ K \in \mathscr{K}([0,1]) \mid C(K) \hookrightarrow X \}.$$

We claim that A is analytic. Before demonstrating this let us make a few observations. $C(K) \hookrightarrow X$ if and only if there exists a continuous one-to-one linear map $F: C(K) \to X$ whose inverse is also continuous.. Also, a continuous map is determined by its values on a dense subset. In particular, a continuous map F on $C(K)$ is determined by its values on the set of all polynomials with rational coefficients on K. Our map F is linear if it respects addition and rational scalar multiplication on a dense subset D of $C(K)$ which is closed under addition and multiplication by rational scalars. Finally, F has continuous inverse provided there is $b > 0$ such that $b \cdot \|x\| \leq \|F(x)\|$ for all $x \in D$. Let $\{f_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ be enumerations of all polynomials with rational coefficients and all rational numbers respectively. We have that $C(K) \hookrightarrow X$ if and only if there exists $F: C(K) \to X$ such that

(1) $\forall n,m \quad F(f_n|_K + f_m|_K) = F(f_n|_K) + F(f_m|_K)$

(2) $\forall n,m \quad F(r_n \cdot f_m|_K) = r_n \cdot F(f_m|_K)$

(3) $\exists\ \alpha,\beta > 0 \quad \forall n \quad \alpha \cdot \|f_n|_K\| \leq \|F(f_n|_K)\| \leq \beta \cdot \|f_n|_K\|$.

Therefore, to verify that A is analytic, for each $n,m,p \in \mathbb{N}$, let

$$B_{n,m,p} = \{(M,\{x_n\}) \in \mathscr{K}([0,1]) \times X^{\mathbb{N}} \mid f_n|_M + f_m|_M = f_p|_M \Rightarrow x_n + x_m = x_p\},$$

$$C_{n,m,p} = \{((M,\{x_n\}) \in \mathscr{K}([0,1]) \times X^{\mathbb{N}} \mid r_n \cdot f_m|_M = f_p|_M \Rightarrow r_n \cdot x_m = x_p\}, \text{ and}$$

$$D_{n,m,p} = \{(M,\{x_n\}) \in \mathscr{K}([0,1]) \times X^{\mathbb{N}} \mid r_n, r_m > 0 \text{ and } r_n \cdot \|f_p|_M\| \leq \|x_p\| \leq r_m \cdot \|f_p|_M\|\}.$$

Let us note that each of the above sets are Borel. We have now that

$$A = \pi_1((\bigcap_{n,m,p} B_{n,m,p}) \cap (\bigcap_{n,m,p} C_{n,m,p}) \cap (\bigcup_{n,m} (\bigcap_{p} D_{n,m,p}))).$$

Therefore A is analytic.

The set $H = \{ M \in \mathscr{K}([0,1]) \mid M \text{ is uncountable} \}$ is analytic [8]. Now if $A \subset H^C$, then, according to theorem 4, there is some countable ordinal β such that for every $M \in A$, the derived set order of M is less than β. However, the derived set order of the ordinal $\omega^{\beta} + 1$ is $\beta+1$ [21]. Thus, the derived set order of H_{α} is $\beta+1$. By assumption, $H_{\alpha} \in A$. Therfore, we have a contradiction. Consequently, A must contain an uncountable element. Q.E.D.

5. <u>Sets with countable sections revisited</u>. We continue our analysis of sets with countable sections.

Theorem 6. (Novikov [17] and Lusin [12]) Let $B \in \mathscr{B}(X\times Y)$ such that $\forall x$ $|B_x| \leq \omega$. Then $B \in \mathscr{G}_{\sigma}$.

Proof. Since B is Borel, there is a continuous bijection $\varphi: H \overset{1-1}{\twoheadrightarrow} B$ where H is a closed subset of $J = \mathbb{N}^{\mathbb{N}}$. Let $M = \{(x,t) \mid \pi_X(\varphi(t)) = x\}$. Note that M is a closed subset of $X\times J$. Define $\Phi: M \overset{1-1}{\twoheadrightarrow} B$ by $\Phi(x,t) = \varphi(t)$. Then Φ is continuous and Φ maps the fibers of M onto the fibers of M onto the fibers of B. Hence, it suffices to show that $M \in (\mathscr{G}(X\times J))_{\sigma}$, since Φ maps Borel graphs to Borel graphs.

