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Some Hausdorff variants of absolute continuity, Banach's condition(S) and Lusin's condition(N)

The old concept of Hausdorff dimensions has been given a revived interest due to its usefulness in the recent development of non-linear dynamic theory and fractal geometry. Thus, it seems worthwhile to use the concept of Hausdorff measures to consider some natural variants of the well-known concepts of the absolute continuity, the Banach's condition (S) and the Lusin's condition (N). To take into account some of the recent works by Foran [2], Iseki [3] and Ene [1], some variants using notions closely related to the Hausdorff measures are also considered. It should be noted, however, that thorough investigations of such variants and their possible applications to dynamic theory and fractal geometry still remain to be done.

First, let us make precise the notions that are closely related to Hausdorff measures. Let E be a set of real numbers. For such positive integer n, and for such positive real number β, let

$\lambda_n^\beta(E) = \inf\{ \sum_{i=1}^{n} |I_i|^\beta$: $\langle I_i \rangle_{i=1}^{n}$ is a sequence of n open intervals which covers E}, and also let

$$\lambda_\infty^\beta(E) = \inf \{ \sum_{i=1}^\infty |I_i|^\beta : \langle I_i \rangle_{i=1}^\infty \text{ is a sequence of open intervals which covers } E\},$$

and

$$\lambda_\omega^\beta(E) = \inf\{\lambda_n^\beta(E): \text{ n is a positive integer}\}.$$

It follows easily that one has the following:

(1) $0 \le \lambda_k^\beta(E) \le \lambda_k^\beta(F)$ whenever $E \subset F$, where k is a positive integer, or ∞, or ω.

(2) $\lambda_\infty^\beta(E) \le \lambda_\omega^\beta(E) \le \lambda_{n+1}^\beta(E) \le \lambda_n^\beta(E)$.

(3) $\lambda_\infty^\beta(\bigcup_i E_i) \le \sum_i \lambda_\infty^\beta(E_i)$.

<u>Remark 1</u>. There are sets E such that each of the strict inequalities in (2) holds. However, for compact set E, one has $\lambda_\omega^\beta(E) = \lambda_\infty^\beta(E)$. Note that (3) does not hold if ∞ is replaced by ω or a positive integer.

<u>Remark 2</u>. The β-dimensional Hausdorff outer measure of a set E is $\Lambda_\beta(E)$, which is defined as

$$\Lambda_\beta(E) = \lim_{\delta\to 0+} [\inf\{ \sum_{i=1}^\infty |I_i|^\beta : \langle I_i\rangle \text{ is a sequence of open intervals which covers E and } |I_i| < \delta \text{ for each } i\}].$$

Thus, we have $\lambda_\infty^\beta(E) \le \Lambda_\beta(E)$. But for $\beta = 1$, one has $\lambda_\infty^1(E) = \Lambda_1(E)$, which is just the Lebesgue outer measure of E. (See Saks [5]). In what follows, in the place of λ_∞^β one can use Λ_β and obtain a related concept. However, we do not consider such related concepts here.

<u>Definition 1</u>. A (real-valued) function f (of a real variable) is said to be $AC_k(\beta,\alpha)$ on a set E, written as $f \in AC_k(\beta,\alpha)$ on E, if for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\sum_i \lambda_k^{\beta}(f[E\cap I_i]) < \epsilon$$

whenever $<I_i>$ is a sequence of non-overlapping closed intervals with endpoints in E and $\sum_i |I_i|^{\alpha} < \delta$. [Here and later on, k denotes a positive integer n, or ∞ or ω, and α and β are positive numbers.]

<u>Remark 3.</u> It follows from (2) that one has

(4) $\quad AC_n(\beta,\alpha) \subset AC_{n+1}(\beta,\alpha) \subset AC_{\omega}(\beta,\alpha) \subset AC_{\infty}(\beta,\alpha)$ on E,

and later on we will see that for some E, the strict inclusions in (4) hold.

