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ON MONOTONIC FUNCTIONS AND REAL NUMBER ORDER

In this note we use the ordering of the real numbers in R to solve several apparently disparate problems in real analysis. In what follows, I will denote the compact interval $[0,1]$, and $C(I)$ will denote the family of continuous real valued functions on I. Moreover, m will denote Lebesgue measure.

These problems are:

Problem 1. Let X be an uncountable closed subset of I. Prove that there exists a continuous nondecreasing function f mapping I onto I such that $f(X) = I$.

Problem 2. Let U be an open subset of I such that $I\setminus U$ is uncountable. Let v be a number such that $0 < v < 1/m(U)$. Prove that there exists a homeomorphism f of I onto I such that $f'(x) = v$ for all $x \in U$.

Problem 3. Let P be a nonvoid perfect subset of I. Let v be a real number such that $0 < v < I/m(I\setminus P)$. Prove that there exists a homeomorphism f of I onto I such that $f'(x) = v$ for all $x \in I\setminus P$, and such that each open interval (c,d) meets $f(P)$ in either the void set or a set with positive Lebesgue measure.

Problem 4. Let F be a nonvoid subset of $C(I)$. Prove that there exists a nondecreasing function $g_0 \in C(I)$ such that g_0 is constant on every interval on which some member of F is constant, and if $g \in C(I)$ enjoys the same property, then g is constant on any interval on which g_0 is constant.

Problem 5. Let X be an uncountable closed subset of I. Prove that there exists an increasing homeomorphism f of the space J of irrational numbers into X such that $X\setminus f(J)$ is a countable set.

Problem 6. Let X be an uncountable closed subset of I. Prove that there is an increasing left continuous function f mapping I into X such that $X\setminus f(I)$ is a countable set. If g is another such function from I to X prove there is a homeomorphism h of I onto I with $(f \circ h)(x) = g(x)$ for $0 < x \leq 1$.

The key to the solution to all these problems is:

Lemma 1. Let (I_n) be a (finite or infinite) sequence of mutually disjoint, closed proper subintervals of I. Then there is a nondecreasing continuous function f of I onto I, constant on each I_n, such that f is not constant on any interval that is not a subinterval of some I_n.

Proof. We say that $x,y \in I$ are equivalent if either $x = y$ or x and y lie in the same interval I_n. Let $[x]$ denote the equivalence class containing x. The set of equivalence classes is totally ordered in the obvious way. Moreover:

1. $[0]$ is the first class and $[1]$ is the last class.
2. There exists a countable set C of classes, such that for any classes a and b $(a < b)$ there is a $c \in C$ such that $a < c < b$. For example, let C be the set of all classes that contain a rational number.
3. Any nonvoid set of classes $\{[x_a]\}$ has a least upper bound. Note that $[y]$ is the least upper bound where y is the least upper bound of the set of numbers $\{x_a\}$ in I.

It follows from properties 1, 2 and 3 that there is an order preserving function g mapping the set of classes onto I such that g is one-to-one and $g([0]) = 0$ and $g([1]) = 1$. For $x \in I$ put $f(x) = g([x])$. Then f is a nondecreasing function mapping I onto I. Because f maps onto I, f must be continuous on I. Clearly f is the desired function.

□

Solution to Problem 1. Because X is uncountable, there is a nonvoid perfect set $P \subset X$ so that $X\setminus P$ is countable. Let f be the function in Lemma 1 where the I_n are the closures of the components of $I\setminus P$. Each I_n meets P at an endpoint, so $f(I_n) \in f(P)$. Finally, $f(P) = f(I)$ and $f(X) = f(I)$. □

This incidentally provides another (albeit inefficient) proof that a closed uncountable subset of R has the same cardinality as R avoiding the usual argument on Cauchy sequences.

Solution to Problem 2. Because $I\backslash U$ is uncountable, there is a nonvoid perfect set $P \subset I\backslash U$ such that $(I\backslash U)\backslash P$ is countable. Let f_1 denote the function in Lemma 1 where the I_n are the closures of the components of $I\backslash P$. For $x \in I$, put $f_2(x) = m((0,x) \cap (I\backslash P))$. Let w be any positive number. For $x \in I$, put $f(x) = (f_1(x) + wf_2(x))/(1 + wm(I\backslash P))$. Then f is an increasing homeomorphism of I onto I and $f' = w/(1 + wm(I\backslash P))$ on $I\backslash P$ and on U. Moreover, $m(U) = m(I\backslash P)$ because U and $I\backslash P$ differ by at most a countable set. Finally, $f' = w/(1 + wm(U))$ on U, and we need only select w so that $w/(1 + wm(U)) = v$. Indeed $w = v/(1 - vm(U))$.

□

In problem 2 we did not allow $v = 1/m(U)$. Note that if $f' = 1/m(U)$ on U, then $m(f(U)) = 1$, $f(U)$ is dense in I. But U might not be dense in I. On the other hand, if U is dense in I, there is a homeomorphism f of I onto I such that $f' = 1/m(U)$ on U. We will not prove this here, because it does not involve our Lemma 1.

Solution to Problem 3. For the perfect set P, let f, f_1, f_2 be as in the solution to Problem 2. Let (c,d) be an open interval that meets $f(P)$. Say $f(a) = c$, $f(b) = d$. Then (a,b) meets P and $f_1(a) < f_1(b)$. Moreover,

$$\begin{aligned} mf((a,b) \cap (I\backslash P)) &= mf_2((a,b) \cap (I\backslash P))\cdot w/(1 + wm(I\backslash P)) \\ &\leq (f_2(b) - f_2(a))w/(1 + wm(I\backslash P)) < f(b) - f(a) = d-c. \end{aligned}$$

It follows that $m((c,d) \cap f(P)) = mf((a,b) \cap P) > 0$. □

Now let I_n be as in Lemma 1 and let P be the closure of the set $I\backslash \cup_n I_n$. Then P is a perfect set. Let f be the function in Problem 3. Then the function $g(x) = m((0,f(x)) \cap f(P))/mf(P)$ also satisfies the conclusion of Lemma 1.