Define $D: \mathscr{P}(X\times J) \to \mathscr{P}(X\times J)$ by

$$D(E) = \bigcup_{x\in X} \{x\} \times E_x',$$

where E_x' is the Cantor–Bendixon derived set of E_x. Since M is closed, M_x is closed for each x. Thus, for each x there is some $\alpha_x < \omega_1$ such that the α_xth derived set of M_x is empty. Consequently, $D^{\omega_1}(M) = \emptyset$. Furthermore, since D is an analytic derivation, there is some $\alpha < \omega_1$ such that $D^{\alpha}(M) = \emptyset$. Also, if $E \in \mathscr{A}(X\times J)$, then $D(E)$ is analytic. Thus, the sets $D^{\tau}(M)$, $\tau \leq \alpha$, are analytic. In addition, $\bigcap_{\tau\leq\alpha} D^{\tau}(M) = \emptyset$. Therefore, applying Novikov's separation principle, there are Borel sets B^{τ}, $\tau \leq \alpha$, such that $\bigcap_{\tau\leq\alpha} B^{\tau} = \emptyset$ and for each $\tau \leq \alpha$, $D^{\tau}(M) \subset B^{\tau}$. For each $\tau<\alpha$, let $A_{\tau} = D^{\tau}(M)\backslash B^{\tau+1}$, and note that each A_{τ} is analytic and each $A_{\tau x}$ consists of isolated points (or is empty). By Theorem 2, $A_\tau \subset G_\tau \in \mathscr{G}_\sigma$. We claim that $M = \bigcup_{\tau<\alpha} A_\tau$, from which it follows that $M \in \mathscr{G}_\sigma$. Clearly, $A_\tau \subset M$ for each $\tau < \alpha$. Thus, suppose $p \in M$. There is some $\tau<\alpha$ such that $p \in D^\tau(M) \setminus D^{\tau+1}(M)$. Hence, $p \notin B^{\tau+1}$. Let γ be the smallest ordinal such that $p \notin B^{\gamma+1}$. Note that $\gamma \leq \tau$. Therefore, $p \in A_\gamma$, and the claim is verified. Q.E.D.

The strongest theorem concerning covering analytic sets with countable sections by countably many graphs is the following theorem first given by Lusin in 1930. We will express this as a faithful separation theorem which is a refinement of the first separation principle. In general this means if A and E are disjoint analytic sets in X×Y and $\forall x\ A_x$ has property P then there is a Borel set B, $A \subset B$, $B \cap E = \emptyset$ and $\forall x\ B_x$ has property P.

Theorem 7. (*ω-Faithful Separation*) (Lusin, 1930 [12], Mauldin, 1978 [15], Maitra, 1980 [13])
Let $A, E \in \mathscr{A}(X\times Y)$ and $\forall x\ |A_x| \leq \omega$ and $A_x \cap E_x = \emptyset$. Then $\exists\ B \in \mathscr{G}_\sigma$ such that $A \subset B$ and $E \cap B = \emptyset$.

In case each B_x is countably infinite , theorem 7 has a particularly nice formulation: B can be expressed as the union of countably many disjoint Borel graphs.

Theorem 8. (*ω-Parametrization theorem*)
Let $B \in \mathscr{B}(X\times Y)$ and $\forall x\ |B_x| = \omega$. Then there is a Borel isomorphism $\Phi: X\times\mathbb{N} \overset{1-1}{\twoheadrightarrow} B$ such that $\forall x\ \Phi(x,\cdot):\mathbb{N} \overset{1-1}{\twoheadrightarrow} \{x\}\times B_x$.