<u>Remark 4</u>. It is clear that $AC_1(1,1)$ is just the absolute continuity in the wide sense (i.e. AC as given in Saks [5]), so that $AC_k(1,1)$ is a generalization of AC when $k \neq 1$, and $AC_k(\beta,\alpha)$ can be thought as a Hausdorff variant of $AC_k(1,1)$. Note also that AC is an additive class, while we will see later that $AC_k(\beta,\alpha)$ may not be so when $k \neq 1$. (Cf. remarks 9 and 10.)

<u>Definition 2</u>. A function f is said to be $D_k(\beta)$ on E, written as $f \in D_k(\beta)$ on E, if for each $\epsilon > 0$ there exists a sequence $<I_i>$ of

non-overlapping closed intervals which covers E and

$$\sum_i \lambda_k^\beta (f[E \cap I_i]) < \epsilon.$$

A function f is said to be $E_k(\beta,\alpha)$ on E, written as $f \in E_k(\beta,\alpha)$ on E, if $f \in D_k(\beta)$ on Z whenever $Z \subset E$ and $\lambda_\infty^\alpha(Z) = 0$.

Remark 5. Similar to (4), one easily has

(5) $$E_n(\beta,\alpha) \subset E_{n+1}(\beta,\alpha) \subset E_\omega(\beta,\alpha) \subset E_\infty(\beta,\alpha) \text{ on } E.$$

Furthermore, we prove

(6) $$AC_k(\beta,\alpha) \subset E_k(\beta,\alpha) \text{ on } E.$$

Proof. Let $f \in AC_k(\beta,\alpha)$ on E and let $Z \subset E$ and $\lambda_\infty^\alpha(Z) = 0$, and $\epsilon > 0$. Since $f \in AC_k(\beta,\alpha)$ on E, it is so on Z. Hence there exist $\delta > 0$ such that

$$\sum \lambda_k^\beta(f[Z \cap I_i]) < \epsilon$$

whenever $\langle I_i \rangle$ is a sequence of non-overlapping intervals with endpoints in Z and $\sum_i |I_i|^\alpha < \delta$. Now, since $\lambda_\infty^\alpha(Z) = 0$, there exists a sequence $\langle J_i \rangle$ of non-overlapping closed intervals which covers Z and $\sum |J_i|^\alpha < \delta$. Of course, we may assume that $J_i \cap Z \neq \phi$ for each i. If I_i is an interval with endpoints in $J_i \cap Z$, one has $\sum |I_i|^\alpha < \delta$, so that $\sum \lambda_k^\beta(f[I_i \cap Z]) < \epsilon$. Since this is true for any choice of I_i with endpoints in $J_i \cap Z$, one concludes that $\sum \lambda_k^\beta(f[J_i \cap Z]) < \epsilon$, completing the proof.

Definition 3. A function f is said to be $S(\beta,\alpha)$ on E, written as $f \in S(\beta,\alpha)$ on E, if for each $\epsilon > 0$ there exists $\delta > 0$ such that

$\lambda_\infty^\beta(f[Z]) < \epsilon$ whenever $Z \subset E$ with $\lambda_\infty^\alpha(Z) < \delta$. A function f is said to be $N(\beta,\alpha)$ on E, written as $f \in N(\beta,\alpha)$ on E, if $\lambda_\infty^\beta(f[Z]) = 0$ whenever $Z \subset E$ with $\lambda_\infty^\alpha(Z) = 0$.

Remark 6. The condition S(1,1) is just the Banach's condition (S), and N(1,1) the Lusin's condition (N), and hence $S(\beta,\alpha)$ and $N(\beta,\alpha)$ can be thought as "Hausdorff" variants of (S) and (N), respectively. When λ_∞^β and/or λ_∞^α above are replaced by Λ_β and/or Λ_α, one has true Hausdorff variants. We leave these kind of variants for interested readers to consider.

