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Clifford B. Cordy, University of California, Santa Barbara, CA 93106

On the Relationship Between the External Intersection Condition and the Intersection Condition for Path Derivatives

1. Introduction. The notion of path derivative was introduced in the paper [B,O,T] to unify the study of certain generalized derivatives. Several natural geometric conditions on path systems have been studied. When the paths within a path system satisfy these conditions, any function that is differentiable with respect to the system will have many of the desirable properties known to be possessed by differentiable functions. Prominent among the conditions on path systems which have been studied are porosity conditions and intersection conditions. Porosity conditions have been studied extensively (see [B,L,P,T], [B,O,T], [B,T], [T]). However, beyond the work [B,O,T] intersection conditions have been studied very little. The two intersection conditions which have proven to be most useful are known as the intersection condition and the external intersection condition.

In the paper [B,O,T] some results are obtained using the intersection condition while others are obtained using the external intersection condition, but the relationship between these conditions is not explored. The purpose of this paper is to make explicit the relationship between these conditions. In particular, it will be shown that if E is a path system which satisfies the external intersection conditon and F is an E-differentiable function, then there is a path system E^* which satisfies both the intersection condition and the external intersection conditions so that F is E^* differentiable and $F'_E \equiv F'_{E^*}$. In other words, when considering path differentiable functions the external intersection condition is stronger than the intersection condition. An example will be constructed showing that the intersection condition is not as strong as the external intersection condition. On the other hand, when studying functions which are not path differentiable the situation seems to be reversed. To illustrate this, some theorems which involve extreme path derivatives and depend on the intersection condition will be given, and then examples will be constructed showing that the external intersection condition is not sufficient for these results.

2. Preliminaries. We will begin with the basic definitions. Let $x \in R$. A path leading to x is a set $E_x \subset R$ such that $x \in E_x$ and x is a point of accumulation of E_x. A system of paths, or path system, is a collection $E = \{E_x : x \in R\}$ where each E_x is a path leading to x. A function F is said to be E-differentiable at x if

$$F'_E(x) = \lim_{\substack{y \to x \\ y \in E_x}} \frac{F(y) - F(x)}{y - x}$$

exists and is finite. In this case $F'_E(x)$ is called the E-derivative of F at x. The extreme E-derivatives $\overline{F}'_E(x)$ and $\underline{F}'_E(x)$ are defined similarly.

2.1 Definition. A path system E will be said to satisfy the condition stated below if there is a positive function δ such that whenever $0 < y - x < \min\{\delta(y), \delta(x)\}$ the paths E_x and E_y intersect in the stated fashion.

(1) intersection condition $E_x \cap E_y \cap [x, y] \neq \phi$

(2) external intersection condition (parameter m)

$$E_x \cap E_y \cap [x - m(y - x), x] \neq \phi \text{ and } E_x \cap E_y \cap [y, y + m(y - x)] \neq \phi$$

In the remainder of this paper when dealing with the external intersection condition any proofs will be given only for the case m=1, but the results hold for any $m > 0$.

3. Results. We will now state our main theorem.

3.1 Theorem. Let E be a system of paths which satisfies the external intersection condition. Suppose F is a function which is E-differentiable on a set A. Then there is a path system E^* which satisfies both the intersection condition and the external intersection condition so that F is E^* differentiable on A and $F'_{E^*}(x) = F'_E(x)$ for $x \in A$.

Before proving this theorem we will need the following lemma.

3.2 Lemma. Let E be a system of paths which satisfies the external intersection condition. Suppose F is a function which is E-differentiable on a set A. Given any $\varepsilon > 0$ there is a positive function δ so that whenever $x,y \in A$ and $0 < y - x < \min\{\delta(y),\delta(x)\}$ each of the following holds:

(a) $$|F'_E(x) - F'_E(y)| < \varepsilon$$

(b) $$\left|\frac{F(x) - F(y)}{x - y} - F'_E(x)\right| < \varepsilon$$

(c) $$\left|\frac{F(x) - F(y)}{x - y} - F'_E(y)\right| < \varepsilon$$

Proof. (a) Let $\varepsilon > 0$ be given. Let δ_1 be the function associated with E by the external intersection condition. Since F is E-differentiable on A, there is a positive function δ_2 so that if $x \in A$, $t \in E_x$ and $0 < |t - x| < 2\delta_2(x)$, then $\left|\frac{F(t) - F(x)}{t - x} - F'_E(x)\right| < \frac{\varepsilon}{4}$.
Define $\delta_3 = \min\{\delta_1, \delta_2\}$.