Proof. Applying Theorem 4 to the sets B and $X\times Y\setminus B$, we have that $B \in \mathscr{G}_\sigma$. Hence, $B = \bigcup_{n=1}^{\infty} B_n$ where each $B_n \in \mathscr{G}$. Let $G_1 = B_1$, and for $n>1$, let $G_n = B_n\setminus \bigcup_{m=1}^{n-1} B_m$. Then $B = \bigcup_{n=1}^{\infty} G_n$, $G_n \cap G_m = \emptyset$ for $m\neq n$, and for each n, $G_n \in \mathscr{G}$. Next, using the fact that each B_x is countably infinite, define the sequence $\{F_n\}_{n=1}^{\infty}$ as follows: for $x\in X$, let $F_{1x} = G_{n_1 x}$ where n_1 is the smallest natural number such that $G_{n_1 x} \neq \emptyset$, and for $m>1$, let $F_{mx} = G_{n_m x}$ where n_m is the smallest natural number such that $n_{m-1} < n_m$ and $G_{n_m x} \neq \emptyset$. We have that $B = \bigcup_{n=1}^{\infty} F_n$, $F_n \cap F_m = \emptyset$ for $m\neq n$, and each F_n is a Borel graph whose projection is X. In fact,

$$F_1 = G_1 \cup [\pi_1^{-1}(X\setminus\pi_1(G_1)) \cap G_2] \cup [\pi_1^{-1}(X\setminus\pi_1(G_1\cup G_2)) \cap G_3] \cup \ldots,$$

and for $n > 1$,

$$F_n = [G_n\setminus\bigcup_{k<n} F_k] \cup [\pi_1^{-1}(X\setminus\pi_1(G_n\setminus\bigcup_{k<n} F_k)) \cap (G_{n+1}\setminus\bigcup_{k<n} F_k)] \cup$$
$$[\pi_1^{-1}(X\setminus\pi_1((G_n\cup G_{n+1})\setminus\bigcup_{k<n} F_k)) \cap (G_{n+2}\setminus\bigcup_{k<n} F_k)] \cup \ldots .$$

Now define $\Phi: X\times\mathbb{N} \to L$ by $\Phi(x,n) = (x,y)$ where $\{y\} = F_{nx}$. Φ is surjective since $\bigcup_{n=1}^{\infty} F_n = B$, and Φ is injective since the F_n's are pairwise disjoint. Furthermore, for each x, the map $\Phi(x,\cdot): \mathbb{N} \to B$ is injective and maps $\mathbb{N}$ onto $\{x\}\times B_x$. Q.E.D.

Problem 1. Let C be a coanalytic subset of $X\times Y$ such that for each x, $|C_x| \leq \omega_1$. Can C be written as the union of countably many coanalytic graphs, or Σ_2^1 or PCA graphs? What role do the axioms of set theory play here?

6. <u>Sets with compact and σ-compact sections, measurable multi-functions</u>. The theory presented for Borel sets with countable sections has some analogs for Borel sets with σ-compact sections. The deepest result

in this direction was obtained by Saint-Raymond (theorem 12 below). Since the techniques are delicate, we will only recall some portion of the methods. We show that a certain crucial portion of Saint-Raymond's argument can be readily obtained from the boundedness principle. First, let us recall Novikov's deep result. Now, in general, $\pi_X(\bigcap_{n=1}^{\infty} E_n) \neq \bigcap_{n=1}^{\infty} \pi_X(E_n)$. However, if each E_{nx} is compact and if for every n, $E_n \subset E_{n+1}$, $\pi_X(\bigcap_{n=1}^{\infty} E_n) = \bigcap_{n=1}^{\infty} \pi_X(E_n)$. Novikov exploited this fact to prove the following theorem.

Theorem 9. (*compact-faithful separation theorem*) (Novikov, 1939 [19]) Let A, E $\in \mathscr{A}(X \times Y)$ and assume that $\forall x$ there is a compact subset K_x of Y such that $A_x \subset K_x$ and $K_x \cap E_x = \emptyset$. Then $\exists$ Borel sets $\{H_n\}$ such that

(1) $\forall n \quad H_n = \bigcup_{i=1}^{k_n} D_{n_i} \times K_{n_i} \qquad K_{n_i}$ compact

(2) $A \subset B = \bigcap H_n$ and $E \cap B = \emptyset$.