THEOREM 1. On any set E, one has

(7) $$AC_\infty(\beta,\alpha) \subset S(\beta,\alpha) \subset E_\infty(\beta,\alpha) = N(\beta,\alpha).$$

PROOF. Let $f \in AC_\infty(\beta,\alpha)$ on E, and let $\epsilon > 0$. Then there exists $\delta > 0$ such that $\sum_i \lambda_\infty^\beta(f[I_i \cap E]) < \epsilon$ whenever $\langle I_i \rangle$ is a sequence of non-overlapping closed intervals with endpoints in E and $\sum_i |I_i|^\alpha < \delta$. Now, let $Z \subset E$ be such that $\lambda_\infty^\alpha(Z) < \delta/2$. Then there exists a sequence $\langle J_i \rangle$ of non-overlapping closed intervals which covers Z and $\sum |J_i|^\alpha < \lambda_\infty^\alpha(Z) + \delta/2 < \delta$. Of course, one can assume that $J_i \cap Z \neq \phi$. Then, letting I_i be an interval with endpoints in $J_i \cap Z$, one has $\sum |I_i|^\alpha \leq \sum |J_i|^\alpha < \delta$, so that $\sum_i \lambda_\infty^\beta(f[I_i \cap Z]) < \epsilon$. Since this is true by any choice of I_i with endpoints in $J_i \cap Z$, one concludes that $\sum \lambda_\infty^\beta(f[J_i \cap Z]) < \epsilon$, and hence $\lambda_\infty^\beta(f[Z]) < \epsilon$ by inequality (3), proving that $f \in S(\beta,\alpha)$ on E, and the proof of $AC_\infty(\beta,\alpha) \subset S(\beta,\alpha)$ is completed. To prove the second inclusion in (7), let $f \in S(\beta,\alpha)$ on E, and let $Z \subset E$ with $\lambda_\infty^\alpha(Z) = 0$, and $\epsilon > 0$. Then there exists $\delta_i > 0$ such that $\lambda_\infty^\beta(f[S]) < \epsilon/2^i$, whenever $S \subset E$ with $\lambda_\infty^\alpha(S) < \delta_i$. In particular, $\lambda_\infty^\beta(f[S]) < \epsilon/2^i$ for each $S \subset Z$. Letting $\langle I_i \rangle$ be any sequence of non-overlapping closed intervals which covers Z, one has

$$\sum_i \lambda_\infty^\beta(f[I_i \cap Z]) \le \sum_{i=1}^\infty \epsilon/2^i = \epsilon,$$

Hence $f \in E_\infty(\beta,\alpha)$, proving $S(\beta,\alpha) \subset E_\infty(\beta,\alpha)$. To prove the last equality in (7), let $f \in E_\infty(\beta,\alpha)$ on E, and let $Z \subset E$ with $\lambda_\infty^\alpha(Z) = 0$. For each $\epsilon > 0$, there exists a sequence of non-overlapping closed intervals which covers Z and $\sum \lambda_\infty^\beta(f[I_i \cap Z]) < \epsilon$. Then by (3) one has $\lambda_\infty^\beta(f[Z]) \le \sum \lambda_\infty^\beta(f[I_i \cap Z]) < \epsilon$. Since ϵ is arbitrary, one has $\lambda_\infty^\beta(f[Z]) = 0$, and hence $f \in N(\beta,\alpha)$. Conversely, let $f \in N(\beta,\alpha)$ on E, and let $Z \subset E$ with $\lambda_\infty^\beta(Z) = 0$, and $\epsilon > 0$. Then $\lambda_\infty^\beta(f[S]) = 0$ for each $S \subset Z$. Covering Z with a sequence $\langle I_i \rangle$ of non-overlapping intervals, one has $\lambda_\infty^\beta(f[I_i \cap Z]) = 0$ so that

$$\sum \lambda_\infty^\beta(f[I_i \cap Z]) = 0 < \epsilon,$$

proving that $f \in E_\infty(\beta,\alpha)$.

<u>Remark 7</u>. We conjecture that there are sets E on which each of the inclusion in (7) is strict. However, we have the following result.

<u>Corollary</u>. Let E be such that $\lambda_\infty^\alpha(E) = 0$. Then

(8) $\quad AC_\infty(\beta,\alpha) = S(\beta,\alpha) = E_\infty(\beta,\alpha) = N(\beta,\alpha)$ on E.

