INROADS

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A COMPARISON OF TWO GENERALIZATIONS OF THE RIEMANN INTEGRAL

When considering one of the many extensions of the Riemann integral on an interval [a,b], it is often desirable for the extended integral to be broad enough to admit every derivative into its class of integrable functions. The Henstock integral is one such generalized Riemann integral. Another such integral was described in a recent article by Michael W. Botsko ([2]).

It is the purpose of this paper to compare these two integrals and to show that the class of functions integrable by Botsko's definition is a proper subset of the class of Henstock-integrable functions.

Botsko called his integral the G-integral, described it as "an easy generalization of the Riemann integral," and suggested that it would be an appropriate integral to be presented to an undergraduate class in advanced calculus.

We now outline the construction of the G-integral. For the definitions to follow, f is assumed to be a real-valued function on the closed interval [a,b].

DEFINITION 1. A real-valued function U on [a,b] is said to be an <u>upper function</u> of f on [a,b] if

(1) $U(a) = 0$ and U is continuous on [a,b], and

(2) $U'(x) \geq f(x)$ for all but a finite number of points of [a,b].

DEFINITION 2. A real-valued function L on [a,b] is said to be a <u>lower function</u> of f on [a,b] if

(1) $L(a) = 0$ and L is continuous on [a,b], and

(2) $L'(x) \leq f(x)$ for all but a finite number of points of [a,b].

If f has at least one upper function and one lower function on [a,b], then it is easily shown that the set

$S = \{\, U(b) \mid U \text{ is an upper function of } f \,\}$

is bounded below, and the set

$T = \{\, L(b) \mid L \text{ is a lower function of } f \,\}$

is bounded above. Then the upper G-integral of f is defined to be the infimum of S and is denoted $G\ \overline{{}_a\int^b} f$, and the lower G-integral of f is defined to be the supremum of T and is denoted by $G\ \underline{{}_a\int^b} f$.

DEFINITION 3. If $G\ \underline{{}_a\int^b} f = G\ \overline{{}_a\int^b}$, then this number is the G-integral of f on [a,b] and is denoted by $G\ {}_a\int^b f$.

Botsko ([2]) shows that if f is Riemann integrable on [a,b], then f is G-integrable on [a,b] and these integrals have the same value. He also states without proof a modification of some results by Katznelson and Stromberg ([4]) and shows that this modified theorem implies that the well-known Dirichlet function, which is of course not Lebesgue integrable, is G-integrable.

The Henstock integral is well known to the readers of this journal. (The definition can be found in, for example, [5].) We will just mention that it is given by Riemann's definition with the positive constant δ replaced by a positive function γ on [a,b], called a gauge.

It is obvious that the G-integral is a simple modification of the Perron integral, in that replacing the word "finite" in Definitions 1 and 2 by "countable" will yield the definition of the Perron integral. Since it is well-known that the Henstock integral is equivalent to the Perron integral, it is clear that any function which is G-integrable is also Henstock integrable.

Our goal here is to show that the G-integral is not as general as the Henstock integral (and hence the Perron integral).

THEOREM: If f is unbounded above on every subinterval of [a,b], then f is not G-integrable on [a,b].

Proof: Suppose such a function f is G-integrable on [a,b]. Let U be an upper function of f. For some subinterval [c,d] of [a,b], $U'(x) \geq f(x)$ for every x in [c,d] and thus U' is unbounded above on every subinterval of [c,d]. But since U' is a Baire 1 function it is continuous on a dense subset of [c,d] ([3],p.46 and [1], p. 117). This implies that there must be a subinterval of [c,d] on which U' is bounded and thus we have a contradiction. Hence f is not G-integrable on [a,b].

We now construct a function which is Henstock integrable but not G-integrable on [0,1].

EXAMPLE: Let f be defined on [0,1] as follows.

f(x) = 0 if x is irrational;

f(x) = n if x = m/n, where m and n are relatively prime natural numbers;

$f(0) = f(1) = 0$.

Clearly, f is unbounded on every subinterval of [0,1] and thus by the theorem is not G-integrable on [0,1]. But since f is constant on all but a countable subset of [0,1], f is Henstock integrable on [0,1]. (See [5] for the proof.) On any interval examples similar to the one above can easily be constructed. From these examples and from the above theorem it follows that over any interval [a,b] the class of G-integrable functions is a proper subset of the class of Henstock integrable functions.

References

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