Moreover $\pi_X(B) = \bigcap \pi_X(H_n)$ is a Borel set.

Corollary 10. If $B \in \mathscr{B}(X \times Y)$ such that $\forall x\ B_x$ is compact, then $\pi_X(B)$ is a Borel set.

Proof. Take $A = B$ and $E = (X \times Y) \setminus B$ in theorem 9.

Let us take a moment here to apply Novikov's theorem to multifunctions. The main fact is that a compact-valued multifunction $F\colon X \to \mathscr{K}(Y)$ is measurable if and only if its "graph" is a Borel set.

Corollary 11. Let $B \subset X \times Y$ with $\forall x\ B_x$ is compact. TFAE

(1) B is a Borel set

(2) $F\colon X \to \mathscr{K}(Y)$ given by $F(x) = B_x$ is Borel measurable.

Moreover, if B is a Borel set, B has a Borel selector.

Proof. Recall that sets of the form $C(U) = \{ K \in \mathscr{K}(Y) \mid K \subset U\}$ and $I(U) = \{ K \in \mathscr{K}(Y) \mid K \cap U \neq \emptyset\}$ where U is open in Y form a subbase for the topology of $\mathscr{K}(Y)$.

(1)→(2). To show that F is Borel measurable it suffices to show that for each open set U of Y each of $F^{-1}(I(U))$ and $F^{-1}(C(U))$ are Borel. Thus, let U be open in Y. Note that $F^{-1}(I(U)) = \pi_X(X\times U \cap B)$ which is Borel by Corollary 10. Next, observe that $F^{-1}(C(U)) = X\setminus F^{-1}(I(Y\setminus U)) = X\setminus \pi_X(X\times(Y\setminus U) \cap B)$ which is Borel. Therefore, F is Borel measurable.

(2)→(1). Since $F:X \to \mathscr{K}(Y)$ is Borel measurable, $Gr(F) = \{(x,y) \mid y \in B_x\} = B$ is Borel. Q.E.D.

Theorem 12. (*σ–compact faithful separation theorem*) (Saint–Raymond, 1976 [20]) Let $A, E \in \mathscr{A}(X\times Y)$ and assume that $\forall x$ there is a σ–compact subset K_x of Y such that $A_x \subset K_x$ and $K_x \cap E_x = \emptyset$. Then there are Borel sets $B_n \in \mathscr{C}$ such that $A \subset B = \bigcup B_n$ and $B \cap E = \emptyset$.

Proof of theorem. In demonstrating this, Saint–Raymond [20,p.392] uses a derivation operator which we define below. Let A and E be two disjoint analytic subsets of X×Y. Let φ be a continuous surjection of some Polish space P onto A.

For each subset Z of P define D(Z) to be the set of points z of Z such that for each neighborhood V of z,

$$\overline{\varphi(V\cap Z)\cap(\{x\}\times Y)} \cap E \neq \phi, \text{ where } x = \pi_X(\varphi(z)).$$

Saint–Raymond then gives the following recursion [20, p.393]

$$Z^0 = P\ ,\quad Z^{\alpha+1} = D(Z^\alpha), \text{ and } \quad Z^\lambda = \bigcap_{\alpha<\lambda} Z^\alpha \quad \text{if } \lambda \text{ is a limit ordinal.}$$

and then proves the following lemma and corollary.

Lemma. If B is a Borel subset of P which contains Z^α, $\alpha < \omega_1$, then there is $H \in \mathscr{C}_\sigma$ containing $\varphi(P\backslash B)$ and disjoint from E.

Corollary. If $\exists\ \alpha < \omega_1$ such that $Z^\alpha = \emptyset$, then there is $H \in \mathscr{C}_\sigma$ such that $A \subset H$ and $H \cap E = \emptyset$.

