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Generalized Integrals as Limits of Riemann-like Sums

1 Introduction

In the early years following the publication of Lebesgue's theory of integration in 1901, attempts were made, without success, to obtain a new integral as a limit of Riemann-like sums. Some partial results were obtained by Borel ([1],[2]), Hahn [11], Lebesgue [18], and others. Fifty years passed before such a definition of the Lebesgue integral was found. The seminal work toward that goal was done by Kurzweil [17] and Henstock ([12],[13]) who, independently introduced an integral (now known as the generalized Riemann integral) ([9],[23]) as a special kind of limit of Riemann sums which, it turned out, is equivalent (see e.g. [13] and [19]) to the Perron and special Denjoy integrals. Then McShane ([21] and [22]) showed that a minor but crucial modification in the Henstock approach would yield the Lebesgue integral.

The purpose of the present paper is to develop the theory of "higher order" generalized Riemann integrals which include the C_nP-integrals of Burkill [4] and which will integrate many everywhere finite generalized derivatives (cf. [6] and [7]).

2 Preliminaries

Let

$$a = x_0 < x_1 < \ldots < x_m = b$$

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be a division (partition) of the interval [a,b], and suppose the numbers z_j are associated with the division by the relation $x_{j-1} \leq z_j \leq x_j$. Such a division, denoted by $\mathcal{D}$ is called a tagged division with tags $z_j, j = 1, 2, \ldots, m$. Suppose, further, there is given a function $\delta(x)$ such that $\delta(x) > 0, x \in [a, b]$. If the tagged division $\mathcal{D}$ has the property that

$$[x_{j-1}, x_j] \subset (z_j - \delta(z_j), z_j + \delta(z_j)), \; j = 1, 2, \ldots, m,$$

then the division is said to be δ-fine. It is known (cf. [12], [23]) that given $\delta(x) > 0$ defined on [a,b], there is a δ-fine tagged division of [a,b].

Definition 2.1 *(cf.[12], [23]) Let f be defined (and finite) on $[a, b]$. The number I is the definite generalized Riemann integral (or the Riemann complete integral) of f on $[a, b]$ if, corresponding to $\epsilon > 0$, there is a $\delta(x) > 0$, $x \in [a, b]$ so that*

$$(2.1) \qquad |I - (\mathcal{D}) \sum_{j=1}^{m} f(z_j)(x_j - x_{j-1})| < \epsilon,$$

for each δ-fine tagged devision $\mathcal{D}$.

It is clear (cf. [23]) that the same integral is obtained by choosing divisions so that $x_0 = a$ is the tag of $[x_0, x_1]$ and x_m is the tag for $[x_{m-1}, x_m]$.

The idea of a restricted tagged division was introduced in [7]. We repeat it here for convenience.

Definition 2.2 *A tagged division of $[a, b]$ will be called a restricted tagged division of $[a, b]$ if it has the form*

$$(2.2) \qquad x_0 = z_1 < x_1 < z_2 < x_2 < \ldots < z_{m-1} < x_{m-1} < z_m = x_m$$

where $x_0 = z_1$ is the tag of $[x_0, x_1], x_m = z_m$ is the tag of $[x_{m-1}, x_m]$ and z_j is the tag of both $[x_{j-1}, z_j]$ and $[z_j, x_j]$ for $j = 2, 3, \ldots, m - 1$. If a restricted tagged division of $[a, b]$ has further the property that $z_j - x_{j-1} = x_j - z_j$, $j = 2, 3, \ldots, m - 1$, the division will be called a restricted symmetric tagged division of $[a, b]$.

It is clear that given $\delta(x) > 0$ defined on [a,b] there exists a δ-fine restricted tagged division of [a,b]. That there exists a δ-fine restricted symmetric tagged division of [a,b] follows from [20].

It is easy to see that if we replace tagged divisions by restricted tagged divisions, the corresponding integral is included in the integral of Definition

