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NEARLY UPPER SEMICONTINUOUS GAUGE FUNCTIONS IN $\mathbb{R}^m$

INTRODUCTION. Using δ-fine partitions for a positive gauge function δ Henstock and Kurzweil defined a generalized Riemann integral, which is equivalent to the Denjoy-Perron integral ([H], [K] and [S], Chapter VIII). In the original definition of this generalized Riemann integral the function δ was a completely arbitrary positive function. P.S. Bullen, in [Q] raised the question of determining how complicated it need be. In [P2] W. Pfeffer proved that this integral can be defined using a function δ that is upper semicontinuous when restricted to a suitable subset whose complement has measure zero. In this proof he first showed that such a δ can be chosen if the integrand is Lebesgue integrable, and then he verified it for each step of the Denjoy-Perron definition.

Since the Denjoy process can be applied only on the real line, he asked whether this theorem remains true for the higher dimensional Henstock-Kurzweil integral [M1], or for its generalizations defined in [M], [JKS] and [P1]. In this paper we give a new proof of the original theorem. This proof avoids the Denjoy process and translates verbatim to the higher dimensional Henstock-Kurzweil integral, and it can be easily applied for other generalized Riemann integrals as well. Our proof is based on the fact that Henstock-Kurzweil integrable functions are Lebesgue measurable, and hence we can use Lusin's theorem. Finally we remark that there exists a Lebesgue integrable f and an $\varepsilon > 0$ so that there exists no Borel measurable gauge function for this ε [FM, Example 1]; that is, one can not expect that δ is upper semicontinuous everywhere.

PRELIMINARIES. By $\mathbb{R}$ we denote the real numbers. By intervals in $\mathbb{R}^m$ we mean sets of type $\prod_{i=1}^{m} [a_i,b_i]$; $a_i,b_i \in \mathbb{R}$, $a_i < b_i$.

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A collection of intervals whose interiors are disjoint is called a non-overlapping collection. By int(E), diam(E), and mes(E) we denote respectively the interior, the diameter and the measure of the set $E \subset \mathbb{R}^m$. For $E_1, E_2 \subset \mathbb{R}^m$ we put $\operatorname{dist}(E_1,E_2) := \inf\{\operatorname{dist}(x,y) : x \in E_1,\ y \in E_2\}$. A function δ on an interval A is called nearly upper semicontinuous if there is a set $H \subset A$ such that $\operatorname{mes}(A\setminus H) = 0$ and $\delta|_H$ is upper semicontinuous.

A subpartition of an interval A is a collection $P = \{(A_1,x_1),\ldots,(A_p,x_p)\}$ where $A_1,\ldots,A_p$ are non-overlapping subintervals of A, and $x_i \in A_i$, $i = 1,\ldots,p$. If, in addition, $\bigcup_{i=1}^{p} A_i = A$, we say that P is a partition of A. Given a function $\delta : A \to (0,+\infty)$ we say that a subpartition P is δ-fine whenever $\operatorname{diam}(A_i) < \delta(x_i)$ for $i = 1,\ldots,p$. If f is a function on an interval $A \subset \mathbb{R}^m$ and $P = \{(A_1,x_1),\ldots,(A_p,x_p)\}$ is a subpartition of A, then we let

$$\sigma(f,P) := \sum_{i=1}^{p} f(x_i)\ \operatorname{mes}(A_i).$$

DEFINITION 1. (Henstock-Kurzweil) A function f on an interval $A \subset \mathbb{R}^m$ is called integrable in A if there is a real number $I =: \int_A f$ with the following property: for every $\varepsilon > 0$ there exists a $\delta : A \to (0,+\infty)$ such that $|\sigma(f,P) - I| < \varepsilon$ for each δ-fine partition P of A.

The reader interested in the properties of this integral can find further references in [P1] and [P2]. We denote the set of Henstock-Kurzweil integrable functions by $\mathcal{R}(A)$. We shall use the property that every $f \in \mathcal{R}(A)$ is Lebesgue measurable; this is a special case of Corollary 4.5 in [P1]. The function δ in Definition 1 is often called a gauge associated with f and ε. We denote by $\Delta(f,A;\varepsilon)$ the family of all gauge functions associated with $f \in \mathcal{R}(A)$ and $\varepsilon > 0$.

THEOREM. For every $A \subset \mathbb{R}^m$, $f \in \mathcal{R}(A)$ and $\varepsilon > 0$ the set $\Delta(f,A;\varepsilon)$ contains a nearly upper semicontinuous gauge function.