Consequently to prove the above theorem, it suffices to show that for some

$\alpha < \omega_1$, $Z^\alpha = \emptyset$ given that for each $x \in X$, the section A_x is contained in a K_σ disjoint from E. In order to prove this, Saint-Raymond gives an indirect argument by showing that if the Z^α are nonempty then there is a compact set K contained in a section of $A \cup E$ and such that no K_σ can contain $K \cap A$ without meeting $K \cap E$. Below we give a different argument which involves the boundedness principle for monotone coanalytic operators and the Baire Category theorem.

Claim 1. D is an analytic operator. Consequently if Z is analytic, then Z^α is analytic for $\alpha < \omega_1$.

Proof. For each $m \in \mathbb{N}$, define the operator $\Lambda_m : 2^P \to 2^P$ as follows:

$$x \in \Lambda_m(Z) \quad \text{IFF}$$

$$x \in \pi_1 \left\{ (z,(z_n),y) \in Z \times Z^{\mathbb{N}} \times E \,\middle|\, \forall n [d(z,z_n) < 1/m \wedge \pi_1(\varphi(z_n)) = \pi_1(\varphi(z))] \wedge \varphi(z_n) \to y \right\},$$

where d is a metric for the topology on P.

We then have

$$z \in D(Z) \quad \text{IFF} \quad \forall m\ z \in \Lambda_m(Z).$$

Consequently, it suffices to show that each Λ_m is analytic. Let ψ be a continuous surjection of some Polish space Q onto E. Fix $m \in N$. For each $k \in N$, set

$$B_k = \left\{ (z,(z_n),w) \in P \times P^{\mathbb{N}} \times Q \,\middle|\, d(z_k,z) < 1/m \right\},$$

$$C_k = \left\{ (z,(z_n),w) \in P \times P^{\mathbb{N}} \times Q \,\middle|\, \pi_1(\varphi(z_k)) = \pi_1(\varphi(z)) \right\} \text{ and}$$

$$D_k = \left\{ (z,(z_n),w) \in P \times P^{\mathbb{N}} \times Q \,\middle|\, \rho(\varphi(z_k),\psi(w)) < 1/k \right\}.$$

For each k, B_k is open, C_k is closed and D_k is open. Next define for each k,

$$f_k\colon P\times P^{\mathbb{N}}\times Q\to P\times P^{\mathbb{N}}\times Q \text{ by}$$

$$f_k(z,(z_n),w) = (z_k,(z_n),w).$$

Note that for each k, f_k is continuous. Now define $\Delta\colon 2^P\to 2^P$ by

$$\Delta(K) = \bigcap_{k=1}^{\infty} (B_k\cap C_k\cap D_k\cap f_k^{-1}(K)).$$

Since for each k, B_k,C_k and D_k are Borel and since for each k, f_k is Borel measurable, it follows that Δ is a Borel operator. Finally,

$$z\in\Lambda_m(Z) \quad \text{IFF} \quad (\exists((z_n),w))(z,(z_n),w)\in\Delta(Z\times P^{\mathbb{N}}\times Q).$$

Therefore, Λ_m is a Σ_1^1 operator. Q.E.D.

Now let Γ be the dual operator of D, i.e., $\Gamma(B) = P\backslash D(P\backslash B)$.

Note that $\forall\alpha < \omega_1$, $\Gamma^{\alpha}(\emptyset) = P\backslash Z^{\alpha}$.

Claim 2. Γ is an inductive, monotone Π_1^1 operator.

Proof. Suppose $B \subset P$. Then $D(P\backslash B) \subset P\backslash B$. Thus,

$$B = P\backslash(P\backslash B) \subset P\backslash D(P\backslash B) = \Gamma(B).$$

Therefore, Γ is inductive.

To show Γ is monotone, suppose that $B \subset C$. Then $P\backslash C \subset P\backslash B$. Hence $D(P\backslash C) \subset D(P\backslash B)$. Thus, $\Gamma(B) = B\backslash D(P\backslash B) \subset C\backslash D(P\backslash C) = \Gamma(C)$.

Lastly, since D is Σ_1^1, Γ is Π_1^1. Q.E.D.

Next, we make use of the Baire Category theorem.

Claim 3. If for each $x\in X$, A_x is contained in a K_σ disjoint from E_x, then for each nonempty $Z \subset P$, $D(Z) \subsetneq Z$.