2.1. That the inclusion goes the other way (i.e. that the integrals are equivalent) follows from the observation (cf. [12], page 84) that if a tag is an interior point of its interval, we may write

$$f(z_j)(x_j - x_{j-1}) = f(z_j)(x_j - z_j) + f(z_j)(z_j - x_{j-1}), \tag{2.3}$$

while if a δ-fine division has tags that are left hand or right hand end points, we can insert additional points to create a δ-fine restricted tagged division so that the corresponding Riemann sum is arbitrarily close to the Riemann sum formed with respect to the original division. For example if the division has the form

$$a = a_0 < a_1 < a_2 < a_3 < a_4 = b, \tag{2.4}$$

where a_0, a_1, a_2, a_4 are tags, we can form a δ-fine restricted tagged division of [a,b] by inserting points b_1, b_2, b_3, b_4 so that

$$a_0 < b_1 < a_1 < b_2 < a_2 < b_3 < a_3 < b_4 < a_4, \tag{2.5}$$

where $a_1 - b_1 = a_2 - b_2 = a_3 - b_3 = b_4 - a_3 < min(\delta(a_1), \delta(a_2), \delta(a_3))$ and a_0, a_1, a_2, a_3, a_4 are tags. Now by choosing $a_1 - b_1$ small enough, the required approximation is accomplished.

3 Higher order generalized Riemann integrals

If f is a finite function defined on [a,b], let two interval functions be defined by $F_\ell(u,v) \equiv F_\ell(f,u,v) \equiv f(v)(v-u)$ and $F_r(u,v) \equiv F_r(f,u,v) \equiv f(u)(v-u)$. In the following we shall introduce a variety of interval functions, all of which, like the above, depend on a given point function f. Our interval functions will be defined on the intervals of restricted tagged divisions of [a,b]. Where the tag of the interval is the right hand end point, we denote the interval function by $\phi_\ell(f,u,v)$, and where the tag is the left hand end point we denote the interval function by $\phi_r(f,u,v)$. It will be convenient to denote a pair of interval functions by a single letter in script letters. For example, we shall write $\mathcal{A}$(u,v) $=\{\phi_\ell(f,u,v), \phi_r(f,u,v)\}$, or, more briefly, $\mathcal{A} = \{\phi_\ell, \phi_r\}$.

Given a restricted tagged division $\mathcal{D}$ of [a,b] and a pair of interval functions (ϕ_ℓ, ϕ_r) we shall consider sums of the form

$$\sum_{i=1}^{m} \phi_r(z_{i-1}, x_{i-1}) + \phi_\ell(x_{i-1}, z_i)$$

. We shall denote such sums by $(\mathcal{D})\sum \phi_\sigma$ where $\sigma = \ell$ or r depending on whether the tag of the interval is the right hand or left hand end point.

In addition we shall require that the interval functions be linear on the set of point functions and have the property that if if $f \geq 0$ and $\epsilon > 0$ then there exists $\delta(x) > 0$ such that

$$(3.1) \qquad |(\mathcal{D})\sum \phi_\sigma| < \epsilon$$

for all δ-fine restricted (restricted symmetric) tagged divisions $\mathcal{D}$ of [a,b]. A pair of interval functions with these properties will be called regular.

We give below the definitions of the generalized Riemann complete integral and the generalized symmetric Riemann complete integral. (cf. [7]).

Definition 3.1 *The number I will be called the generalized Riemann complete (generalized symmetric Riemann complete) integral of f with respect to the pair of regular interval functions $\mathcal{A}(u,v) = \{\phi_\ell(u,v), \phi_r(u,v)\}$ on $[a,b]$ if, corresponding to $\epsilon > 0$, there is a function $\delta(x) > 0$ so that*

$$(3.2) \qquad |I - (\mathcal{D})\sum(\phi_\sigma + F_\sigma)| < \epsilon$$

for all δ-fine restricted (restricted symmetric) tagged divisions $\mathcal{D}$ where $\sigma = \ell$ or r, depending on whether the tag of the interval is the right hand or left hand end point.

The notation for these integrals is

$$I = (GRC, \mathcal{A})\int_a^b f(t)dt \quad and \quad I = (GSRC, \mathcal{A})\int_a^b f(t)dt,$$

respectively.

4 Properties of the integral

Let $\mathcal{A} = \{\phi_\ell, \phi_r\}$ be two interval functions defined on the intervals of restricted tagged divisions of an interval [a,b] and let $\delta(x) > 0$ be defined on [a,b]. We note that if

$$\mathcal{D}_1 : [z_1, x_1], [x_1, z_2], \ldots, [z_{m-1}, x_{m-1}], [x_{m-1}, z_m]$$

is a δ-fine restricted tagged division of [a,c] and

$$\mathcal{D}_2 : [z_m, x_m], [x_m, z_{m+1}], \ldots, [z_{q-1}, x_{q-1}], [x_{q-1}, z_q]$$

is a δ-fine restricted tagged division of [c,b], then

$$\mathcal{D}_3 : [z_1, x_1], [x_1, z_2], \ldots, [x_{m-1}, z_m], [z_m, x_m], \ldots, [x_{q-1}, z_q]$$

is a δ-fine restricted tagged division of [a,b].