PROOF. Since it is easy to show by a compactness argument that if $\varepsilon > 0$ and $f \in \mathcal{R}(A)$, then $\Delta(f,A;\varepsilon) \neq \phi$, (See e.g. [P1], Proposition 2.4.) we can choose a function $\delta_0 \in \Delta(f,A;\varepsilon/2)$. Plainly we may assume that δ_0 is bounded on A.

By Lusin's theorem we can choose pairwise disjoint closed sets $F_i \subset A$, $i = 1,\ldots$ so that $f|_{F_i}$ is continuous and $\operatorname{mes}(A \setminus \bigcup_{i=1}^{+\infty} F_i) = 0$. We shall define a nearly upper semicontinuous gauge function $\delta \in \Delta(f,A;\varepsilon)$ as follows. For $x \in F_j$ $(j = 1,2,\ldots)$ we let $\delta(x) := \min\{1/j, \max\{\delta_0(x), \limsup_{\substack{y\to x\\ y\in F_j}} \delta_0(y)\}\}$. It is obvious that $\delta|_{F_j}$ is upper semicontinuous. If $x \in A \setminus \bigcup_{i=1}^{+\infty} F_i$, then we put $\delta(x) := \delta_0(x)$. If

$$\delta'(x) = \begin{cases} \delta(x) & \text{if } x \in \bigcup_{i=1}^{+\infty} F_i \\ 0 & \text{otherwise}, \end{cases}$$

then $\delta' = \delta$ almost everywhere. We show that δ' is upper semicontinuous and hence δ is nearly upper semicontinuous. Suppose that $\lim_{n\to+\infty} x_n = x$ and for every i the number of n's with $x_n \in F_i$ is finite. Since $\delta(x) \leq 1/i$ for every $x \in F_i$, it follows that $\lim_{n\to+\infty} \delta'(x_n) = 0$. Hence we obtain that δ' is upper semicontinuous at the points of $A \setminus \bigcup_{i=1}^{+\infty} F_i$. If $x \in F_j$, then every sequence $x_n \to x$ $(n \to +\infty)$ can be divided into two subsequences x_{k_n} $(n = 1,\ldots)$ and x_{ℓ_n} $(n = 1,\ldots)$ so that x_{k_n} consists of those elements of x_n which belong to F_j and the remaining terms of x_n are in x_{ℓ_n}. Since the sets F_i are pairwise disjoint and closed, the sequence x_{ℓ_n} can contain only finitely many terms belonging to a fixed F_i $(i = 1,\ldots,i\neq j)$ and hence the preceding argument shows that $\lim_{n\to\infty} \delta'(x_{\ell_n}) = 0$. Since δ' is non-negative and since $\delta'|_{F_j}$ is upper semicontinuous, we conclude that $\limsup_{n\to\infty} \delta'(x_{k_n}) \leq \delta'(x)$; that is, we have proved that δ' is upper semicontinuous.

We have yet to show that $\delta \in \Delta(f,A;\varepsilon)$. Suppose that $P = \{(A_1,x_1),\dots,(A_p,x_p)\}$ is a δ-fine subpartition of A. If $x_k \in A \setminus \bigcup_{i=1}^{+\infty} F_i$, then we let $x'_k := x_k$. Suppose that $x_k \in F_j$. If $\delta(x_k) > \delta_0(x_k)$, then

$$\limsup_{\substack{y \to x_k \\ y \in F_j}} \delta_0(y) > \delta(x_k).$$

Since $f|_{F_j}$ is continuous, we can choose an $x'_k \in F_j$ so that

$$|f(x'_k) - f(x_k)| < \varepsilon/(2\,\mathrm{mes}(A)) \quad \text{and}$$

$$\delta_0(x'_k) > \delta(x_k) \ (> \mathrm{diam}(A_k)).$$

Hence $P' := \{(A_1,x'_1),\dots,(A_p,x'_p)\}$ is a δ_0-fine subpartition of A. Thus $|\sigma(f,P') - \int_A f| < \varepsilon/2$.

We also have

$$|\sigma(f,P') - \sigma(f,P)| \le \sum_{i=1}^{p} |f(x'_i) - f(x_i)| \cdot \mathrm{mes}(A_i) <$$

$$< (\varepsilon/(2\,\mathrm{mes}(A))) \sum_{i=1}^{p} \mathrm{mes}(A_i) = \varepsilon/2.$$

Hence we proved that $|\sigma(f,P) - \int_A f| < \varepsilon$; that is, $\delta \in \Delta(f,A;\varepsilon)$. This completes the proof of the Theorem.

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