Proof. Fix $x\in X$ such that $\varphi(Z)_x \neq \emptyset$. There is a sequence of compact sets $\{K_n\}_{n=1}^{\infty}$ such that $A_x \subset \bigcup_{n=1}^{\infty} K_n$ and $(\bigcup K_n) \cap E_x = \emptyset$. Thus,

$$\varphi^{-1}(A_x) \subset \varphi^{-1}(\bigcup K_n) = \bigcup \varphi^{-1}(K_n)$$

Since $\varphi^{-1}(A_x) = \varphi^{-1}(\{x\}\times Y)$, $\varphi^{-1}(A_x)$ is a closed subset of P. Also note that for each n, $\varphi^{-1}(K_n)$ is closed. Now set

$$C = \overline{Z \cap \varphi^{-1}(A_x)} .$$

Since $\varphi(Z)_x \neq \emptyset$, $C \neq \emptyset$. Furthermore, C is Polish and $C \subset \bigcup \varphi^{-1}(K_n)$. Therefore by the Baire Category Theorem, there is $n \in N$ such that $\operatorname{int}_C \varphi^{-1}(K_n) \neq \emptyset$. Consequently, there is an open subset V of P such that $C \cap V \neq \emptyset$ and $C \cap V \subset \varphi^{-1}(K_n)$. Choose $z \in Z \cap V \cap \varphi^{-1}(A_x)$. Since $\overline{\varphi(Z \cap V) \cap (\{x\} \times Y)} \subset K_n$, $z \notin D(Z)$. Thus $D(Z) \subsetneq Z$. Q.E.D.

Claim 4. If for each $x \in X$, A_x is contained in a K_σ disjoint from E_x, then there is $\alpha < \omega_1$ such that $\Gamma^\alpha(\emptyset) = P$.

Proof. Since Γ is an inductive, monotone, coanalytic operator, $|\Gamma| \leq \omega_1$ [4,p.59]. Thus $\Gamma(\bigcup_{\alpha<\omega_1} \Gamma^\alpha(\emptyset)) = \bigcup_{\alpha<\omega_1} \Gamma^\alpha(\emptyset)$. Consequently by the claim, $\bigcup_{\alpha<\omega_1} \Gamma^\alpha(\emptyset) = P$. By the boundedness principle, there is $\alpha<\omega_1$ such that $P \subset \Gamma^\alpha(\emptyset)$. Hence $\Gamma^\alpha(\emptyset) = P$. Q.E.D.

An immediate consequence of claim 4 is: If for each $x \in X$, A_x is contained in a K_σ disjoint from E_x, then there is $\alpha < \omega_1$ such that $Z^\alpha = \emptyset$. This completes the proof of the σ–compact faithful separation theorem.

In order to raise an unsolved problem concerning selectors, let us recall a basic selection theorem.

Theorem 13. The space of compact subsets of Y, $\mathcal{K}(Y)$, has a Baire class 1 selector.

Proof. By embedding Y in $[0,1]^\omega$, it suffices to prove the result for $\mathcal{K}([0,1]^\omega)$. Let φ: $[0,1] \to [0,1]^\omega$ be a continuous map of $[0,1]$ onto $[0,1]^\omega$. Also, let s: $\mathcal{K}([0,1]) \to [0,1]$ be a continuous selector for $\mathcal{K}([0,1])$. Define $\hat{s}$: $\mathcal{K}([0,1]^\omega) \to [0,1]^\omega$ by $\hat{s}(K) = \varphi(s(\varphi^{-1}(K)))$. $\hat{s}$ is a Baire class

1 selector for $\mathcal{K}([0,1]^{\omega})$. Q.E.D.

A natural question which naturally arises is how many disjoint selectors are there for the uncountable compact sets? We can formulate this question as follows:

Problem 2. Let $B \in \mathcal{B}([0,1]^2)$ such that $\forall x\ B_x$ is compact and uncountable. Does B have 2^{ω} pairwise disjoint Borel selectors? (B does have $\aleph_1$ pairwise disjoint selectors [16].) In particular, what about the σ-compact set B constructed in [16]?