<u>Proof</u>. By (7), it suffices to show that $N(\beta,\alpha) \subset AC_\infty(\beta,\alpha)$ on E. Let $f \in N(\beta,\alpha)$ on E, and let $\epsilon > 0$. Since $\lambda_\infty^\alpha(E) = 0$, one has $\lambda_\infty^\alpha(Z) = 0$ for all $Z \subset E$ and hence $\lambda_\infty^\beta(f[Z]) = 0$ for all $Z \subset E$. Let $\langle I_i \rangle$ be any sequence of non-overlapping closed intervals with end points in E. Then $\lambda_\infty^\beta(f[I_i \cap Z) = 0$ for each i, so that $\sum \lambda_\infty^\beta(f[I_i \cap E)) = 0 < \epsilon$, completing the proof.

<u>Remark 8</u>. Now, we come to see how the notions developed here are related to some of the works done recently by Foran [2], Iseki [3] and Ene [1]. First, we prove the following characterization of $AC_n(\beta,\alpha)$.

<u>Theorem 2</u>. For a function f to be $AC_n(\beta,\alpha)$ on E it is necessary and sufficient that for each $\epsilon > 0$ there exists $\delta > 0$ such that for each sequence $\langle I_i \rangle$ of non-overlapping intervals with end points in E and $\sum |I_i|^\alpha < \delta$, one has intervals J_{ij} for $j = 1,2,3, \cdots, n$ such that

$$(9) \qquad B(f;E\cap(\cup I_i)) \subset \bigcup_i \left[\bigcup_{j=1}^{n} (I_i \times J_{ij}) \right]$$

and

$$(10) \qquad \sum_i \left[\sum_{j=1}^{n} |J_{ij}|^\beta \right] < \epsilon.$$

[Here, B(f;A) denotes the graph of f on A.]

Proof. Let $f \in AC_n(\beta,\alpha)$ on E, and let $\epsilon > 0$. Then there exists $\delta > 0$ each that

$$\sum \lambda_n^\beta(f[I_i \cap E]) < \epsilon/2$$

whenever $\langle I_i \rangle$ is a sequence of non-overlapping closed intervals with end points in E and $\sum |I_i|^\alpha < \delta$. For such sequence $\langle I_i \rangle$, there exist intervals J_{ij} for $j = 1,2,3,\cdots,n$ such that $\langle J_{ij} \rangle_{j=1}^\infty$ covers $f[I_i \cap E]$ and $\sum_{j=1}^n |J_{ij}|^\beta < \lambda_n^\beta(f[I_i \cap E]) + \epsilon/2^{i+1}$, $i = 1,2,3, \cdots$. Then (9) and (10) hold. Hence the condition is necessary. That the condition is sufficient follows from the fact that

$$\lambda_n^\beta(S) \le \sum_{j=1}^n |J_i|^\beta \text{ whenever } \langle J_i \rangle_{j=1}^n \text{ covers } S.$$

Remark 9. If the "n" in the condition of Theorem 2 is replaced by "∞" and "some positive integer n_j (depending on j)", respectively, then one obtains a characterization of $AC_\infty(\beta,\alpha)$ and $AC_\omega(\beta,\alpha)$, respectively.

Remark 10. In Foran [2], a function f is said to be AC(n) on E if for each $\epsilon > 0$ there exists $\delta > 0$ such that for each sequence $\langle I_i \rangle$ of non-overlapping intervals with $I_i \cap E \ne \phi$ and $\sum |I_i| < \delta$, one has intervals J_{ij} for $j = 1,2,3, \cdots$, n such that (9) and (10) hold. Thus, we have $AC_n(1,1) = AC(n)$ (and hence $AC_n(\beta,\alpha)$ is a variant of Foran's condition AC(n)). Then by Ene's work in [1], one has for $(\beta,\alpha) = (1,1)$ that $AC_n(\beta,\alpha) \subsetneq AC_{n+1}(\beta,\alpha)$, and

$AC_n(\beta,\alpha)$ is not an additive class for $n \neq 1$ on some set E. We think that all these hold even when $(\beta,\alpha) \neq (1,1)$, but we are not going to investigate them here.