Suppose $x,y \in A$ and $0 < y - x < \min\{\delta_3(y),\delta_3(x)\}$. By the choice of δ_1 there are points a and b so that $a \in E_x \cap E_y \cap [2x - y, x]$ and $b \in E_x \cap E_y \cap [y, 2y - x]$.

We will now show that $\left|\frac{F(b) - F(a)}{b - a} - F'_E(x)\right| < \frac{\varepsilon}{2}$.

If $x = a$, this inequality follows immediately from the choice of δ_2, so we will assume $x \neq a$.
Now use the fact that $\frac{b - x}{b - a} + \frac{x - a}{b - a} = 1$ to compute

$$\left|\frac{F(b) - F(a)}{b - a} - F'_E(x)\right| \leq \left|\frac{F(b) - F(x)}{b - x} - F'_E(x)\right|\left|\frac{b - x}{b - a}\right| + \left|\frac{F(x) - F(a)}{x - a} - F'_E(x)\right|\left|\frac{x - a}{b - a}\right|$$
$$< \varepsilon/4 + \varepsilon/4 = \varepsilon/2.$$

It follows similarly that $\left|\frac{F(b) - F(a)}{b - a} - F'_E(y)\right| < \frac{\varepsilon}{2}$.

Combining these two inequalities we get $|F'_E(x) - F'_E(y)| < \varepsilon$ which proves (a).

To prove (b), we will use the result (a) to pick a positive function δ_4 so that if $x,y \in A$ and $0 < y - x < \min\{\delta_4(y),\delta_4(x)\}$, then $|F'_E(x) - F'_E(y)| < \frac{\varepsilon}{4}$.
Define $\delta = \min\{\delta_1,\delta_2,\delta_4\}$.

Suppose $0 < y - x < \min\{\delta(y),\delta(x)\}$. By the choice of δ_1 there is a point $b \in E_x \cap E_y \cap [y, 2y - x]$. If $y = b$, then by the choice of δ_2

$$\left|\frac{F(y) - F(x)}{y - x} - F'_E(x)\right| < \frac{\varepsilon}{4},$$

so we will assume $y \neq b$. Now we compute,

$$\left|\frac{F(y) - F(x)}{y - x} - F'_E(x)\right| \le \left|\frac{F(y) - F(b)}{y - b} - F'_E(x)\right|\left|\frac{y - b}{y - x}\right| + \left|\frac{F(b) - F(x)}{b - x} - F'_E(x)\right|\left|\frac{b - x}{y - x}\right|$$

$$\le \left|\frac{F(y) - F(b)}{y - b} - F'_E(y)\right| + |F'_E(y) - F'_E(x)|$$

$$+ \left|\frac{F(b) - F(x)}{b - x} - F'_E(x)\right| 2$$

$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{2\varepsilon}{4} = \varepsilon.$$

This proves (b). The inequality (c) is proven similarly.

Proof of Theorem 3.1. For each positive integer n let $\varepsilon_n = 2^{-n}$ and take δ_n to be the positive function corresponding to ε_n given by lemma 3.2. We can assume without loss of generality that $\{\delta_n\}$ is a strictly decreasing sequence of functions tending to zero.

For each $x \in A$ define the set

$$B_x = \bigcup_{n \geq 1} \{t \in R: \delta_{n+1}(x) \leq |t - x| < \delta_n(x) \text{ and } |t - x| < \delta_n(t)\}.$$

The path system $E^* = \{E_x^*: x \in R\}$ is then defined by,

$$E_x^* = \begin{cases} E_x \cup B_x & \text{if } x \in A \\ R & \text{if } x \notin A. \end{cases}$$

The fact that E^* satisfies the external intersection condition follows immediately from the fact that E does. We will now define the positive function δ which will be used to show that E^* satisfies the intersection condition. Set $\delta = \min\{\delta_0,\delta_1\}$ where δ_0 is the function associated with E^* by the external intersection condition and δ_1 is the function described above.