7. Parametrizations: Filling up sets with selectors. In Theorem 8 we showed that if $B \in \mathcal{B}(X \times Y)$ and for every x, $|B_x| = \omega$, then B has a Borel parametrization, i.e., a Borel measurable coding of disjoint Borel selectors of B which fill up B. It is natural to ask whether there is an analogous result with each B_x uncountable. In other words, if $B \in \mathcal{B}(X \times Y)$ and $\forall x\ B_x$ is uncountable is there a Borel map Φ of $X \times J$ onto B such that for each x, $\Phi(x,\cdot)$ maps J onto B? If Φ exists, then for each $\sigma \in J$, $\Phi(X \times \{\sigma\})$ is a Borel graph. Φ is a Borel measurable coding of a family of pairwise disjoint selectors filling up B. Now, in general, this is not possible. In [9], Kallman and Mauldin gave an example of a Borel set $B \subset [0,1] \times [0,1]$ such that for each x, B_x is an uncountable G_δ set and yet B does not even have a Borel selector. However, necessary and sufficient conditions for the existence of a Borel parametrization have been given:

Theorem 14. (*Parametrization theorem*) (Mauldin, 1979 [14])
Let $B \in \mathscr{B}(X \times Y)$. TFAE

(1) $\exists$ Borel set $M \subset B$ such that $\forall x$ M_x is a Cantor set.

(2) $\exists$ a Borel map $\Phi: X \times [0,1] \overset{1-1}{\twoheadrightarrow} B$ such that

$$\forall x\ \Phi(x,\cdot): [0,1] \overset{1-1}{\twoheadrightarrow} B_x.$$

(3) $\exists$ atomless conditional probability distribution $x \to \mu_x \in \Pr(Y)$

$$\forall x\ \mu_x(B_x) = 1.$$

In [16], Mauldin gave an example of a σ-compact subset B of $[0,1] \times [0,1]$ such that for each x, B_x is uncountable and yet B does not contain a Borel set each section of which is an uncountable compact set. According to 1 above, this set does not have a Borel parametrization.

8. <u>One-to-one selections and parametrizations</u>. Let us motivate this section by slightly modifing a set considered by Hadamard [1].

$H = \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid x,y$ are transcendental and x,y are not algebraically related$\}$

Does H contain a Borel graph? This is the question considered by Hadamard. It can be answered affirmatively on the basis of several theorems. In fact , H has a Borel parametrization. This is a corollary of Theorem 14. Does H contain a Borel isomorphism? The answer is yes. It follows from the theorem of Debs and Saint-Raymond, theorem 18.

Problem 3. Does H have a parametrization of Borel isomorphisms? (open)

Before stating the results of Debs and Saint-Raymond, let us give some measure theoretic results concerning one-to-one selections.

Theorem 15. (Graf and Mauldin, 1985 [7])
Let $X = Y = [0,1]$. Let $B \in \mathcal{B}(X\times Y)$ such that $\forall x\ \forall y\ |B_x|, |B^y| > \omega$. $\exists\ C, D \subset [0,1]$ such that $\lambda(C) = \lambda(D) = 1$ and a Borel isomorphism $\varphi: C \twoheadrightarrow D$ such that $\mathrm{Gr}\,\varphi \subset B$.

In order to give an example to show that the conclusion of theorem 15 is the most one could hope for in the direction, we need the next lemma.

Lemma 16. Let $h: A \subset [0,1] \to \mathcal{K}_n = \left\{ K \in \mathcal{K}([0,1]) : \lambda(K) \geq 1 - 1/n \right\}$ be Borel measurable and $x \in h(x)\ \forall x \in A$. Then h is not onto.

