Real Analysis Exchange Vol. 13 (1987-88)

Liu Genqian, Qingyang Teachers Training School, Gansu, China.

THE MEASURABILITY OF δ IN HENSTOCK INTEGRATION

Bullen [1] posed the question as the title states. This question has already been considered though implicitly in other papers. We shall use an idea in [3] together with a technique in [4] to prove the result in the affirmative.

A function f is said to be Henstock integrable on $[a,b]$ if there exists a number A such that for every $\varepsilon > 0$ there is a strictly positive function δ such that whenever a division D given by

$$a = x_0 < x_1 < \dots < x_n = b \quad \text{and} \quad \xi_1, \xi_2, \dots, \xi_n$$

satisfies $\xi_i - \delta(\xi_i) < x_{i-1} \leq \xi_i \leq x_i < \xi_i + \delta(\xi_i)$ for $i = 1,2,\dots,n$ we have

$$\left| \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) - A \right| < \varepsilon,$$

or alternatively,

$$\left| \sum f(\xi)(v - u) - A \right| < \varepsilon$$

where $[u,v]$ denotes a typical interval in D with $\xi \in [u,v] \subset (\xi - \delta(\xi), \xi + \delta(\xi))$. For such divisions we write $D = \{[u,v]; \xi\}$ and say that D is δ-fine. If F is the primitive of f, we often write $A = F(a,b) = F(b) - F(a)$. Next, a sequence of functions f_n is said to be control-convergent to f on $[a,b]$ if the following conditions are satisfied.

(i) $f_n(x) \to f(x)$ almost everywhere in $[a,b]$ as $n \to \infty$ where each f_n is Henstock integrable on $[a,b]$;

(ii) the primitives F_n of f_n are ACG_* uniformly in n, that is $[a,b]$ is the union of a sequence of closed sets X_i such that on each X_i the functions F_n are $AC_*(X_i)$ uniformly in n;

(iii) the primitives F_n converge uniformly on $[a,b]$.

Theorem. Let f be Henstock integrable on $[a,b]$. Then for every $\varepsilon > 0$ there exists a strictly positive, measurable function δ such that for any δ-fine division $D = \{[u,v]; \xi\}$ we have

$$|\Sigma f(\xi)(v - u) - F(a,b)| < \varepsilon.$$

Proof. Since f is Henstock integrable on $[a,b]$, it follows from [2] that there is a sequence of step functions f_n control-convergent to f on $[a,b]$. We assume that $f_n(x) \longrightarrow f(x)$ everywhere as $n \longrightarrow \infty$ except in a set Z of measure zero. Given $\varepsilon > 0$, since each f_n is Riemann integrable on $[a,b]$, there is a constant $\delta_n > 0$ such that for any δ_n-fine division $D = \{[u,v]; \xi\}$ we have

$$|\Sigma f_n(\xi)(v - u) - F_n(a,b)| < \varepsilon\, 2^{-n-1}.$$

Here F_n is the primitive of f_n and we assume $\delta_{n+1} \leq \delta_n$ for all n.

In view of [3; Lemma], the sequence F_n is oscillation convergent, that is, we can write $[a,b] = \cup_{i=1}^{\infty} X_i$ where each X_i is closed such that for each i and for every $\varepsilon > 0$ there is an integer N such that for every partial division of $[a,b]$ given by

$$a \leq a_1 < b_1 \leq a_2 < b_2 \leq \ldots \leq a_p < b_p \leq b$$

with $a_1, b_1, a_2, b_2, \ldots, a_p, b_p$ belonging to X_i, we have

$$\sum_{k=1}^{p} \omega(F_n - F_m; [a_k,b_k]) < \varepsilon \quad \text{whenever} \quad n, m \geq N$$

where ω denotes the oscillation of $F_n - F_m$ over $[a_k,b_k]$. Note that F_n converges uniformly to F on $[a,b]$. It follows that there is a subsequence $F_{n(i,j)}$ of F_n such that for any partial division of $[a,b]$ as given above we have

$$\sum_{k=1}^{p} \omega(F_{n(i,j)} - F; [a_k,b_k]) < \varepsilon\, 2^{-i-j}.$$

We may assume that for each i, $F_{n(i,j)}$ is a subsequence of $F_{n(i-1,j)}$.

Now consider $f_{n(j)} = f_{n(j,j)}$ for $j = 1,2,\ldots$ and define δ on $[a,b]$ with respect to f as follows. Let $Y_i = X_i - (X_1 \cup X_2 \cup \ldots \cup X_{i-1})$, $i = 1,2,\ldots$. For $\xi \in [a,b]$ and $\xi \in Y_i - Z$ there is $j(\xi)$ such that

$$|f_{n(j)}(\xi) - f(\xi)| < \varepsilon/(b-a) \quad \text{whenever} \quad j \geq j(\xi)$$

and concurrently, if $j(\xi) \neq 1$,

$$|f_{n(j)}(\xi) - f(\xi)| \geq \varepsilon/(b-a) \quad \text{when} \quad j = j(\xi) - 1.$$

Thus define $\delta(\xi) = \delta_{n(j(\xi))}$ when $j(\xi) \geq i$ and $\delta(\xi) = \delta_{n(i)}$ when $j(\xi) < i$. Therefore we have defined $\delta(\xi)$ for $\xi \in [a,b] - Z$.