Suppose $0 < y - x < \min\{\delta(y),\delta(x)\}$. If either $x \notin A$ or $y \notin A$, then the paths E_x^* and E_y^* clearly intersect as desired, so we will assume $x \in A$ and $y \in A$. Pick positive integers n and m so that $\delta_{n+1}(x) \leq y\text{-}x < \delta_n(x)$ and $\delta_{m+1}(y) \leq y\text{-}x < \delta_m(y)$. Consider the following cases.

Case 1: $n > m$.

Since $\{\delta_k(x)\}$ is a decreasing sequence $\delta_n(x) < \delta_m(x)$, so $|y - x| < \delta_m(x)$. Therefore $x \in E_y^*$, so $E_x^* \cap E_y^* \cap [x, y] \neq \phi$.

Case 2: $n = m$.

Reasoning as in case 1 we get $x \in E_y^*$ and $y \in E_x^*$.

Case 3: $n < m$.

Reversing the roles of x and y and applying case 1 we get $y \in E_x^*$, so $E_x^* \cap E_y^* \cap [x, y] \neq \phi$.

In each of these cases E_x^* and E_y^* intersect as desired so E^* satisfies the intersection condition.

We will now show that F is E^*-differentiable on A. Let $\varepsilon > 0$ and $x \in A$ be given. Pick n so $\varepsilon_n = 2^{-n} \leq \varepsilon$. Suppose $t \in B_x$ and $0 < |t - x| < \delta_n(x)$. By the choice of B_x we have $|t - x| < \delta_n(t)$, so by the choice of δ_n we have $\left|\frac{F(t) - F(x)}{t - x} - F'_E(x)\right| < \varepsilon_n \leq \varepsilon$.

Thus $\lim\limits_{\substack{t \to x \\ t \in B_x}} \frac{F(t) - F(x)}{t - x} = F'_E(x)$. Since $E^*_x = E_x \cup B_x$ it follows that

$$\lim_{\substack{t \to x \\ t \in E^*_x}} \frac{F(t) - F(x)}{t - x} = F'_E(x).$$

Remark. The above proof shows that E^* satisfies a rather special case of the intersection condition. In particular, when $0 < y - x < \min\{\delta(y),\delta(x)\}$ either $y \in E^*_x$ or $x \in E^*_y$. Perhaps this condition can be further exploited in cases where the intersection condition is not strong enough. If F is E-differentiable on [0,1] and it is possible to pick E^* and δ so that when $0 < y - x < \min\{\delta(y),\delta(x)\}$ both $y \in E^*_x$ and $x \in E^*_y$, then the function F actually has a composite derivative (as studied by O'Malley in [O], [O,W]). In fact, the existence of such a δ is equivalent to the existence of a composite derivative. It would therefore be of interest to find out if it is possible to choose such a δ under the hypotheses of theorem 3.1.

Theorem 3.1 can easily be used to obtain some new results since any theorem which holds for path systems that satisfy the intersection condition also holds for path systems satisfying the external intersection condition as long as the functions under consideration are assumed to be path differentiable. We now state some of these results which were originally stated for the intersection condition in [B,J] and [B,O,T].

3.3 Corollary. Let E and E^* be path systems both of which satisfy the external intersection condition. If F is both E and E^* differentiable on a set A, then the set $\{x \in A: F'_E(x) \neq F'_{E^*}(x)\}$ is at most denumerable.

3.4 Corollary. Let E be a system of paths which satisfies the external intersection condition. If the function F is continuous and E-differentiable on an interval I_0, then F is differentiable in the ordinary sense on a residual subset of I_0.

3.5 Corollary. Let E be a system of paths which is bilateral (i.e. for each x, x is a bilateral point of accumulation of E_x) and satisfies the external intersection condition. If F is E-differentiable, then F'_E has the Darboux and the Denjoy properties.

3.6 Corollary. Let E be a system of paths which satisfies the external intersection condition. Suppose F is E-differentiable on a measurable set A and $|F'_E| \leq M$ on A. Then F(A) is a measurable set and $\lambda(F(A)) < M\lambda(A)$.