Solution to Problem 4. Let $g \gg f$ mean that g is constant on any interval on which f is constant. Let $g \gg F$ mean $g \gg f$ for all $f \in F$. For $g \in C(I)$, let

$$E_g = \{x \in I\colon g \text{ is constant on no neighborhood of } x\}.$$

Then E_g is evidently a perfect set. Let $E = \bigcap_{f \in F} E_f$. Then E is a closed set. If E is countable, let $g_0(x) = 0$ for $x \in I$, and let P be the void set. If E is uncountable, let $P \subset E$ be the perfect set for which $E \backslash P$ is countable, and let g_0 be the function in Lemma 1 where the I_n are the closures of the components of $I \backslash P$. Then $E_{g_0} = P \subset E$ and hence $g_0 \gg F$.

Let $g \in C(I)$ such that $g \gg g_0$ does not hold. There is an open interval U on which g_0 is constant but g is not. Then E_g meets U, E_{g_0} does not meet U, and (because E_g is a perfect set) $E_g \cap U$ is uncountable. But $E \backslash E_{g_0} = E \backslash P$ is countable. Thus U contains only countably many points in E, so there is some $x \in (E_g \backslash E) \cap U$. There is an $f \in F$ such that $x \notin E_f$. So f is constant on some neighborhood of x but g is not. Hence $g \gg f$ does not hold, and $g \gg F$ does not hold. □

In particular, if $f \in C(I)$ there is a nondecreasing function $g \in C(I)$ such that f and g are constant on the same intervals.

Solution to Problem 5. Let $P \subset X$ be the perfect set such that $X \backslash P$ is countable. Let f_1 be the function in Lemma 1 where the I_n are the closures of the components of $I \backslash P$. Then f_1 is a nondecreasing continuous, closed function mapping I onto I. Put

$$Y = \{y \in (0,1)\colon y \text{ is irrational and } f_1^{-1}(y) \text{ is a singleton set}\}.$$

Then $(0,1) \backslash Y$ is a countable dense subset of $(0,1)$, $f_1^{-1}(Y) \subset P \subset X$ and the sets $P \backslash f_1^{-1}(Y)$ and $X \backslash f_1^{-1}(Y)$ are countable. There is an increasing homeomorphism f_2 of R onto $(0,1)$ that maps the set of rational numbers onto $(0,1) \backslash Y$. Finally, f_1 is closed and continuous, so f_1^{-1} is an increasing homeomorphism of Y onto $f_1^{-1}(Y)$ and $f_1^{-1} \circ f_2$ is an increasing homeomorphism of J onto $f_1^{-1}(Y)$. □

It follows that if X_1 and X_2 are uncountable compact subsets of R, there exist countable sets E_1 and E_2 such that $X_1 \backslash E_1$ is homeomorphic to $X_2 \backslash E_2$. This will not work in general in R^2. Let

$$Y_1 = \{(x,y): x^2 + y^2 \leq 1\} \quad \text{and} \quad Y_2 = \{(x,y): (x-2)^2 + (y-2)^2 \leq 1\}.$$

Let $X_1 = Y_1$ and $X_2 = Y_1 \cup Y_2$. Then for any countable sets E_1 and E_2 in R^2 the set $X_1 \backslash E_1$ must be connected but the set $X_2 \backslash E_2$ must not be connected. So $X_1 \backslash E_1$ and $X_2 \backslash E_2$ cannot be homeomorphic.

Solution to Problem 6. Let P be the perfect set for which $P \subset X$ and $X \backslash P$ is countable. Let f_1 denote the function in Lemma 1 where the I_n are the closures of the components of $I \backslash P$. For each $x \in I$, let $f(x)$ be the smallest $y \in I$ for which $f_1(y) = x$. It follows routinely that f is a strictly increasing function mapping I into P and f is left continuous. Moreover, $P \backslash f(I)$ and $X \backslash f(I)$ are countable.

Now suppose g is another strictly increasing left continuous function mapping I into X such that $X \backslash g(I)$ is countable. Then every point in $g(0,1)$ is a (left) condensation point of $g(0,1)$, so $g(0,1) \subset P$. Likewise $f(0,1) \subset P$, and indeed $P \backslash (f(0,1) \cap g(0,1))$ is countable. Thus $f(0,1) \cap g(0,1)$ is a dense subset of P. Let S be a countable dense subset of $f(0,1) \cap g(0,1)$. Then $f^{-1}(S)$ and $g^{-1}(S)$ are countable dense subsets of $(0,1)$. The mapping $x \mapsto f^{-1}(g(x))$ is an order preserving mapping of $g^{-1}(S)$ onto $f^{-1}(S)$. There is an increasing homeomorphism h of I onto I such that $h(x) = f^{-1}(g(x))$ for $x \in g^{-1}(S)$. Finally, $f \circ h = g$ on a dense subset of I, and (because f and g are left continuous) $(f \circ h)(x) = g(x)$ for $0 < x \leq 1$. □

It is worth noting that f and g completely determine h. Moreover, $f(0,1) = g(0,1)$ necessarily. However, we leave the proof.

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