Don Spear, Department of Mathematics, University of North Texas, Denton, TX 76203-5116

Pitt's dimensionless Cantor set

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At the annual AMS meeting in January of 1988, L.D. Pitt proposed during an informal conversation over coffee with Mauldin an example of a dimensionless Cantor set, K. In *Dimension und äußeres Maß* [2], Hausdorff defined the dimension of a set A to be the class of all Hausdorff functions such as φ for which $0 < \mathcal{H}^{\varphi}(A) < \infty$. A function φ is a Hausdorff function if (i) $\varphi\colon [0,\delta] \to \mathbb{R}$ for some $\delta > 0$, (ii) φ is non-decreasing, (iii) $\varphi(0) = 0$, and (iv) $\varphi(t)\downarrow 0$ as $t\downarrow 0$. Here, K is dimensionless means, if φ is a Hausdorff function, $x \in K$ and $r > 0$, then $K \cap B_r(x)$ is either non-σ-finite or of zero measure with respect to $\mathcal{H}^{\varphi}$. It is not known if such a Hausdorff function exists.

Pitt proved that if x is not isolated from the left in K, then $\dim_{\mathcal{H}} K \cap [0,x] = \alpha(x)$ with, for x in (0,1], $\alpha(x) = \ln(2)/(\ln(2/x))$ and $\alpha(0) = 0$. Extending this we show

Theorem. If $x \in K$ and $r > 0$, then $K \cap B_r(x)$ is either non-σ-finite or of zero measure with respect to $\mathcal{H}^{\varphi}$ for which $\varphi(t) = t^{\gamma}\cdot L(t)$ where $\gamma \geq 0$ and L is slowly varying (L is slowly varying if $\lim_{t\downarrow 0} L(ct)/L(t) = 1$ for any $c > 0$).

Start the construction of K by setting $J_{\emptyset} = [0,1]$. Assume on the p^{th} level that, for $\sigma \in \{0,1\}^p$, $J_{\sigma} = [a_{\sigma}, b_{\sigma}]$

has been constructed with midpoint m_σ and length l_σ. For σ a finite sequence and τ any sequence, let $\sigma*\tau$ denote the concatenation of σ and τ. Remove the open interval $(m_\sigma - l_\sigma(1-m_\sigma)/2,\ m_\sigma + l_\sigma(1-m_\sigma)/2)$ from J_σ. Denote by $J_{\sigma*0}$ and $J_{\sigma*1}$, respectively, the left and right intervals that remain. For $\theta \in \bigcup_{n=0}^{\infty}\{0,1\}^n$, define $K_\theta = \bigcap_{p=1}^{\infty} \bigcup_{\sigma\in\{0,1\}^p} J_{\theta*\sigma}$ and $K = K_\emptyset$. To prove our theorem, we use three lemmas.

Lemma 1. For $\theta \in \{0,1\}^p$, lower and upper bounds of the Hausdorff dimension are

$$\underline{\lim}_{j\to\infty} \frac{\log 2^{p-j}}{-\log s_\theta(j)} \leq \dim_{\mathcal{H}} K_\theta \leq \underline{\lim}_{j\to\infty} \frac{\log 2^{p-j}}{-\log u_\theta(j)}$$

where, for $i \geq p$, $u_\theta(i) = \max\{l_\zeta : \zeta \in \{0,1\}^i,\ J_\zeta \subseteq J_\theta\}$ and $s_\theta(i) = \min\{l_\zeta : \zeta \in \{0,1\}^i,\ J_\zeta \subseteq J_\theta\}$.

Lemma 2. Suppose θ is an element of $\{0,1\}^p$. If $b_\theta < z$, then $\mathcal{H}^{\alpha(z)}(K_\theta) = 0$ and, if $z < b_\theta$, then $\mathcal{H}^{\alpha(z)}(K_\theta) = \infty$.

Suppose that φ is a Hausdorff function and $0 \leq x < y \leq 1$. Assume y is a limit point of K from the left. There are three cases to consider.

(a) For any $0 < \beta < \alpha(y)$, $\underline{\lim}_{t\downarrow 0}\ t^\beta \varphi(t)^{-1} > 0$.

(b) For some $0 < \gamma < \alpha(y)$, $\lim_{t\downarrow 0}\ t^\gamma \varphi(t)^{-1} = 0$.

(c) For any $0 < \gamma < \alpha(y)$, $\overline{\lim}_{t\downarrow 0}\ t^\gamma \varphi(t)^{-1} > 0$

and

for some $0 < \beta < \alpha(y)$, $\underline{\lim}_{t\downarrow 0}\ t^\beta \varphi(t)^{-1} = 0$.

Lemma 3. If (a) holds, then $K \cap [x,y]$ has zero $\mathcal{H}^{\varphi}$ measure. If (b) holds, then $K \cap [x,y]$ is of non-σ-finite measure with respect to $\mathcal{H}^{\varphi}$.

References

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