Theorem 3.1 can be thought of as giving an "improvement" of the path system E. That is, under the hypotheses of the theorem E can be "improved" giving a path system E^*, which also satisfies the intersection condition, without altering the path derivative of F. We will now show that it is not always possible to improve a path system which satisfies the intersection condition to a path system that satisfies the external intersection condition in the sense of Theorem 3.1. Towards this end we give a theorem from [B,O,T].

3.7 Theorem. Let E be a system of paths which satisfies the external intersection condition. If F is E-differentiable then F'_E is a Baire 1 function.

The following example shows that it is possible to have a system of paths E which satisfies the intersection condition and an E-differentiable function F so that F'_E is not a Baire 1 function. Clearly, such a path system can not be improved to one which satisfies the external intersection condition without changing the path derivative of F.

3.8 Example. Let P_0 be a perfect subset of a Hamel basis which contains a rational number (see [J]). For each integer $n \neq 0$ define $P_n = P_0 + \frac{1}{n}$. By the choice of P_0, the sets P_n are pairwise disjont, and the set $P = \bigcup_{n \in R} P_n$ is closed.

Define a function F on P by

$$F(x) = \begin{cases} 0 & \text{if } x \in P_0 \\ \frac{1}{n} & \text{if } x \in P_n \text{ for some } n \neq 0. \end{cases}$$

Since F is continuous on P, F can be extended to a continuous function on R by defining F linearly on each component of $R \setminus P$.

Let $\{x_k\}$ denote the sequence of one sided limit points of P_0. Define the path system E as follows

$$E_x = \begin{cases} R & \text{if } x \notin P \\ P_n & \text{if } x \in P_n \text{ for some } n \neq 0 \\ P_0 & \text{if } x \in P_0 \setminus \{x_k\} \\ \{x\} \cup \{x + \frac{1}{n} : n \neq 0\} & \text{if } x = x_k \text{ for some } k. \end{cases}$$

It is evident that for $x \in P_0$, $F'_E(x) = \begin{cases} 0 \text{ if } x \in P_0 \setminus \{x_k\} \\ 1 \text{ if } x = x_k \text{ for some } k \end{cases}$.

Thus since $\{x_k\}$ and $P_0 \setminus \{x_k\}$ are each dense in P_0, F'_E has no relative points of continuity in P_0. This means F'_E is not a Baire 1 function.

It remains to be shown that E satisfies the intersection condition. Define the associated function δ by

$$\delta(x) = \begin{cases} \min\{\text{dist}(P_k, P_n) : |k| \leq |n|, k \neq n\} & \text{if } x \in P_n, n \neq 0 \\ \min\{|x_j - x| : j < k\} & \text{if } x = x_k \text{ for some } k \\ 1 & \text{if } x \in P_0 \setminus \{x_k\} \\ \text{dist}(x, P) & \text{if } x \notin P \end{cases}$$

Suppose $0 < y - x < \min\{\delta(y),\delta(x)\}$. By the choice of δ it is impossible to have $x \in P$ and $y \notin P$ (or vice-versa) or ot have $x \in P_m$ and $y \in P_n$ with $m \neq n$. Thus the only possible cases are: (a) $x \notin P$ and $y \notin P$, (b) for some $n \neq 0$, $x \in P_n$ and $y \in P_n$, (c) $x \in P_0$ and $y \in P_0$. In the cases (a) and (b) it is trivial to show that $E_x \cap E_y \cap [x, y] \neq \phi$, so we will only consider (c). There are two possible subcases, (i) $x \in P_0 \backslash \{x_k\}$ and $y \in P_0 \backslash \{x_k\}$, (ii) either $x \in \{x_k\}$ or $y \in \{x_k\}$ but not both. In case (i) $x, y \in E_x \cap E_y$. In the second case either $x \in E_y$ or $y \in E_x$ depending on whether $x \in \{x_k\}$ or $y \in \{x_k\}$. Thus in any of the above cases $E_x \cap E_y \cap [x, y] \neq \phi$ so E satisfies the intersection condition.