Then it is easy to prove the following theorems. (Regularity of (ϕ_ℓ, ϕ_r) is not required.)

Theorem 4.1 *If f is generalized Riemann complete integrable on $[a, b]$ with respect to $\mathcal{A} = \{\phi_\ell, \phi_r\}$, and if $[c, d] \subset [a, b]$, then f is generalized Riemann integrable on $[c, d]$ with respect to $\mathcal{A}$.*

Theorem 4.2 *If f is generalized Riemann complete (generalized symmetric Riemann complete) integrable with respect to $\mathcal{A} = \{\phi_\ell, \phi_r\}$ on $[a, c]$ and $[c, b]$ with integrals I_1 and I_2, respectively, where $a < c < b$ then f is generalized Riemann complete (generalized symmetric Riemann complete) integrable with respect to $\mathcal{A}$ on $[a, b]$ with integral $I = I_1 + I_2$.*

In order to prove a monotone convergence theorem we need the following result.

Theorem 4.3 *If $f \geq 0$ on $[a, b]$ and f is generalized Riemann complete (generalized symmetric Riemann complete) integrable (with respect to a pair of regular interval functions), then f is generalized Riemann integrable and the values of the integrals are the same (and non-negative).*

Proof. If

$$F(b) - F(a) = (GRC, \mathcal{A}) \int_a^b f(t)\, dt$$

then, for arbitrary $\epsilon > 0$,

$$|F(b) - F(a) - (\mathcal{D}) \sum(\phi_\sigma + F_\sigma)| < \epsilon/2$$

and

$$|(\mathcal{D}) \sum \phi_\sigma| < \epsilon/2$$

for sums over suitably chosen divisions D. It follows that

$$|F(b) - F(a) - (\mathcal{D}) \sum F_\sigma| \leq$$

$$|F(b) - F(a) - (\mathcal{D}) \sum(\phi_\sigma + F_\sigma)| + |(\mathcal{D}) \sum \phi_\sigma| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Since $\epsilon > 0$ is arbitrary, the result follows.

Theorem 4.4 *(Monotone convergence theorem) Let $\{f_n(x)\}$ be a sequence of (Lebesque) measurable functions that are generalized Riemann complete (generalized symmetric Riemann complete) integrable with respect to a pair of regular interval functions $\mathcal{A}(u,v) = \{\phi_\ell(u,v), \phi_r(u,v)\}$ on $[a,b]$. Suppose that $f_n(x) \leq f_{n+1}(x)$ and that f_n converges to a finite limit function$f(x)$. If the sequence of integrals $(GRC, \mathcal{A}) \int_a^b f_n(x)\,dx$ converges to a number I, then $f(x)$ is generalized Riemann complete (generalized symmetric Riemann complete) integrable with respect to the pair of regular interval functions $A(u,v)$ and*

$$(GRC, \mathcal{A}) \int_a^b f(x)\,dx = I.$$

Proof. For fixed positive integer N, n>N, let

$$g_n(x) = f_n(x) - f_N(x). \tag{4.1}$$

Then $g_n(x) \geq 0$ and by Theorem 4.3, $g_n(x)$ is generalized Riemann integrable. Since $g_n(x) \leq g_{n+1}(x)$ and $\lim_{n\to\infty} g_n(x) = f(x) - f_N(x)$, the result follows from the monotone convergence theorem for the generalized Riemann integral.

5 A-variational equivalence

Let H and f be given point functions, defined and finite on a finite interval [a,b], and let $\mathcal{A} \equiv \{\phi_\ell, \phi_r\} \equiv \{\phi_\ell(f,u,v), \phi_r(f,u,v)\}$ be two regular interval functions (depending on f) defined on the subintervals of [a,b]. Let $\mathcal{D}$ denote a restricted (restricted symmetric) tagged division of type (2.1) and define an interval function d(u,v) on the intervals of $\mathcal{D}$ as follows:

$$d(u,v) = |H(v) - H(u) - \phi_\ell(f,u,v) - f(v)(v-u)|,$$

if [u,v] is one of the intervals of $\mathcal{D}$ of the form $[x_i, z_{i+1}], i = 1, 2, \ldots, m-1$, and

$$|d(u,v) = |H(v) - H(u) - \phi_r(f,u,v) - f(u)(v-u)|$$

if [u,v] is one of the intervals of $\mathcal{D}$ of the form $[z_i, x_i], i = 1, 2, \ldots, m-1$. Now consider sums of the form $\sum d(u,v)$ over restricted (restricted symmetric) tagged divisions of [a,b] where [u,v] has alternatively the form $[z_i, x_i]$ and $[x_i, z_{i+1}], i = 1, 2, \ldots, m-1$. Let $\delta = \delta(x) > 0$ and define V by

$$V = V(H, \mathcal{A}, f; \delta; [a,b]) \equiv sup \sum d(u,v)$$

where the supremum is taken over all δ-fine restricted (restricted symmetric) tagged divisions of [a,b].