Proof. Suppose h is onto. For each $t > 1 - 1/n$, $t \neq 1$, let $E_t = \{ K \in \mathcal{K}_n \mid \lambda(K) = t\}$. We claim that $\lambda(h^{-1}(E_t)) \geq 1-t$ for each t. Suppose $\lambda(h^{-1}(E_t)) < t$. There is $K \subset [0,1]/h^{-1}(E_t)$ such that $\lambda(K) = t$. By the surjectivity of h, there is x such that $h(x) = K$. But, $x \in h^{-1}(E_t)$, a contradiction. Thus, the claim holds. Since the uncountable collection $\{E_t\}$ consists of pairwise disjoint Borel sets, the uncountable collection $\{h^{-1}(E_t)\}$ consists of pairwise disjoint λ-measurable sets of positive measure. This is a contradiction. Therefore, h is not onto. Q.E.D.

Theorem 17. There is a Borel subset B of $[0,1]^2$ such that $\forall x\ \forall y\ \lambda(B_x) = \lambda(B^y) = 1$ and B does not contain the graph of a Borel isomorphism of $[0,1]$ onto $[0,1]$. Indeed, B does not contain the graph of a Borel surjection of $[0,1]$ onto $[0,1]$.

Proof. Let $\langle F_n\rangle_{n=1}^{\infty}: [0,1] \twoheadrightarrow \prod_{n=1}^{\infty} \mathcal{K}_n$ be a Borel isomorphism. For each n, let $R_n = \bigcup_{y\in[0,1]} F_n(y)\times\{y\}$. Then for each n, R_n is Borel. Consequently $R = \bigcup_{n=1}^{\infty} R_n \in \mathcal{B}([0,1]^2)$. Furthermore, for each $y \in [0,1]$ and $n \in \mathbb{N}$ we have

$\lambda(R^y) \geq \lambda(F_n(y)) \geq 1 - 1/n$, and hence, $\lambda(R^y) = 1$. Let $A = \{x: \lambda(R_x) = 1\}$. Then the Borel ssubset A of [0,1] has Lebesgue measure one. Now suppose g is a Borel map of A onto [0,1] such that $\forall x\ (x,g(x)) \in R$. For each n, let $A_n = \{x \in A : x \in F_n(g(x))\}$. By lemma 16, the map $F_n \circ g|_{A_n}$ is not onto. For each n, choose $K_n \in \mathcal{K}_n \setminus F_n \circ g|_{A_n}$. Let $g(x) = y$ where $\forall n\ F_n(y) = K_n$. Now $(x,y) \in R_n$, for some n. Thus, $x \in A_n$ and $F_n(g(x)) = K_n$, a contradiction. Thus, R does not contain the graph of a Borel map of A onto [0,1]. Finally, to complete the construction of the example, let θ be a Borel isomorphism of [0,1] onto A. Let $B = (\theta \times \mathrm{id})^{-1}(R)$. Clearly, the Borel set B has all the required properties. Q.E.D.

Mauldin raised the possibility that the category version of the preceding theorem may have a different answer. If $B \in \mathcal{B}([0,1]^2)$ and $\forall x\ \forall y\ B_x$ and B^y are comeager, then does B contain the graph of a Borel isomorphism? The answer is yes, and in [5], Debs and Saint-Raymond prove the following theorem.

Thereom 18. (Debs and Saint-Raymond, 1989 [5]) Let X,Y be compact perfect metric spaces. $B \in \mathcal{B}(X \times Y)$ with $\forall x\ \forall y\ B_x$ and B^y are dense G_δ sets. Then B does contain the graph of a Borel isomorphism.

Remarkably, this result depends on X and Y being compact. Specifically:

Example 1. (Debs and Saint-Raymond, 1989 [5]) $\exists\ G_\delta$ set $G \subset N^N \times 2^N$ with all fibers both ways dense and such that G contains no Borel isomorphism.

Example 2. (Debs and Saint-Raymond, 1989 [5]) $\exists\ B \in \mathcal{B}(2^N \times 2^N)$ with $\forall x\ B_x$ is a dense G_δ and $\forall y\ B^y$ is residual and B contains no Borel isomorphism.

Problem 4. Let X, Y be compact perfect metric spaces. $B \in \mathcal{B}(X \times Y)$ with $\forall x\ \forall y\ B_x$ and B^y are dense G_δ sets. Does B have a Borel parametrization of Borel isomorphisms of X onto Y?

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