Now that we have established the relationship between the external intersection condition and the intersection condition for path differentiable functions, we will consider the situation when the functions being studied are not assumed to be path differentiable. Although no general relationship has been established in this case, it appears that the intersection condition is more useful than the external intersection condition when studying functions by means of their extreme path derivatives. To illustrate this situation we will give three theorems which rely on the use of the intersection condition, and we will give corresponding examples which show the external intersection condition does not suffice in these situations. The first of these theorems appears in [B,O,T].

3.9 Theorem. Let E be a system of paths that satisfies the intersection condition. If everywhere in a set A one of the extreme derivatives $\overline{F}'_E(x)$ or $\underline{F}'_E(x)$ is finite, then F is VBG on A.

As the following example shows this result does not hold for the external intersection condition.

3.10 Example. Let F be a function which attains every real value on every subinterval of R. Clearly, such a function cannot be VBG. Define a path system E as follows, for each $x \in R$ set

$$E_x = \{x\} \cup \{t: \frac{F(t) - F(x)}{t - x} \geq 0\}.$$

From this choice of E it is evident that $\underline{F}'_E(x) = 0$ for each $x \in R$. For this example, it is unnecessary to define the δ function since the paths of any two points will intersect as desired. To see this, let $x,y \in R$ be given. Suppose $x < y$. Since F attains every real value in each interval, there is a point $t \in (y, 2y - x]$ so that $F(t) > F(y)$ and $F(t) > F(x)$. Now $t \in E_x \cap E_y \cap [y, 2y - x]$. Similarly, $E_x \cap E_y \cap [2x - y, x] \neq \phi$, so E satisfies the external intersection condition.

Corollary 3.3 is derived from the following theorem which appears in [B,O,T].

3.11. Theorem. Let E and E^* be path systems both of which satisfy the intersection condition. Then for any function F the set $\{x: \overline{F}'_E(x) < \underline{F}'_{E*}(x)\}$ is at most denumerable.

This theorem is used in [B,J] to obtain the following.

3.12 Theorem. Let E be a system of paths satisfying the intersection condition. Suppose F is a function which satisfies the inequalities $|\overline{F}'_E| \leq M$ and $|\underline{F}'_E| \leq M$ on a measurable set A. Then the set F(A) is measurable and $\lambda(F(A)) \leq M\lambda(A)$.

We will now discuss an example which shows that neither of these theorems hold if the intersection condition is replaced by the external intersection condition.

3.13 Example. There is a function F defined on [0,1], a perfect set $P \subset [0, 1]$ and a path system E satisfying the external intersection condition for which each of the following holds:

(1) F is approximately differentiable a.e. on P with $F'_{ap}(x) = 2$ a.e. on P.

(2) For each $x \in P$, $-1 \leq \underline{F}'_E(x) \leq \overline{F}'_E(x) \leq 1$

(3) $0 < \lambda(F(P)) = 2\lambda(P)$.

Before discussing the example, we will consider the consequences of (1), (2), and (3). Since the approximate derivative F'_{ap} exists a.e. on P, there is a path system E^* which satisfies the external intersection condition so that F'_{E*} exists and equals F'_{ap} wherever the latter exists. Thus the fact that $\{x \in P: \overline{F}'_E(x) \leq 1 < 2 = F'_{E*}(x)\}$ has positive measure shows that the conclusion of theorem 3.11 does not hold for the external intersection condition. The combination of (2) and (3) shows that the conclusion of theorem 3.12 does not hold for the external intersection condition.

We will now outline the construction of the example. For complete details see [C]. Let P be a Cantor set with $\lambda(P) = \frac{1}{2}$. Define F on P by $F(x) = 2x$. F is defined on $[0,1]\setminus P$ so that F oscillates between 0 and 2 on each interval contiguous to P. The path system E is defined as follows,

$$E_x = \begin{cases} \{x\} \cup \{t: \left|\dfrac{F(t) - F(x)}{t - x}\right| \leq 1\} & \text{if } x \in P \\ R & \text{if } x \in [0,1]\setminus P \end{cases}.$$

From these choices of F, P, and E it is clear that (1), (2), and (3) all hold. The difficulty is in showing that E satisfies the external intersection condition. This is made possible by ensuring that F oscillates rapidly enough on the intervals contiguous to P.

The results presented here were obtained while working on my dissertation under the direction of A.M. Bruckner.

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