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Harry I. Miller and Lejla Miller, D. Tucovica 8, 71000 Sarajevo, Yugoslavia.

A RESULT ABOUT POROUS SETS AND DIFFERENCE SETS

Introduction. The results presented in this paper evolved out of conversations that the first author had with Professor C.E. Weil during a brief visit to Michigan State University in the fall of 1986.

In this paper we will show that if $A \subset \mathbb{R}$ has the Baire property and is of the second category, then there exists a porous set P, $P \subset A$, such that $D(P) = \{p - p' : p, p' \in P\}$, the difference set of P, contains an interval.

A set E, $E \subset \mathbb{R}$, is said to be porous at a point x, if there exists a constant c, $0 < c \leq 1$, and a sequence of intervals $\{I_n\}$, each containing x, whose lengths tend to zero as n tends to infinity, such that each interval I_n contains an interval J_n that is disjoint from E and $m(J_n)/m(I_n) \geq c$ for each n. Here m denotes Lebesgue measure. A set P, $P \subset \mathbb{R}$, is called porous if it is porous at each of its points.

Results. A classical result of Piccard (See [1].) states that if $A \subset \mathbb{R}$ is a Baire set and is of the second category, then $D(A)$ contains an interval.

We will need the following lemma which is slightly stronger than Piccard's theorem. The proof of this lemma is easy and is therefore omitted.

Lemma. If Q is a set of the first Baire category and $a < b$, then $D((a,b)\backslash Q) = D((a,b)) = (a-b, b-a)$.

Before presenting the proof of the theorem mentioned in the introduction we mention that C, the Cantor set is porous and $D(C) = [-1,1]$. Utz [3] has a very nice geometric proof of this last fact.

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Theorem. If $B \subset \mathbb{R}$ has the Baire property and is of the second category, then there exists a porous set P, $P \subset B$ such that $D(P) = \{p - p' : p, p' \in P\}$ contains an interval.

Proof. If $B \subset \mathbb{R}$ has the Baire property and is of the second category, then there exists an open interval $J = (b,c)$ and a sequence of closed, nowhere dense subsets of $\mathbb{R}$, $\{F_n\}_{n=1}^{\infty}$, such that $B \supset A$, where $A = J \setminus \bigcup_{n=1}^{\infty} F_n$. By our lemma we have $D(A) = (-a,a)$, where $a = c - b$.

Examine $J \setminus F_1$. $J \setminus F_1$ is open and can therefore be expressed as an at most countable union of pairwise disjoint open sub-intervals of J, call them $\{I_{i1}\}$. $D(J\setminus F_1) = (-a,a)$. Furthermore, $D(J\setminus F_1) = D(\bigcup_i I_{i1}) = \bigcup_{i,j} (I_{i1} - I_{j1})$ where for any two subsets D and E of the real line, $D - E$ denotes the set $\{d - e; d \in D, e \in E\}$; i.e., the algebraic difference of D and E. Notice that $D - E$ is an open interval if D and E are open intervals.

By the proceeding, the collection of sets $\{I_{i1} - I_{j1}\}_{i,j}$ forms an open cover for each closed interval that is contained in $(-a,a)$. Let e be a fixed positive real number that is smaller than a.

$\{I_{i1} - I_{j1}\}_{i,j}$ is an open cover of $[-a + e, a - e]$ and therefore, by the Heine Borel Covering Theorem, finitely many of the open intervals $I_{i1} - I_{j1}$ cover the interval $[-a + e, a - e]$. Consequently, there exists a natural number n_1 such that each of the sets in the finite subcover can be expressed as the algebraic difference of two of the sets in the collection $\{I_{i1}\}_{i=1}^{n_1}$.

There exists a natural number m_1 and a sequence of intervals $\{J_{i1}\}_{i=1}^{n_1}$ with the following properties. Each J_{i1} is a closed interval that is contained in I_{i1}, whose endpoints are of the form $k/3^{m_1}$ and such that the difference set of the interior of $\bigcup_{i=1}^{n_1} J_{i1}$ contains $[-a + e, a - e]$. That such a sequence of intervals exists is an easy exercise that is left to the reader.

Now subdivide each of the intervals J_{i1} into closed intervals of length 3^{-m_1}. Remove the open middle ninth of each of these intervals of length 3^{-m_1}, obtaining intervals of length $4\cdot 3^{-(m_1+2)}$. The union of the interiors of these closed intervals (of length $4\cdot 3^{-(m_1+2)}$) is an open set, call it

G_2. By an argument like the one used by Utz in [3] it is easy to show that $D(G_2)$ contains $[-a + e, a - e]$.

Since G_2 is open and F_2 is nowhere dense, it follows from our lemma that $D(G_2\backslash F_2)$ contains $[-a + e, a - e]$. Furthermore, since F_2 is closed, $G_2\backslash F_2$ is open and can therefore be expressed as an at most countable union of pairwise disjoint open intervals, call them $\{I_{i2}\}$.

$$D(G_2\backslash F_2) = D(\bigcup_{i=1}^{\infty} I_{i2}) = \bigcup_{i,j} (I_{i2} - I_{j2}).$$

Proceeding as before, by the Heine Borel Covering Theorem finitely many of the open intervals $I_{i2} - I_{j2}$ cover the interval $[-a + e, a - e]$. Therefore there exists a natural number n_2 such that each of the sets in the finite subcover can be expressed as the algebraic difference of two of the sets in the collection $\{I_{i2}\}_{i=1}^{n_2}$.

