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HENSTOCK INTEGRALS AND LUSIN'S CONDITION (N)

The theory of Henstock integrals [2] has been studied by many. One effort is to prove directly that the primitive of a Henstock integrable function satisfies the property ACG_*. (See [1,3,5].) In this note we show the connection between Henstock integrals and Lusin's condition (N), whereby we give another direct proof of the above result and that Henstock and Denjoy integrals are equivalent.

A real valued function f is said to be Henstock integrable to A on $[a,b]$ if for every $\varepsilon > 0$ there exists δ a positive function such that for any division D given by

$$a = x_0 < x_1 < \ldots x_n = b, \ \{\xi_1, \xi_2, \ldots, \xi_n\},$$

satisfying $0 \le x_i - \xi_i < \delta(\xi_i)$ and $0 \le \xi_i - x_{i-1} < \delta(\xi_i)$, $i = 1,2,\ldots,n$ we have

$$\left| \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) - A \right| < \varepsilon,$$

or alternatively

$$\left| \sum f(\xi)(v - u) - A \right| < \varepsilon$$

where $[u,v]$ denotes a typical interval in D with $\xi - \delta(\xi) < u \le \xi \le v < \xi + \delta(\xi)$. We call such a division $D = \{[u,v]; \xi\}$ a δ-fine division. In what follows we use F to denote the primitive f and write $F(u,v) = F(v) - F(u)$. A function F is said to satisfy Lusin's condition (N) if $|F(E)| = 0$ whenever $|E| = 0$. Here $|E|$ denotes the Lebesgue measure of E. For other definitions see [1] or [4].

Theorem 1. If f is Henstock integrable on $[a,b]$ with primitive F, then F satisfies Lusin's condition (N).

Proof. Suppose $E \subset [a,b]$ and its measure $|E| = 0$. Put $f^*(x) = f(x)$ when $x \in [a,b] \setminus E$ and 0 when $x \in E$. Thus f^* is also Henstock integrable on $[a,b]$ with primitive F. That is, given $\varepsilon > 0$ there is δ a positive function such that for any δ-fine division $D = \{[u,v]; \xi\}$ we have

$$| \sum \{F(u,v) - f^*(\xi)(v - u)\}| < \varepsilon.$$

By Henstock's lemma [2; Theorem 5], the partial sum $\sum_1$ of the above $\sum$ in which $\xi \in E$ satisfies

$$\sum_1 |F(u,v)| < 2\varepsilon.$$

For each $\xi \in E$ there is a shrinking family of intervals $[u,v]$ such that $\xi \in [u,v] \subset (\xi-\delta(\xi), \xi+\delta(\xi))$. Let M be the collection of all such intervals for all $\xi \in E$. Obviously M forms a Vitali covering of E. Since F is continuous, $F([u,v]) = [F(u^*), F(v^*)]$ where $F(u^*)$ and $F(v^*)$ are respectively the infimum and the supremum of $F(x)$ for $x \in [u,v]$. In view of the continuity of F again, the family of all intervals $F([u,v])$ for $[u,v] \in M$ forms a Vitali covering of $F(E)$. Thus by the Vitali covering theorem, there is a finite sum $\sum_2$ such that

$$\begin{aligned} |F(E)| &< \textstyle\sum_2 |F([u,v])| + \varepsilon \\ &= \textstyle\sum_2 \{F(v^*) - F(u^*)\} + \varepsilon \\ &\leq \textstyle\sum_2 |F(v^*) - F(\xi)| + \sum_2 |F(\xi) - F(u^*)| + \varepsilon \\ &< 5\varepsilon. \end{aligned}$$

Since ε is arbitrary, $|F(E)| = 0$. Hence F satisfies Lusin's condition (N).

Theorem 2. If f is Henstock integrable on $[a,b]$ with primitive F, then F satisfies ACG_*.

Proof. Note that F is VBG_*. (See [1] or [5].) Therefore the result follows from Theorem 1 and Saks [4; p. 228 Theorem 6.8].

Using Theorem 2 with the rest of the proof following [5], we obtain

Theorem 3. The Henstock integral and the restricted Denjoy integral are equivalent.

References

[1] R. Gordon, Equivalence of the generalized Riemann and restricted Denjoy integrals, Real Analysis Exchange 12(1986-87), 551-574.

[2] R. Henstock, A Riemann-type integral of Lebesgue power, Canadian J. Math. 20(1968), 79-87.

[3] Y. Kubota, An elementary theory of the special Denjoy integral, Math. Japonica 24(1980), 507-520.

[4] S. Saks, Theory of the integral, Warsaw, 1937.

[5] P.Y. Lee and W. Naak-in, A direct proof that Henstock and Denjoy integrals are equivalent, Bull. Malaysian Math. Soc. (2) 5(1982), 43-47.

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