Theorem 3. For each pair of positive integers n, m, one has

$$AC_n(\beta,\alpha) + AC_m(\beta,\alpha) \subset AC_{nm}(\beta,\alpha) \text{ on every set E.}$$

Proof. First, we prove the case $\beta \geq 1$. Let $f \in AC_n(\beta,\alpha)$, $g \in AC_m(\beta,\alpha)$, $h = f + g$ on E, and let $\epsilon > 0$. Let δ_f be the δ determined by the $\epsilon_f = \epsilon/_{n2^\beta}$ and the fact that $f \in AC_n(\beta,\alpha)$ in Theorem 2, and similarly δ_g by $\epsilon_g = \epsilon/m2^\beta$. Then take $\delta_h = \min\{\delta_f,\delta_g\}$. One has $\delta_h > 0$. Now, let $\langle I_i \rangle$ be a sequence of non-overlapping closed intervals with endpoints in E and $\sum |I_i|^\alpha < \delta_h$. Then for each i there exist intervals F_{ij} and G_{ip} for $j = 1,2,3, \cdots, n$ and $p = 1,2,3, \cdots, m$ such that

$$B(f;E\cap[\bigcup_i I_i)) \subset \bigcup_i[\bigcup_{j=1}^{n} F_{ij}] \text{ and } \sum_i [\sum_{j=1}^{n} |F_{ij}|^\beta)] < \epsilon_f,$$

and similar for G_{ip} and g. Now, let

$H_{ijp} = [a_{ij} + c_{ip}, b_{ij} + d_{ip}]$, where $[a_{ij}, b_{ij}] = F_{ij}$, $[c_{ip}, d_{ip}] = G_{ip}$ for $j = 1,2,3, \cdots, n$; $p = 1,2,3, \cdots,m$, and $i = 1,2,3,\cdots$. Then one has

$$B(h;E\cap[\bigcup_i I_i]) \subset \bigcup_i[\bigcup_{p=1}^{m} \bigcup_{j=1}^{n} I_i \times H_{ijp}]$$

and

$$\sum_i [\sum_{j,p} |H_{ijp}|^\beta] \leq \sum_i [\sum_{j,p} 2^{\beta-1}(|F_{ij}|^\beta + |G_{ip}|^\beta)]$$
$$< n2^{\beta-1} \epsilon_f + m2^{\beta-1} \epsilon_g = \epsilon.$$

Hence by theorem 2, $h \in AC_{nm}(\beta,\alpha)$ on E, completing the proof for the case $\beta \geq 1$. For the case $0 < \beta < 1$, the proof is similar, noting that in this case one uses the inequality $(|a| + |b|)^{\beta} \leq |a|^{\beta} + |b|^{\beta}$, which holds for $0 < \beta < 1$. [In the case $\beta \geq 1$, one uses the fact that

$$(|a| + |b|)^{\beta} \leq 2^{\beta-1} (|a|^{\beta} + |b|^{\beta}).]$$

Corollary. Letting $AC_u(\beta,\alpha) = \cup\{AC_n(\beta,\alpha):$ n is a positive integer$\}$, one has that on every set E, $AC_u(\beta,\alpha)$ is an additive class (and in fact is a linear space).

Remark 11. From (4), we have on every set E

$$AC_u(\beta,\alpha) \subset AC_{\omega}(\beta,\alpha) \subset AC_{\infty}(\beta,\alpha). \tag{11}$$

However, there are some sets E on which $AC_{\infty}(\beta,\alpha)$ is not an additive class and hence $AC_u(\beta,\alpha) \neq AC_{\infty}(\beta,\alpha)$. [A similar situation may hold for $AC_{\omega}(\beta,\alpha)$ but we will not consider it here.]

Proof. Mazurkiewicz [4] has constructed a continuous function f which is N(1,1) on [0,1], but f+g is not N(1,1) on [0,1] for any non-constant linear function g. Taking E to be the projection of the set Q in [4] on the x-axis, one sees that $|E| = 0$, and hence by the corollary to theorem 1, one sees that the function f is $AC_{\infty}(1,1)$ on E while f+g is not $AC_{\infty}(1,1)$ on E for any non-constant linear function g. Since every linear function g is $AC_{\infty}(1,1)$, one sees that on E, $AC_{\infty}(1,1)$ is not an additive class. [For

$(\beta,\alpha) \neq (1,1)$, examples remain to be done!]