Definition 5.1 *The function H is $\mathcal{A}$-variationally ($\mathcal{A}$-symmetric variationally) equivalent to the function f if given $\epsilon > 0$ there exists $\delta \equiv \delta(x) > 0$ such that*

$$V = V(H, \mathcal{A}, f, \delta, [a, b]) < \epsilon,$$

where V is defined with respect to δ-fine restricted (restricted symmetric) divisions of $[a, b]$.

It is easy to see that an equivalent definition of $\mathcal{A}$-variational ($\mathcal{A}$-symmetric variational) equivalence is the following (cf. [14]).

Definition 5.2 *The function H is $\mathcal{A}$-variationally ($\mathcal{A}$-symmetric variationally) equivalent to the function f if, given $\epsilon > 0$, there exists $\delta \equiv \delta(x) > 0$ and a monotone increasing function χ such that*

$$0 = \chi(a) < \chi(b) < \epsilon$$

and

$$d(u, v) \leq \chi(v) - \chi(u),$$

for intervals (u,v) in all δ-fine restricted (restricted symmetric) tagged divisions.

If H is $\mathcal{A}$-variationally ($\mathcal{A}$-symmetric variationally) equivalent to f on [a,b] the difference H(b) - H(a) will be called the $\mathcal{A}$-variational ($\mathcal{A}$-symmetric variational) integral of f on [a,b].

Theorem 5.1 *If H is a point function defined on $[a, b]$ and H is $\mathcal{A}$-variationally ($\mathcal{A}$-symmetric variationally) equivalent to the function f on $[a, b]$, then the generalized Riemann complete (generalized symmetric Riemann complete) integral of f with respect to $\mathcal{A} \equiv \{\phi_\ell, \phi_r\} \equiv \{\phi_\ell(f, u, v), \phi_r(f, u, v)\}$ exists and equals the variational integral $H(b)$ - $H(a)$. Conversely, if the generalized Riemann complete (generalized symmetric Riemann complete) integral of f with respect to $\mathcal{A} = (\phi_\ell, \phi_r) = \{\phi_\ell(f, u, v), \phi_r(f, u, v)\}$ exists, then so does the corresponding $\mathcal{A}$-variational ($\mathcal{A}$-symmetric variational) integral and they are equal.*

Proof. Let $\epsilon > 0$ be given. Then since H is $\mathcal{A}$-variationally ($\mathcal{A}$-symmetric variationally) equivalent to f, there exists $\delta(x) > 0$ such that

$|H(b) - H(a) - (\mathcal{D}) \sum(\phi_\sigma + F_\sigma)| =$

$$|(\mathcal{D})\sum(H(v)-H(u)-\phi_\sigma(f,u,v)-F_\sigma(u,v)|\le$$

$$(\mathcal{D})\sum|(H(v)-H(u)-\phi_\sigma(f,u,v)-F_\sigma(u,v)|\le\chi(b)-\chi(a)<\epsilon,$$

for all δ-fine divisions $\mathcal{D}$, and this proves the first part of the theorem.

Now let

$$H(u,v)=(GRC,\mathcal{A})\int_u^v f(t)\,dt, \tag{5.1}$$

and the proof of the second part of the theorem follows from Definitions (3.1) and (5.1).

6 The C_nP-integral and the generalized Riemann complete integral.

The C_nP-integral of a finite point function may be defined in the following way (cf.[4]). Assuming the $C_{n-1}P$-integral has been defined, where the C_0P-integral is the Perron integral, we define the n^{th} Cesàro integral mean of a function g by

$$C_n(g,x,x+h)=n/h^n\int_x^{x+h}(x+h-\xi)^{n-1}g(\xi)\,d\xi,$$

the integral being a $C_{n-1}P$-integral. (We take $C_0(g,x,x+h)$ to mean g(x+h).) If $C_n(g,x_0,x_0+h)\to g(x_0)$, as $h\to 0$, g is said to be C_n-continuous at x_0. If

$$lim\left(\frac{C_n(g,x,x+h)-g(x)}{h/(n+1)}\right)$$

as $h\to 0$ exists, it is denoted by $C_nDg(x)$. Otherwise the upper and lower limits of the expressions are denoted by $C_n\overline{D}g(x)$ and $C_n\underline{D}g(x)$, and called the upper and lower C_n-derivatives, respectively.