There exists a natural number m_2, $m_2 > m_1 + 2$, and a sequence $\{J_{i2}\}_{i=1}^{n_2}$ of intervals with the following properties. Each J_{i2} is a closed interval that is contained in I_{i2}, whose endpoints are of the form $k/3^{m_2}$ and such that the difference set of the interior of the set $\bigcup_{i=1}^{n_2} J_{i2}$ contains the interval $[-a + e, a - e]$; again this is an easy exercise, left to the reader.

Proceeding as before, we subdivide each of the closed intervals J_{i2} into closed intervals of length 3^{-m_2}. Remove the open middle ninth of each of these intervals of length 3^{-m_2}; obtaining closed intervals of length $4\cdot 3^{-(m_2+2)}$. The union of the interiors of these intervals of length $4\cdot 3^{-(m_2+2)}$ is an open set, call it G_3. Again, by an argument like the one used by Utz in [3] it follows that $D(G_3)$ contains $[-a + e, a - e]$.

Continue this process by induction, obtaining sequences

$$\{n_i\}_{i=1}^{\infty}, \quad \{m_i\}_{i=1}^{\infty}, \quad \left\{\{I_{ij}\}_{i=1}^{n_j}\right\}_{j=1}^{\infty}, \quad \left\{\{J_{ij}\}_{i=1}^{n_j}\right\}_{j=1}^{\infty},$$

and $\{G_i\}_{i=2}^{\infty}$ that satisfy the following conditions. $m_{i+1} > m_i + 2$ for each i. For each j, $\{I_{ij}\}_{i=1}^{n_j}$ is a sequence of pairwise disjoint open intervals such that $\bigcup [I_{ij} - I_{kj} : i, k \in \{1,2,\dots,n_j\}]$ contains $[-a + e, a - e]$.

For each j, $\{J_{ij}\}_{i=1}^{n_j}$ is a sequence of closed intervals satisfying: The endpoints of J_{ij} are of the form $k/3^{m_j}$, $J_{ij} \subset I_{ij}$ and the difference set of the interior of the set $\bigcup_{i=1}^{n_j} J_{ij}$ contains the interval $[-a + e, a - e]$. In addition, each G_j is formed as follows. Subdivide each of the closed intervals $J_{i,j-1}$ into closed intervals of length $3^{-m_{j-1}}$. Remove the open middle ninth of each of these intervals of length $3^{-m_{j-1}}$; obtaining closed intervals of length $4 \cdot 3^{-(m_{j-1}+2)}$. G_j is defined to be the union of the interiors of these intervals of length $4 \cdot 3^{-(m_{j-1}+2)}$. G_j is open for each j and $D(G_j)$ contains the closed interval $[-a + e, a - e]$.

Finally we have

$$B \supset J\backslash F_1 \supset \bigcup_{i=1}^{n_1} I_{i1} \supset \bigcup_{i=1}^{n_1} J_{i1} \supset G_2 \supset G_2/F_2 \supset$$

$$\supset \bigcup_{i=1}^{n_2} I_{i2} \supset \bigcup_{i=1}^{n_2} J_{i2} \supset G_3 \supset G_3\backslash F_3 \quad \cdots \quad .$$

Let $P = \bigcap_{j=1}^{\infty} \left(\bigcup_{i=1}^{n_j} J_{ij} \right)$. Then P is a compact subset of B. Furthermore, since $\bigcup_{i=1}^{n_j} J_{ij} \supset P$ for each $j \geq 2$, P is porous. Finally, if $t \in [-a + e, a - e]$, for each j, there exists $x_j, y_j \in \bigcup_{i=1}^{n_j} J_{ij}$ such that $x_j - y_j = t$. By a double application of the Bolzano-Weierstrass Theorem, there exists a subsequence $\{j_k\}$ of the natural numbers such that $\lim_{k\to\infty} y_{j_k} = x$ and $\lim_{k\to\infty} y_{j_k} = y$. Clearly $x - y = t$. Furthermore, by the definition of P, $x, y \in P$ and therefore $D(P) \supset [-a + e, a - e]$, completing the proof.

We will complete the paper with a few observations.

<u>Remark</u>. The result presented here is related to a theorem of Tkadlec in [2]. Our proof made use of our lemma in conjunction with the fact that every Baire set of the second category contains an open interval less a first category set. The analogue of these statement, for sets of positive measure, is not true, i.e. a set of positive measure need not contain an open interval less a set of measure zero. Therefore our proof can not be used to get a measure theoretic analogue of our theorem. However, we conjecture that

every set of positive measure contains a porous set whose difference set contains an interval.

We remark that the set P constructed above is in fact uniformly porous, i.e. in the definition of porous the same number c can be used for each point of P. Also, in our construction any fraction smaller than one third could be used in place of one ninth when removing middle intervals.

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References

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