<u>Theorem 4</u>. For a function to be $E_n(\beta,\alpha)$ on E it is sufficient that for each $Z \subset E$ with $\lambda_\infty^\alpha(Z) = 0$ and for each $\epsilon > 0$ there exists a sequence $\langle I_i \rangle$ of non-overlapping closed intervals which covers Z, and for each i there exists n intervals J_{ij} for $j = 1,2,3, \cdots, n$ such that

$$B(f;Z) \subset \bigcup_i \bigcup_j^n [I_i \times J_{ij}]$$

and

$$\sum_i [\sum_{j=1}^n (|I_i|^\alpha + |J_{ij}|^\beta] < \epsilon.$$

For $\alpha = 1$, the condition is also necessary.

<u>Proof</u>. To prove that the condition is necessary for $\alpha = 1$, let $f \in E_n(\beta,\alpha)$ on E and let $Z \subset E$ with $\lambda_\infty^\alpha(Z) = 0$ and $\epsilon > 0$. Since $\lambda_\infty^\alpha(Z) = 0$, there exists a sequence $\langle K_i \rangle$ of non-overlapping closed intervals whose union contains Z such that $\sum |K_i|^\alpha < \epsilon/2n$. For each i, since $\lambda_\infty^\alpha(Z \cap K_i) = 0$, one has that $f \in D_n(\beta)$ on $Z \cap K_i$ so that there exists a sequence $\langle K_{ij} \rangle_j$ of non-overlapping closed intervals which covers $Z \cap K_i$ and

$$\sum_j \lambda_n^\beta(f[K_{ij} \cap (Z \cap K_i)]) < \epsilon/2^{i+2}.$$

Of course, we may assume that $K_{ij} \subset K_i$ for all j. Now, listing $\langle K_{ij} \rangle_{i,j}$ as $\langle I_p \rangle_p$, one has that $\langle I_p \rangle$ is a sequence of non-overlapping closed intervals which covers Z and

$$\sum_p \lambda_n^\beta(f[I_p \cap Z]) < \frac{\epsilon}{4},$$

and $\sum |I_p|^\alpha \le \sum |K_i|^\alpha < \epsilon/2n$ since $\alpha = 1$.

For each p, there exist intervals J_{pj} for $j = 1,2,3, \cdots, n$ which cover $f[I_p \cap Z]$ and $\sum_{j=1}^{n} |J_{pj}|^\beta < \lambda_n^\beta(f[I_p \cap Z]) + \epsilon/2^{p+2}$, and hence

$$B(f;Z) \subset \bigcup_p [\bigcup_j (I_p \times J_{pj})]$$

and

$$\sum_p [\sum_{j=1}^{n} (|I_p|^\alpha + |J_{pj}|^\beta)] \le n \sum_p |I_p|^\alpha + \sum_p |J_p|^\beta \le \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon,$$

completing the proof of necessary condition for $\alpha = 1$. That the condition is sufficient follows easily from the fact

$\lambda_n^\beta(S) \le \sum_{i=1}^{n} |J_i|^\beta$ whenever $\langle J_i \rangle_{i=1}^{n}$ is n intervals which covers S.

Remark 12. When $\alpha \ne 1$, whether the condition is still necessary remains to be seen.

Remark 13. In Iseki [3], a function f is said to be dwindle on a set E if for each $\epsilon > 0$ there exists a sequence $\langle I_i \rangle$ of open intervals which covers E and such that $\sum \text{diam}(f[I_i \cap E]) < \epsilon$; and a function f is said to be continuous (M) on E if it is dwindle on every null subset of E. It follows from theorem 2 in [3] that a function which is dwindle on E must be $D_1(1)$ on E, and for continuous functions, being dwindle on E is equivalent to being $D_1(1)$ on E. Furthermore, by theorem 7 in [3], a function is continuous (M) on E if and only if it is $E_1(1,1)$ on E.