Definition 6.1 *$M(x)$ and $m(x)$ are C_n-major and minor functions, respectively, of $f(x)$ in $[a,b]$ if*

$$M(x)\ and\ m(x)\ are\ C_n-continuous, x\in[a,b]; \tag{6.1}$$

$$M(a)=m(a)=0; \tag{6.2}$$

$$C_n\underline{D}M(x)\ge f(x)\ge C_n\overline{D}m(x), x\in[a,b]. \tag{6.3}$$

It can be shown [4] that for any major or minor functions M(x) and m(x), respectively, the difference M(x) - m(x) is monotonic increasing, and so M(b) $\geq$ m(b) and $+\infty > inf\ M(b) \geq sup\ m(b) > -\infty$. If inf M(b) = sup m(b), this value is defined to be the C_nP-integral of f on [a,b], denoted by

$$(C_nP \int_a^b f(t)\, dt.$$

Theorem 6.1 *Suppose f is finite and C_nP-integrable on $[a,b]$, and let*

$$F(x) = C_nP \int_a^x f(t)\, dt.$$

Then $F(x)$ is $\mathcal{A}$-variationally equivalent to f on $[a,b]$ where $\mathcal{A} \equiv \mathcal{A}^{(n)} \equiv (\phi_\ell, \phi_r)$ is defined for $C_{n-1}P$-integrable F by

(6.4) $\phi_\ell \equiv \phi_\ell(f,n,u,v) = (n+1)C_n(F,v,u) - nF(v) - F(u) \equiv L(F),$

and

(6.5)$\phi_r \equiv \phi_r(f,n,u,v) = -(n+1)C_n(F,u,v) + nF(u) + F(v) \equiv R(F).$

<u>Proof.</u> Let $\overline{M}(x)$ be a C_n-major function of f(x) on [a,b]. Let $\eta > 0$ be arbitrary. Then by (6.3) we have

$$C_n\underline{D}\overline{M}(x) > f(x) - \eta, \quad x \in [a,b],$$

and it is easy to show that there exists $\delta_1(x) > 0$ such that $M(x) = \overline{M}(x) + \eta x$ satisfies the inequalities

(6.6) $$M(u) \geq M(x) + R(M,x,u) + f(x)(u-x),$$

if $0 \leq u - x \leq \delta_1(x)$, and

(6.7) $$M(t) \leq M(x) - L(M,x,t) + f(x)(t-x),$$

if $0 \geq t - x \geq -\delta_1(x)$.

Similarly, if $\overline{m}(x)$ is a C_n-minor function and $m(x) = \overline{m}(x) - \eta x$, it can be shown that there exists $\delta_2(x) > 0$ such that

(6.8) $$m(u) \leq m(x) + R(m,x,u) + f(x)(u-x)$$

if $0 \leq u - x \leq \delta_2(x)$, and

(6.9) $$m(t) \geq m(x) - L(m,x,t) + f(x)(t-x),$$

if $0 \geq t - x \geq -\delta_2(x)$. It is also clear that (6.6)-(6.9) are satisfied for $\delta(x)$ = min $(\delta_1(x), \delta_2(x))$. Now if $0 \leq u - x \leq \delta(x)$,then

$F(u) - F(x) = (M(u) - M(x)) - (M(u) - F(u)) + (M(x) - F(x)) \geq$

$R(M, x, u) + f(x)(u - x) - (M(u) - F(u)) + (M(x) - F(x)) =$

$\phi_r(f, n, x, u) + f(x)(u-x) - (M(u) - F(u)) + (M(x) - F(x)) + R(M-F, x, u) \geq$