Next, consider $\xi \in Z$. Let $Z_{ij} = Z \cap Y_i \cap S_j$ for $i, j = 1,2,\ldots$ where

$$S_j = \left\{x \in [a,b];\ j - 1 \leq |f(x)| < j\right\}$$

Note that, in view of the controlled convergence, F is $AC_*(X_i)$ for each i. Therefore for every i and j there is $\eta_{ij} < \varepsilon j^{-1} 2^{-i-j}$ such that for every finite or infinite sequence of nonoverlapping intervals $\left\{I_k^{(ij)}\right\}$ with endpoints a_k and b_k of $I_k^{(ij)}$ belonging to X_i and

$$\sum_k |b_k - a_k| < \eta_{ij} \quad \text{we have} \quad \sum_k \omega\left(F;\ I_k^{(ij)}\right) < \varepsilon\ 2^{-i-j}.$$

Now for fixed i and j take $I_k^{(ij)}$, $k = 1,2,\ldots$, with endpoints a_k and b_k belonging to X_i such that

$$\bigcup_k I_k^{(ij)} \supset Z_{ij} \quad \text{and} \quad \sum_k |b_k - a_k| < \eta_{ij}.$$

For $\xi \in Z_{ij}$ where ξ is a limit point of X_i on both sides and ξ is not an endpoint of any $I_k^{(ij)}$, we define $\delta(\xi)$ such that $(\xi - \delta(\xi), \xi + \delta(\xi)) \subset I_k^{(ij)}$ for some k. The set of remaining ξ in Z_{ij} not yet defined is countable, say, $\xi_1, \xi_2, \ldots$. Note that if $\xi_p \in Z_{ij}$, then ξ_p is either not a limit point of X_i or an endpoint of some $I_k^{(ij)}$. Here we have used the fact that a closed set is the union of a perfect set and a countable set

At such ξ_p, $p = 1,2,\ldots$, since F is continuous there, there is $\delta_p > 0$ such that whenever $\xi_p - \delta_p < u \le \xi_p \le v < \xi_p + \delta_p$ we have

$$|F(v) - F(u)| < \varepsilon\, 2^{-i-j-p} \quad \text{and} \quad |f(\xi_p)(v - u)| < \varepsilon\, 2^{-i-j-p}.$$

Finally, define $\delta(\xi_p) = \delta_p$ for $p = 1,2,\ldots$ and we have defined a strictly positive function δ on $[a,b]$.

For any δ-fine division $D = \{[u,v];\ \xi\}$ we have

$$\begin{aligned} |\Sigma\, f(\xi)(v - u) - F(a,b)| &\le |\Sigma_1\, f(\xi)(v - u) - \Sigma_1\, f_{n(j(\xi))}(\xi)(v - u)| \\ &+ |\Sigma_1\, f_{n(j(\xi))}(\xi)(v - u) - \Sigma_1\, F_{n(j(\xi))}(u,v)| \\ &+ |\Sigma_1\, F_{n(j(\xi))}(u,v) - \Sigma_1\, F(u,v)| \\ &+ |\Sigma_2\, f(\xi)(v - u)| + |\Sigma_2\, F(u,v)| \end{aligned}$$

where Σ_1 denotes the partial sum of Σ for which $\xi \in [a,b] - Z$ and $\Sigma_2 = \Sigma - \Sigma_1$, that is, the sum for which $\xi \in Z$. The first term on the right side of the above inequality is less than ε, and so is the second term. It follows from the oscillation convergence as in [3] that the third term is also less than ε. The fourth and fifth terms are less than ε because of F being $AC_*(X_i)$ and continuous and by the definition of δ relative to f on the sets Z_{ij}. Hence f is also Henstock integrable on $[a,b]$ with the given function δ.

We shall now show that the above δ is measurable. Since Z is of measure zero, it suffices to show that δ is measurable on $[a,b] - Z$. Let M_i denote the set of all integers $n(j(\xi))$ for which $\xi \in Y_i - Z$ and $j(\xi) \ge i$. For each $p \in M_i$, let E_p denote the set of all $\xi \in Y_i - Z$ such that

$$|f_{n(j)}(\xi) - f(\xi)| < \varepsilon/(b - a) \quad \text{whenever} \quad n(j) \ge p$$

and concurrently, if $p = n(j(\xi))$ and $j(\xi) \ne 1$,

$$|f_{n(j(\xi)-1)} - f(\xi)| \ge \varepsilon/(b - a).$$

Obviously, E_p is a measurable set. Note that δ takes constant value δ_p on E_p. On the other hand, δ takes constant value $\delta_{n(i)}$ on $(Y_i - Z) - \cup \{E_p;\ p \in M_i\}$ and therefore δ as a function restricted to $Y_i - Z$ is measurable. Since $Y_1 - Z, Y_2 - Z, \ldots$ are pairwise disjoint, δ is measurable on their union which is $[a,b] - Z$.

References

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Received September 22, 1987