Remark 14. In Ene [1], a function f is said to be E(n) on E if for each null subset S of E and for each $\epsilon > 0$ there exist rectangles $D_{ij} = I_i \times J_{ij}$ with $j = 1,2,3, \cdots, n$, where $\langle I_i \rangle$ is sequence of non-overlapping intervals, $I_i \cap S \neq \phi$ such that

$$B(f;S) \subset \bigcup_i \left[\bigcup_{j=1}^{n} D_{ij} \right]$$

and

$$\sum_i \left[\sum_{j=1}^{n} \operatorname{diam}(D_{ij}) \right] < \epsilon .$$

It follows easily from theorem 4 that a function is E(n) on E if and only if it is $E_n(1,1)$ on E. Thus, the concept of $E_n(\beta,\alpha)$ is a variant of Ene's E(n), and also of Iseki's continuous (M) as given in remark 13.

Definition 4. For a function property P on sets, we say that a function f is generalized P on E, writing as $f \in GP$ on E, if E can be written as the union of countably many sets on each of which f is P. If each of the countably many sets can be taken as closed set, we say that f is closed generalized P on E, and written as $f \in [GP]$ on E. (Thus, we have properties like $GN(\beta,\alpha)$, $GS(\beta,\alpha)$, $GE_k(\beta,\alpha)$, $GAC_k(\beta,\alpha)$, $[GN(\beta,\alpha)]$, etc.).

Remark 15. It is easily seen that $GN(\beta,\alpha) = N(\beta,\alpha)$. Thus, it follows easily from the inclusions in (4) to (7) and in (11), one has on every set E and for every (β,α) the following:

(12) $GAC_n \subset GAC_{n+1} \subset GAC_u \subset GAC_\omega \subset GAC_\infty \subset GS \subset E_\infty = N = GN,$

(13) $GE_n \subset GE_{n+1} \subset GE_u \subset GE_\omega \subset GE_\infty = E_\infty = N = GN$,

(14) $GAC_k \subset GE_k \subset N = GN$,

and similar inclusions hold for the closed generalized properties, too. Furthermore, one has

(15) $[GP] \subset GP$ for every property P.

Remark 16. It follows from the corollary to theorem 3 that the class $GAC_u(\beta,\alpha)$ is an additive class (and in fact a linear space) of functions. So is the class $GAC_1(\beta,\alpha)$. However, we think that none of the other classes listed in (12) to (14) is additive, in general. (Cf. remarks 4, 10, 11, remark 13 and [3], and also remark 18 below.)

Remark 17. For continuous functions, the classes $GAC_u(1,1)$ and $GE_u(1,1)$ are just the $\mathcal{F}$ in [2] and $\mathcal{E}$ in [1], respectively. For $k \neq \infty$, we have both $GAC_k(\beta,\alpha) \subset GS(\beta,\alpha)$ and $GAC_k(\beta,\alpha) \subset GE_k(\beta,\alpha)$. But the relation between $GS(\beta,\alpha)$ and $GE_k(\beta,\alpha)$ is not clear, and hence some characterizations of their intersection might be of some interest. In particular, one may investigate whether $GS(\beta,\alpha) \cap GE_\omega(\beta,\alpha)$ is additive or not.

Remark 18. We do not investigate classes larger than $N(\beta,\alpha)$ here. However, it is worthwhile to notice (cf Saks [5], chapter IX) that for continuous functions one has $GS(1,1) \subset GN(1,1) = N(1,1) \subset S(1,1) + S(1,1)$ on every interval, and hence in particular one has $S(1,1) \subsetneq GS(1,1) \subsetneq N(1,1)$ and none of them is an additive class.

Remark 19. We end this note by observing some trivial relations between various classes with different indices (β,α). Many interesting questions can be raised and answers to those questions still remain to be investigated. First, note that

(16) $N(\beta,\alpha_1) \subset N(\beta,\alpha_2)$ whenever $\alpha_2 < \alpha_1$.

(17) $N(\beta_2,\alpha) \subset N(\beta_1,\alpha)$ whenever $\beta_2 < \beta_1$.

Hence for $\alpha < \beta$, one has $N(\alpha,\alpha) \cap N(\beta,\beta) \subset N(\beta,\alpha)$,

and $N(\alpha,\alpha) \cup N(\beta,\beta) \supset N(\alpha,\beta)$.

Could any of the inclusions be strict? Similar inclusions and questions can be considered for the classes S, E_k, AC_k, [GS], etc. Also, the question of how to characterize a small class within a larger class remains to be done.

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