$\phi_r(f, n, x, u) + f(x)(u - x) + (n+1)\{(M(x) - m(x)) - (M(u) - m(u))\},$

since M - F is increasing and so

$R(M - F, x, u) \geq$

$-(n+1)(M-F)(u) + n(M-F)(x) + (M-F)(u) = n[(M-F)(x) - (M-F)(u)],$

and

$$(M - F)(x) - (M - F)(u) \geq (M - m)(x) - (M - m)(u).$$

In a similar way we obtain for $0 \leq u - x \leq \delta(x)$,

$F(u) - F(x) = (m(u) - m(x)) - (m(u) - F(u)) + (m(x) - F(x)) \leq$

$\phi_r(f, n, x, u) + f(x)(u - x) + (n+1)\{(M(u) - m(u)) - (M(x) - m(x))\}.$

We have therefore shown that if $0 \leq u - x \leq \delta(x)$, then

$$|F(u) - F(x) - \phi_r(f, n, x, u) - f(x)(u - x)| \leq$$

$$(n+1)[(M(u) - m(u)) - (M(x) - m(x))].$$

It follows in a similar way that if $0 \leq x - t \leq \delta(x)$ then

$$|F(x) - F(t) - \phi_\ell(f, n, t, x) - f(x)(x - t)| \leq$$

$$(n+1)[(M(x) - m(x)) - (M(t) - m(t))].$$

Now, given $\epsilon > 0$, we may choose C_nP-major and minor functions M(x) and m(x) so that (6.6)-(6.9) are satisfied and so that

$$M(b) - m(b) < \epsilon/(n+1)$$

Thus

$$V(F, \mathcal{A}, f; \delta; [a, b]) \leq (n+1)[M(b) - m(b)] \leq \epsilon$$

This shows that F is $\mathcal{A}$-variationally equivalent to f on [a,b].

Now we show that $\mathcal{A} = (\phi_\ell, \phi_r)$ as defined above is regular. The condition of linearity is obviously satisfied. Moreover it is known that if $f \geq 0$ and C_nP-integrable, then f is generalized Riemann integrable [4]. It follows that given $\epsilon > 0$, there exists $\delta > 0$, such that for all δ-fine restricted tagged divisions $\mathcal{D}$ of [a,b] we have at the same time

$$|I - (\mathcal{D}) \sum (\phi_\sigma + F_\sigma)| < \epsilon/2$$

and

$$|(I - (\mathcal{D}) \sum F_\sigma)| < \epsilon/2$$

and this implies that for these divisions $|(\mathcal{D}) \sum \phi_\sigma| < \epsilon$.

Theorem 6.2 *If f is finite and C_nP-integrable on $[a,b]$ then f is generalized Riemann complete integrable with respect to the pair of interval functions $\mathcal{A} = \{\phi_\ell, \phi_r\}$ defined in Theorem 6.1, and the integrals are equal.*

Proof. The proof follows from Theorem 6.1 and 5.1.

Corollary 6.1 *If f is generalized Riemann integrable then f is GRC- integrable and the integrals are equal.*

7 The P^n-and $\mathcal{P}^n$-integrals.

Since the P^n- and $\mathcal{P}^n$-integrals ([8],[15],[16]) are n-fold integrals while the generalized Riemann complete integral represents a one- step integration process, no direct comparison between the P^n- or $\mathcal{P}^n$-integral and the GRC-integral can be expected.

On the other hand since $\mathcal{P}^n$-integrability on [a,b] implies $C_{n-1}P$-integrability on any closed sub-interval $[\alpha, \beta]$ (Theorem 11.1, [15]), we have by Theorem 6.2 the following:

Theorem 7.1 *If f(x) is finite and $\mathcal{P}^{n+1}$-integrable over $(a_i; x)$ and $[\alpha, \beta]$ is any closed sub-interval of (a, a_{n+1}), then f(x) is GRC-integrable with respect to an appropriate pair of interval functions $\mathcal{A}^{(n)} = \{\phi_\ell, \phi_r\}$ over $[\alpha, \beta]$. If*

$$F(x) = (-1)^s \int_{(a_i)}^{x} f(x)\, d_{n+1}x, \qquad a_s \leq x < a_{s+1},$$

then F(x) has generalized (Peano) derivatives([10], [23]), $F_{(k)}(x), 1 \leq k \leq n$, in $[\alpha, \beta]$, and

$$F_{(n)}(\beta) - F_{(n)}(\alpha) = (GRC, \mathcal{A}^{(n)}) \int_{\alpha}^{\beta} f(x)\, dx.$$

(It has been shown moreover that the GRC-integral integrates everywhere finite Peano and C_nP-derivatives [7].)

8 The SC_nP-integrals

Because the SC_nP-integrals ([3],[5]) are defined only almost everywhere, they do not appear to have a definition in terms of Riemann sums.

On the other hand it was shown in [7] that the GSRC-integral integrates everywhere finite de la Vallée Poussin and SC_n-derivatives.

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