Density Continuous Functions

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Let $\mathbf{R}_O$ and $\mathbf{R}_D$ denote the real numbers with the ordinary topology and the density topology, respectively. There are four ways that a function can be continuous using these two topologies. They are denoted by

$$C_{XY} = \{f:\mathbf{R}_X \to \mathbf{R}_Y\}.$$

So, for example, C_{OO} is the set of functions continuous in the usual sense and C_{DO} is the set of approximately continuous functions. It is not hard to prove that C_{OD} consists precisely of the constant functions. The functions in C_{DD} are called the *density continuous* functions. It is also clear from the definitions that

$$(1) \qquad C_{OD} \subset C_{OO} \subset C_{DO} \supset C_{DD}.$$

Of course, the ordinary continuous and the approximately continuous functions have been studied extensively, but less is known about the density continuous functions. It turns out that their structure is more difficult than the other two classes, as the following statements show.

Example 1. ([CLO1]) There exists a C^∞ function which is not density continuous.

Example 2. ([CL1]) There is an $f \in C_{DD} \cap C^\infty$ such that $f(x) + x \notin C_{DD}$. Therefore, C_{DD} is not closed under pointwise addition.

Example 3. ([KO1]) C_{DD} is not closed under uniform convergence, or even equal convergence.

In fact the techniques used in these examples can be used to establish the following theorem.

Theorem 1. ([CLO1; CLO2]) With the uniform topology, C_{DD} is first category in itself and $C_{DD} \cap C_{OO}$ is first category in C_{OO}.

Since, as noted above, C_{OD} only contains the constant functions, Theorem 1 implies that first and third containments in equation (1) are proper and first category. (That the second containment is proper and first category is well known.) This can be construed to mean that C_{DD} is a small space in the topological sense, making it harder to study its structure.

However, there are a few facts known about the structure of this class of functions. The final containment in (1) shows that $C_{DD} \subset$ Baire 1. A stronger result can be proved.

Theorem 2. ([CLO1]) $C_{DD} \subset$ Baire*1.

Theorem 3. C_{DD} is a lattice.

Theorem 2 has strong consequences, such as the following statements.

Theorem 4. ([CLO1]) The set of points on which density continuous functions can be discontinuous are precisely the nowhere dense F_σ sets.

Theorem 5. ([CL2]) If $A \in F_\sigma \cap G_\delta$ and is density closed, then there is a function $f \in C_{DD}$ such that $A = \{x: f(x) = 0\}$.

Corollary 1. If $f \in C_{DD}$, $\alpha \in \mathbf{R}$ and $A = \{x: f(x) > \alpha\} \neq \varnothing$, then $\mathrm{int}(A)$ is dense in A.

None of these results can be taken as a characterization of the functions in C_{DD}, however, because all these properties are also shared by all functions in C_{OO}.

There are some criteria for determining whether a function is density continuous.

Theorem 6. ([CL1]) If f is locally convex, then f is density continuous.

Corollary 2. ([CL1]) If $f(x) = x^a$ for some $a \in \mathbf{R}$, then f is density continuous.

Corollary 3. ([CL1]) Real analytic functions are density continuous.

It seems likely that the algebraic functions are also density continuous, but this is still unknown.

Ostaszewski [KO2] has studied C_{DD} as a semigroup under composition. We denote this semigroup by SD. He proved that SD has the inner automorphism property; i. e., if ϕ is any isomorphism of SD onto itself, then there must exist an $h \in C_{DD}$ such that $\phi(f) = h^{-1} \circ f \circ h$, for all $f \in$ SD. These researches let naturally to the question of whether the density topology is generated; i. e., do the level sets of functions in C_{DD} form a subbase for the density topology? The following theorem, which is a consequence of Theorem 3 and Corollary 1, contains the answer.

Theorem 7. ([CL2]) The density topology is not generated.

This result is a bit surprising because apparently all previous examples of non-generated spaces with the semigroup of their selfmaps having the inner automorphism property were either quite complicated or somewhat unnatural. For some other examples, see the survey concerning topological semigroups by K. Magill [KM1].

References

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